# Reductions

How do we show that a problem

is easy?

Reduce each instance to one (or more)

instances of a known easy problem.

$$I_1 \in P_1 \qquad R(I_1) = I_2 \in P_2$$

To solve $I_1$ in $P_2$:

1. Apply $R$ to turn $I_1$ into $I_2 \in P_2$

2. Apply algorithm for $P_2$ to $I_2$.

Cost = cost of applying $R$ plus

cost of applying $P_2$ algorithm.

Examples

Reduction to some form of matrix multiplication:

Transitive closure

Context-free language recognition

Reduction to linear programming

Reduction to network flow

etc.

Cost of reduction?

linear time, quadratic time, ...

Positive use of reduction:

$$? \implies easy$$

Reduce a problem of unknown complexity

to an easy problem

"Negative" use of reduction:

How do we show that a problem of

unknown complexity is hard?

Reduce a hard problem to it

$$hard \implies ?$$

If the questionable problem were easy, so would be

the hard problem   XX

Reductions in both directions show

computational equivalence (up to the

cost of the reduction)

$$P_1 \Longleftrightarrow P_2$$

both are easy or both are hard

transitivity of reductions

$$P_1 \Rightarrow P_2 \Rightarrow P_3$$
$$R_1 \qquad R_2$$

gives a reduction from $P_1$ to $P_3$

Cost is cost of $R_2(R_1())$

both p-time, overall p-time
   - linear                linear

$$R_1 \qquad R_2$$

$$n \qquad a_n \qquad b_n$$

$$a_n \qquad (ab)_n$$

$$n \qquad a_n^2 \qquad b_n^2$$

$$a_n^2 \qquad b(a_n^2)^2 = b_a^2 n^4$$

$$a_n^k \qquad b_n^l$$

$$b(a_n^k)^l = b_a^l n^{kl}$$

Satisfiability: Is a Boolean (logical) function true for some choice of variable assignments?

$$(x \vee y) \wedge (\bar{x} \vee \bar{y}) \qquad \text{sat}: x=1, y=0$$

$\wedge$    and

$\vee$    or

$-$    not

$x$    variable

$x, \bar{x}$    literal

$(x \vee \bar{y} \vee z)$   clause: disjunction ("or") of literals

$(x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$   conjunctive normal form:
           conjunct ("and") of clauses

$x \vee \bar{x}$          tautology: true for all choices
                      of variables

$F$ is sat iff $\bar{F}$ is not a tautology:
                 can be falsified

# Reduction of CNF sat to 3-CNF sat
## (at most 3 literals/clause)

$$(x \vee y \vee z \vee w \vee u \vee v) \Rightarrow$$

$$(x \vee y \vee a) \wedge (\bar{a} \vee z \vee b) \wedge (\bar{b} \vee w \vee c)$$

$$\wedge (\bar{c} \vee u \vee v)$$

Needs $k-3$ extra vars per clause of length $k$.

Graph coloring reducible to sat,

and vice-versa  (p-time reductions)

Must phrase graph coloring as a yes-no
question: can graph $G$ be colored with
$k$ colors?

$G$: $n$ vertices, $m$ edges

$F$: $nk$ variables $x_{ij}$, one per vertex per color

$x_{ij}$ true iff vertex $i$ colored color $j$

Classes:

Each vertex colored
$$(x_{i1} \vee x_{i2} \vee \cdots \vee x_{ik}) \quad i \in V \qquad\qquad n$$

[ No vertex colored twice
$$(\bar{x}_{ij} \vee \bar{x}_{il}) \quad i \in V, \; j \neq l \text{ colors} \qquad n\binom{k}{2} \;]$$
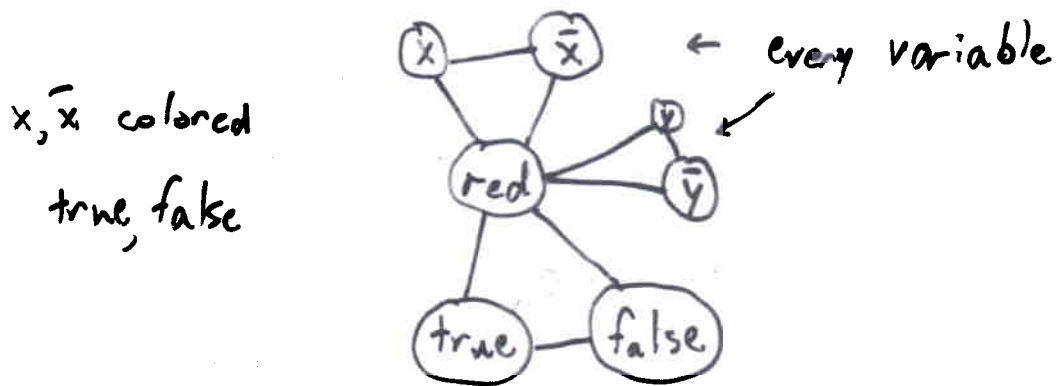
No adjacent vertices the same color

$$(\bar{x}_{il} \vee \bar{x}_{jl}) \quad (i,j) \in E, \; l \text{ a color} \quad mk$$

# literals $= nk + 2n\binom{k}{2} + 2mk$

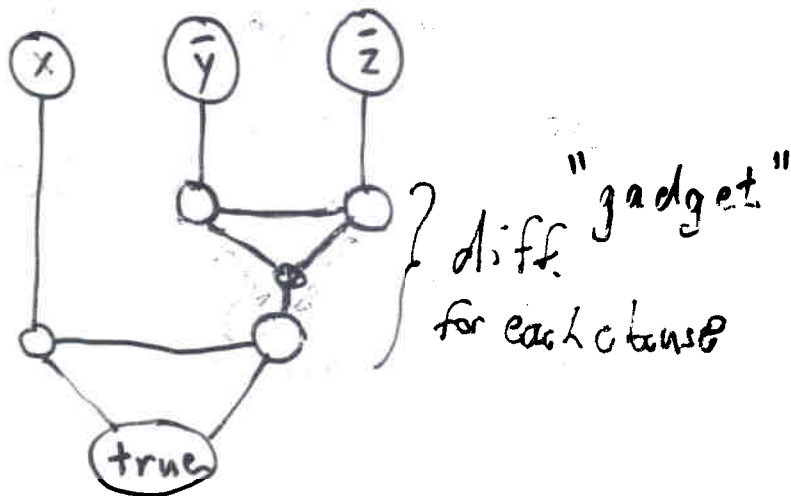Vice-versa: ( 3 - sat )

Reduction to 3 - coloring

Vertices: $x, \bar{x}$, true, false, red, 5 per clause

x, x̄ colored
true, false



← every variable

Clause $x \vee \bar{y} \vee \bar{z}$



} diff. "gadget"
for each clause

Colorable iff formula satisfiable

$$\text{Clique} \Rightarrow \text{Sat}$$

$$X_{ij} \qquad i \in V, \quad 1 \le j \le k$$

$$x_{ij} \text{ true} = \text{vertex } i \text{ is } j^{th} \text{ in clique}$$

$$i, \ell \in V \text{ not adj.}$$

$$(\bar{x}_{ij} \vee \hat{x}_{\ell j'}) \quad \forall j, j'$$

$$(\bar{x}_{ij} \vee \hat{x}_{ij'}) \quad i \ne i' \quad \forall j$$

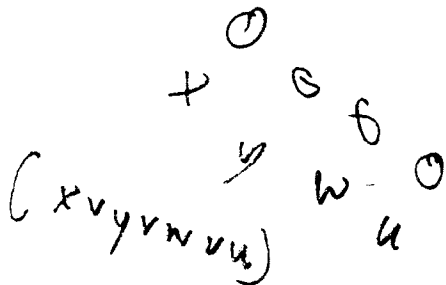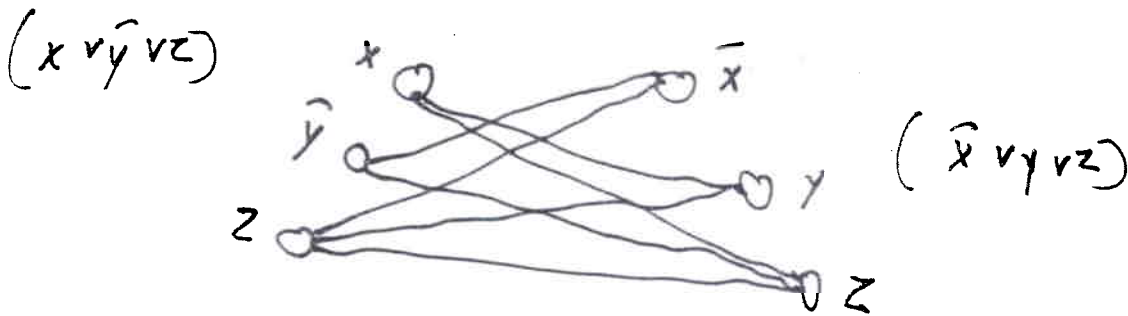$$(x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}) \quad \forall j$$

Sat $\Rightarrow$ Clique

Given a graph, are there $k$ pairwise adjacent vertices?

One vertex per literal occurrence, two vertices in different clauses joined by an edge if compatible (not $x, \bar{x}$)

$k = $ # clauses

$(x \vee \bar{y} \vee z)$



$(\bar{x} \vee y \vee z)$

$(x \vee y \vee w \vee u_4)$

Clique $\Longleftrightarrow$ Independent set

Are there $k$ pairwise nonadjacent vertices?

Complement graph

Clique $\Longleftrightarrow$ Vertex cover

Are there $k$ vertices "covering" all edges

$S$ a vertex cover in $G$ iff

$V-S$ is an independent set in $G$ iff

$V-S$ is a clique in $\bar{G}$