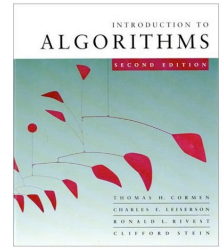
How to Multiply

integers, matrices, and polynomials

COS 423
Spring 2007

slides by Kevin Wayne

Convolution and FFT



Chapter 30

Fourier Analysis

Fourier theorem. [Fourier, Dirichlet, Riemann] Any periodic function can be expressed as the sum of a series of sinusoids.

sufficiently smooth

$$y(t) = \frac{2}{\pi} \sum_{k=1}^{N} \frac{\sin kt}{k} \qquad N = 100$$

Euler's Identity

Sinusoids. Sum of sines and cosines.

$$e^{ix} = \cos x + i \sin x$$

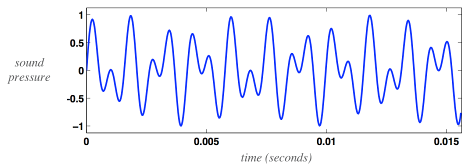
Euler's identity

Sinusoids. Sum of complex exponentials.

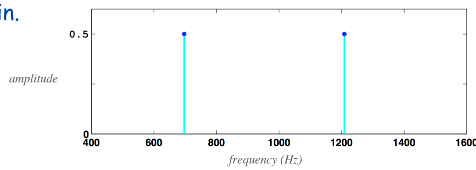
Time Domain vs. Frequency Domain

Signal. [touch tone button 1] $y(t) = \frac{1}{2}\sin(2\pi \cdot 697\ t) + \frac{1}{2}\sin(2\pi \cdot 1209\ t)$

Time domain.



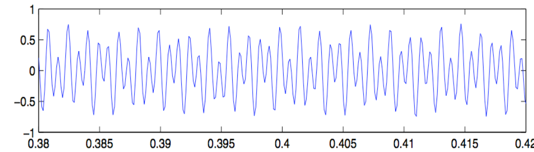
Frequency domain.



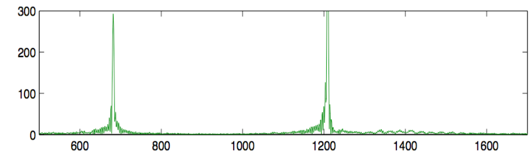
Reference: Cleve Moler, Numerical Computing with MATLAB

Time Domain vs. Frequency Domain

Signal. [recording, 8192 samples per second]



Magnitude of discrete Fourier transform.



Reference: Cleve Moler, Numerical Computing with MATLAB

Fast Fourier Transform

FFT. Fast way to convert between time-domain and frequency-domain.

Alternate viewpoint. Fast way to multiply and evaluate polynomials.

↖ we take this approach

> *" If you speed up any nontrivial algorithm by a factor of a million or so the world will beat a path towards finding useful applications for it. "* — Numerical Recipes

Fast Fourier Transform: Applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry…
- Digital media. [DVD, JPEG, MP3, H.264]
- Medical diagnostics. [MRI, CT, PET scans, ultrasound]
- Numerical solutions to Poisson's equation.
- Shor's quantum factoring algorithm.
- …

> *" The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. "* — Charles van Loan

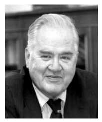
Fast Fourier Transform: Brief History

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge-König (1924). Laid theoretical groundwork.

Danielson-Lanczos (1942). Efficient algorithm, x-ray crystallography.

Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.



Importance not fully realized until advent of digital computers.

Polynomials: Coefficient Representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add. $O(n)$ arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate. $O(n)$ using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1})) \cdots))$$

Multiply (convolve). $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

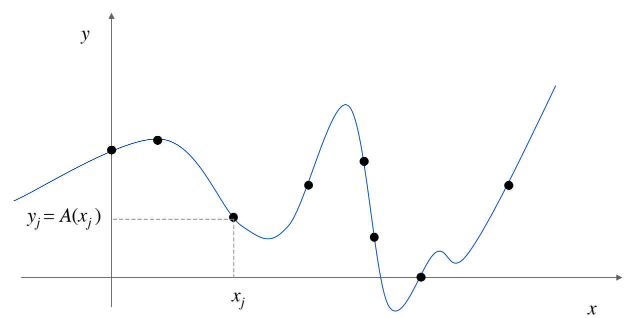
A Modest PhD Dissertation Title

> *" New Proof of the Theorem That Every Algebraic Rational Integral Function In One Variable can be Resolved into Real Factors of the First or the Second Degree. "*
>
> *— PhD dissertation, 1799 the University of Helmstedt*

Polynomials: Point-Value Representation

Fundamental theorem of algebra. [Gauss, PhD thesis] A degree n polynomial with complex coefficients has exactly n complex roots.

Corollary. A degree n-1 polynomial $A(x)$ is uniquely specified by its evaluation at n distinct values of x.



Polynomials: Point-Value Representation

Polynomial. [point-value representation]

$$A(x): (x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$
$$B(x): (x_0, z_0), \ldots, (x_{n-1}, z_{n-1})$$

Add. $O(n)$ arithmetic operations.

$$A(x)+B(x): (x_0, y_0+z_0), \ldots, (x_{n-1}, y_{n-1}+z_{n-1})$$

Multiply (convolve). $O(n)$, but need $2n$-1 points.

$$A(x) \times B(x): (x_0, y_0 \times z_0), \ldots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

Evaluate. $O(n^2)$ using Lagrange's formula.

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k}(x - x_j)}{\prod_{j \neq k}(x_k - x_j)}$$

Converting Between Two Representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

Goal. Efficient conversion between two representations $\Rightarrow$ all ops fast.

$a_0, a_1, \ldots, a_{n-1}$ ⇄ $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$

coefficient representation *point-value representation*

Converting Between Two Representations: Brute Force

Coefficient $\Rightarrow$ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Running time. $O(n^2)$ for matrix-vector multiply (or n Horner's).

Converting Between Two Representations: Brute Force

Point-value $\Rightarrow$ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible iff x_i distinct

Running time. $O(n^3)$ for Gaussian elimination.

or $O(n^{2.376})$ via fast matrix multiplication

Divide-and-Conquer

Decimation in frequency. Break up polynomial into low and high powers.

- $A(x) \quad = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{low}(x) \quad = a_0 + a_1x + a_2x^2 + a_3x^3$.
- $A_{high}(x) = a_4 + a_5x + a_6x^2 + a_7x^3$.
- $A(x) = A_{low}(x) + x^4 A_{high}(x)$.

Decimation in time. Break polynomial up into even and odd powers.

- $A(x) \quad = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

Coefficient to Point-Value Representation: Intuition

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1x + ... + a_{n-1}x^{n-1}$, evaluate it at n distinct points $x_0, ..., x_{n-1}$.

we get to choose which ones!

Divide. Break polynomial up into even and odd powers.

- $A(x) \quad = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

Intuition. Choose two points to be ± 1.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.

Can evaluate polynomial of degree $\le n$ at 2 points by evaluating two polynomials of degree $\le \frac{1}{2}n$ at 1 point.

Coefficient to Point-Value Representation: Intuition

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- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

Intuition. Choose four complex points to be $\pm 1, \pm i$.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.
- $A(i) = A_{even}(-1) + i\, A_{odd}(-1)$.
- $A(-i) = A_{even}(-1) - i\, A_{odd}(-1)$.

Can evaluate polynomial of degree $\le n$ at 4 points by evaluating two polynomials of degree $\le \frac{1}{2}n$ at 2 points.

Discrete Fourier Transform

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1x + ... + a_{n-1}x^{n-1}$, evaluate it at n distinct points $x_0, ..., x_{n-1}$.

Key idea. Choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

DFT

Fourier matrix F_n

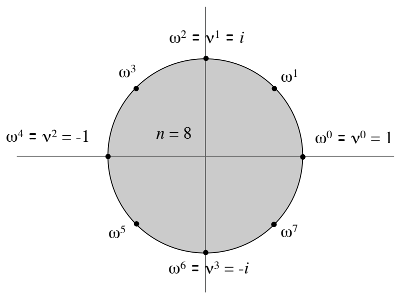
Roots of Unity

Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = \omega^2 = e^{4\pi i / n}$.



Fast Fourier Transform

Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + \ldots + a_{n-1} x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break up polynomial into even and odd powers.

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n-2} x^{n/2 - 1}$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n-1} x^{n/2 - 1}$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

Conquer. Evaluate $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}n^{th}$ roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

Combine.

- $A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$
- $A(\omega^{k+\frac{1}{2}n}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\nu^k = (\omega^k)^2$

$\nu^k = (\omega^{k+\frac{1}{2}n})^2$ $\quad$ $\omega^{k+\frac{1}{2}n} = -\omega^k$

FFT Algorithm

```
fft(n, a0,a1,…,an-1) {
   if (n == 1) return a0

   (e0,e1,…,en/2-1) ← FFT(n/2, a0,a2,a4,…,an-2)
   (d0,d1,…,dn/2-1) ← FFT(n/2, a1,a3,a5,…,an-1)

   for k = 0 to n/2 - 1 {
      ωk ← e2πik/n
      yk     ← ek + ωk dk
      yk+n/2 ← ek - ωk dk
   }

   return (y0,y1,…,yn-1)
}
```

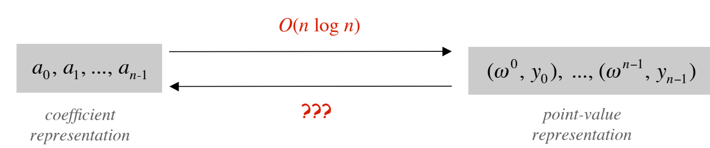

FFT Summary

Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the n^{th} roots of unity in $O(n \log n)$ steps.

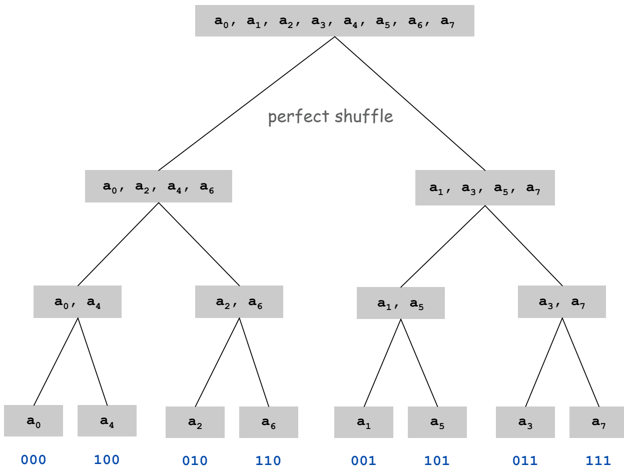
assumes n is a power of 2

Running time.

$$T(n) = 2T(n/2) + \Theta(n) \;\Rightarrow\; T(n) = \Theta(n \log n)$$



Recursion Tree



bit-reversed order

Fourier Matrix Decomposition

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_4 = \begin{bmatrix} \omega^0 & 0 & 0 & 0 \\ 0 & \omega^1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{bmatrix} \qquad a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$y = F_n a = \begin{bmatrix} I_{n/2} & D_{n/2} \\ I_{n/2} & -D_{n/2} \end{bmatrix} \begin{bmatrix} F_{n/2}\, a_{even} \\ F_{n/2}\, a_{odd} \end{bmatrix}$$

Inverse FFT

Inverse Discrete Fourier Transform

Point-value $\Rightarrow$ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

↑ Inverse DFT ↑ Fourier matrix inverse $(F_n)^{-1}$

Inverse DFT

Claim. Inverse of Fourier matrix F_n is given by following formula.

$$G_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

$\frac{1}{\sqrt{n}} F_n$ is unitary

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide by n).

Inverse FFT: Proof of Correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n\, G_n\right)_{k k'} = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{kj}\,\omega^{-jk'} = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{(k-k')j} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \Rightarrow$ series sums to n.
- Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)(1 + x + x^2 + ... + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \Rightarrow$ series sums to 0. ▪

Inverse FFT: Algorithm

```
ifft(n, a0,a1,…,an-1) {
   if (n == 1) return a0

   (e0,e1,…,en/2-1) ← FFT(n/2, a0,a2,a4,…,an-2)
   (d0,d1,…,dn/2-1) ← FFT(n/2, a1,a3,a5,…,an-1)

   for k = 0 to n/2 - 1 {
      ωk ← e-2πik/n
      yk+n/2 ← (ek + ωk dk) / n
      yk+n/2 ← (ek - ωk dk) / n
   }

   return (y0,y1,…,yn-1)
}
```


Inverse FFT Summary

Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the n^{th} roots of unity in $O(n \log n)$ steps.

assumes n is a power of 2

$a_0, a_1, \ldots, a_{n-1}$ — $O(n \log n)$ → $(\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$

$a_0, a_1, \ldots, a_{n-1}$ ← $O(n \log n)$ — $(\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$

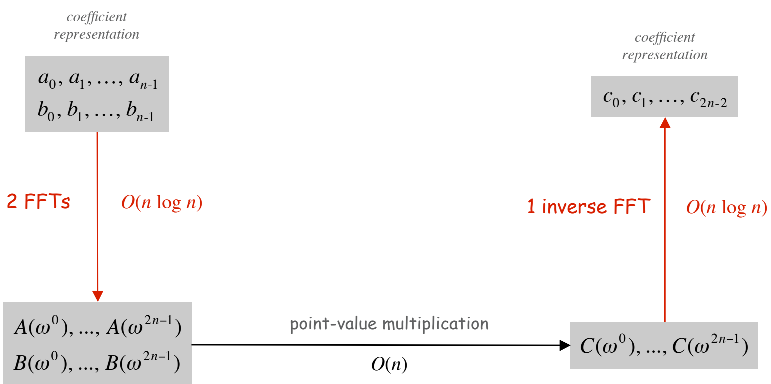
coefficient representation

point-value representation

Polynomial Multiplication

Theorem. Can multiply two degree n-1 polynomials in $O(n \log n)$ steps.

pad with 0s to make n a power of 2



FFT in Practice ?

April 24, 2007

FFT in Practice

Fastest Fourier transform in the West. [Frigo and Johnson]

- Optimized C library.
- Features: DFT, DCT, real, complex, any size, any dimension.
- Won Wilkinson Prize '99.
- Portable, competitive with vendor-tuned code.

Implementation details.

- Instead of executing predetermined algorithm, it evaluates your hardware and uses a special-purpose compiler to generate an optimized algorithm catered to "shape" of the problem.
- Core algorithm is nonrecursive version of Cooley-Tukey.
- $O(n \log n)$, even for prime sizes.

FFTW

http://www.fftw.org

Integer Arithmetic

Integer Multiplication, Redux

Integer multiplication. Given two n bit integers $a = a_{n-1} \ldots a_1 a_0$ and $b = b_{n-1} \ldots b_1 b_0$, compute their product ab.

Convolution algorithm.

- Form two polynomials.
- Note: $a = A(2)$, $b = B(2)$.
- Compute $C(x) = A(x)\ B(x)$.
- Evaluate $C(2) = ab$.
- Running time: $O(n \log n)$ complex arithmetic operations.

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Theory. [Schönhage-Strassen 1971] $O(n \log n \log \log n)$ bit operations.

Integer Multiplication, Redux

Integer multiplication. Given two n bit integers $a = a_{n-1} \ldots a_1 a_0$ and $b = b_{n-1} \ldots b_1 b_0$, compute their product ab.

"the fastest bignum library on the planet"

Practice. [GNU Multiple Precision Arithmetic Library]
It uses brute force, Karatsuba, and FFT, depending on the size of n.

GMP
«Arithmetic without limitations»

http://gmplib.org

Integer Arithmetic

Fundamental open question. What is complexity of arithmetic?

operation	upper bound	lower bound
addition	$O(n)$	$\Omega(n)$
multiplication	$O(n \log n \log \log n)$	$\Omega(n)$
division	$O(n \log n \log \log n)$	$\Omega(n)$

Factoring

Factoring. Given an n-bit integer, find its prime factorization.

$2^{67}-1 = 147573952589676412927 = 193707721 \times 761838257287$

a disproof of Mersenne's conjecture that $2^{67} - 1$ is prime

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740375634795617128280467960974295731425931888892312890849
362326389727650340282662768919964196251178439958943305021
275853701189680982867331732731089309005525051168770632990
72396380786710086096962537934650563796359
```

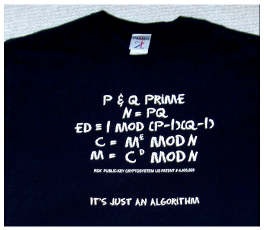
RSA-704
($30,000 prize if you can factor)

Factoring and RSA

Primality. Given an n-bit integer, is it prime?
Factoring. Given an n-bit integer, find its prime factorization.

Significance. Efficient primality testing $\Rightarrow$ can implement RSA.
Significance. Efficient factoring $\Rightarrow$ can break RSA.

Theorem. Poly-time algorithm for primality testing.



Shor's Algorithm

Shor's algorithm. Can factor an n-bit integer in $O(n^3)$ time on a quantum computer.

algorithm uses quantum QFT !

Ramification. At least one of the following is wrong:

- RSA is secure.
- Textbook quantum mechanics.
- Extended Church-Turing thesis.

Peter Shor

Shor's Factoring Algorithm

Period finding.

2^i	1	2	4	8	16	32	64	128	...
2^i mod 15	1	2	4	8	1	2	4	8	...
2^i mod 21	1	2	4	8	16	11	1	2	...

period = 4
period = 6

Theorem. [Euler] Let p and q be prime, and let $n = p\,q$. Then, the following sequence repeats with a period divisible by $(p\text{-}1)\,(q\text{-}1)$:

$$x \bmod n,\ x^2 \bmod n,\ x^3 \bmod n,\ x^4 \bmod n,\ \ldots$$

Consequence. If we can learn something about the period of the sequence, we can learn something about the divisors of $(p\text{-}1)\,(q\text{-}1)$.

use random values of x to get divisors of $(p\text{-}1)\,(q\text{-}1)$,
from this, can get the divisors of $n = p\,q$