Top-Down Analysis of Path Compression:

Deriving the Inverse-Ackermann Bound Naturally (and Easily)

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Universität des Saarlandes

Theorem: (Tarjan 1975)

Any sequence of m Union, Find operations in a universe of n elements that uses linking by rank and path compression takes time at most

$$O(m \cdot \alpha(m,n) + n) .$$

Ackermann function - Wikipedia, the free encyclopedia - Mozilla Firefox

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http://en.wikipedia.org/wiki/Ackerman's_function

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A two-parameter variation of the inverse Ackermann function can be defined as follows:

$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log_2 n\}.$$

This function arises in more precise analyses of the algorithms mentioned above, and gives a more refined time bound. In the disjoint-set data structure, *m* represents the number of operations while *n* represents the number of elements; in the minimum spanning tree algorithm, *m* represents the number of edges while *n* represents the number of vertices. Several slightly different definitions of α(*m*, *n*) exist; for example, $\log_2 n$ is sometimes replaced by *n*, and the floor function is sometimes replaced by a ceiling.

Fertig

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Definition and properties

[edit]

The Ackermann function is defined recursively for non-negative integers *m* and *n* as follows:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

The Ackermann function can be calculated by a simple function based directly on the definition:

Fertig

I am not smart enough to understand this easily.

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I am not smart enough to come up with proofs (or even reproduce proofs) involving the inverse Ackermann function

based on this definition.

What do I tell my students ?

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$\alpha(m,n)$ grows veeeeery slowly ....

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Let's move on to the next subject !

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Warm-up example: Partial sum problem in the semi-group setting

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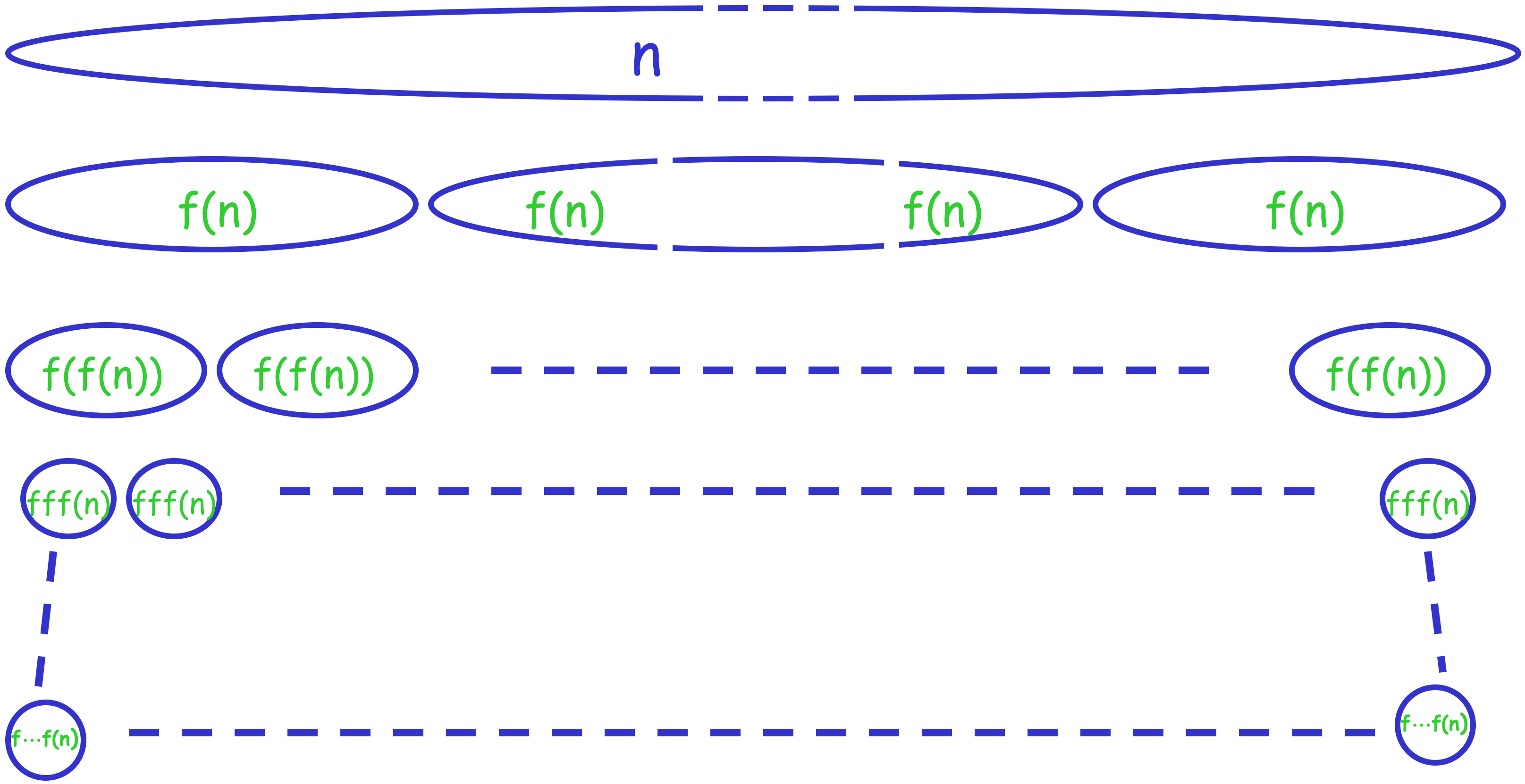
Warm-up example: Partial sum problem in the semi-group setting

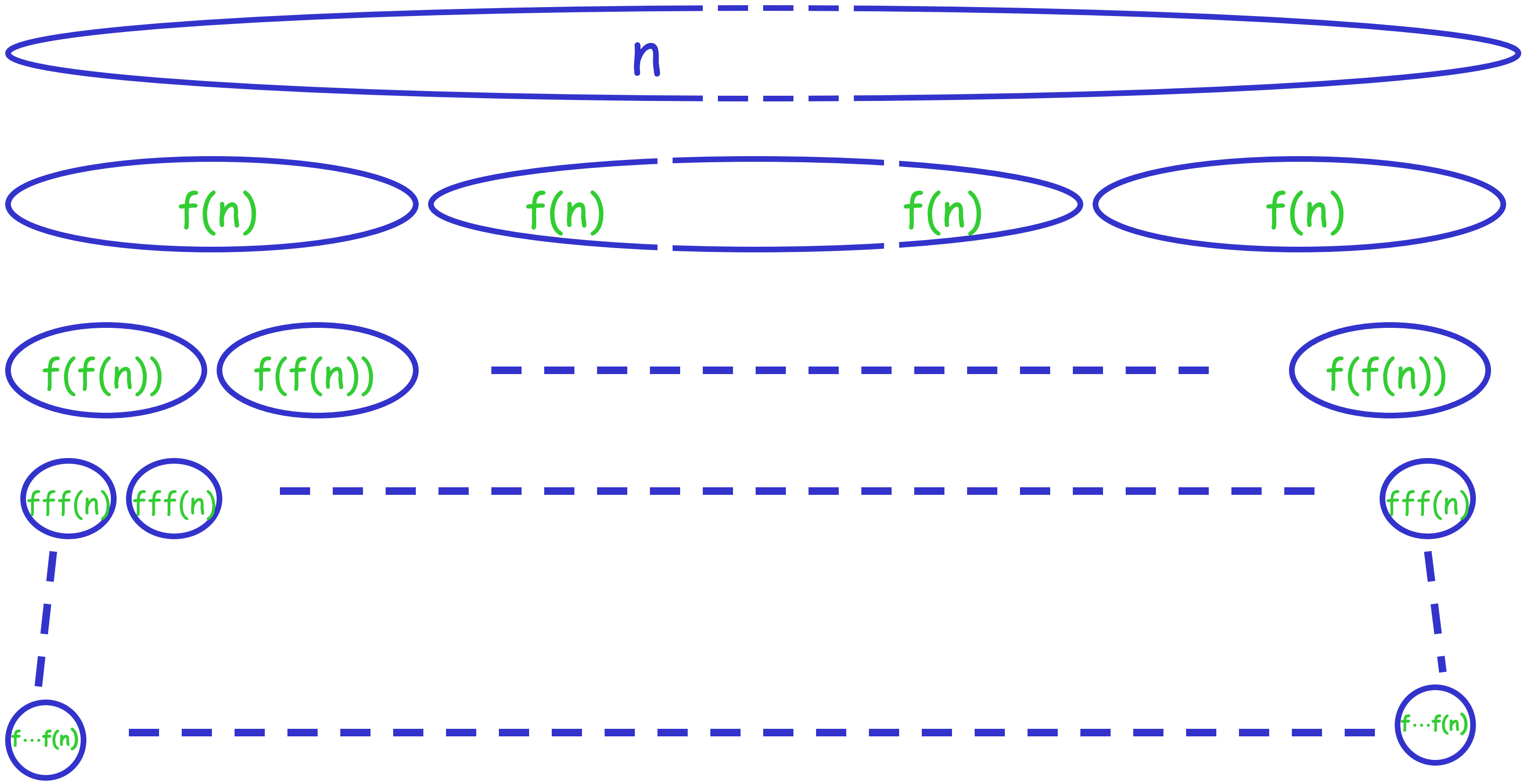
Main example: Union Find with Path Compression

Divide-and-Conquer Recurrences, Baby Version



Recurrence:

$$X(n) \leq \begin{cases} 0 & \text{if } n \leq 1 \\ a \cdot n + \dfrac{n}{f(n)} \cdot X(f(n)) & \text{if } n > 1 \end{cases}$$



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Solution: $X(n) \le a \cdot n \cdot f^{*}(n)$

$$f^*(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + f^*(\ f(n)\ ) & \text{if } n > 1 \end{cases}$$

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Properties: 1) $f^*(\,f(n)\,) = f^*(n) - 1$

2) f a "nice" compaction (i.e. $f(n) < n$)

$\Rightarrow$ f^* a "nice" compaction and

f^* "much smaller" than f

Examples for f^* :

$f(n)$	$f^*(n)$
$n-1$	$n-1$
$n-2$	$n/2$
$n-c$	n/c
$n/2$	$\log_2 n$
n/c	$\log_c n$
$\sqrt{n}$	$\log \log n$
$\log n$	$\log^* n$

Partial sum problem in the semi-group setting

Data: $A_1, A_2, \cdots, A_n \in$ "Semigroup" $(G,+)$

Query: i,j **Answer**: $A_i + A_{i+1} + \cdots + A_j$

"partial sum"

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$S_0(n) = \binom{n+1}{2}$

Example semi-groups $(G,+)$:

$(\mathbb{R},$ max$)$

$(\mathbb{R}^n,$ componentwise-max$)$

$($ $d \times d$ matrices, mult $)$

Claim: $S_1(n) =$

Claim: $S_1(n) = n \log_2 n$

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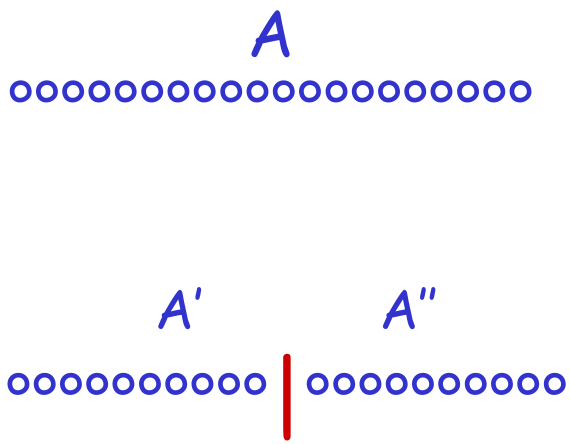
"1-op-structure"

case $n=1$: trivial

case $n \geq 2$: use recursive construction


A

partition A-sequence into
2 subsequences A' and A''
of length $n/2$ each



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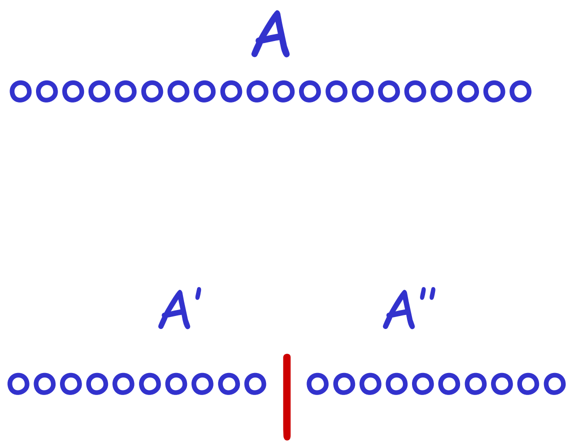

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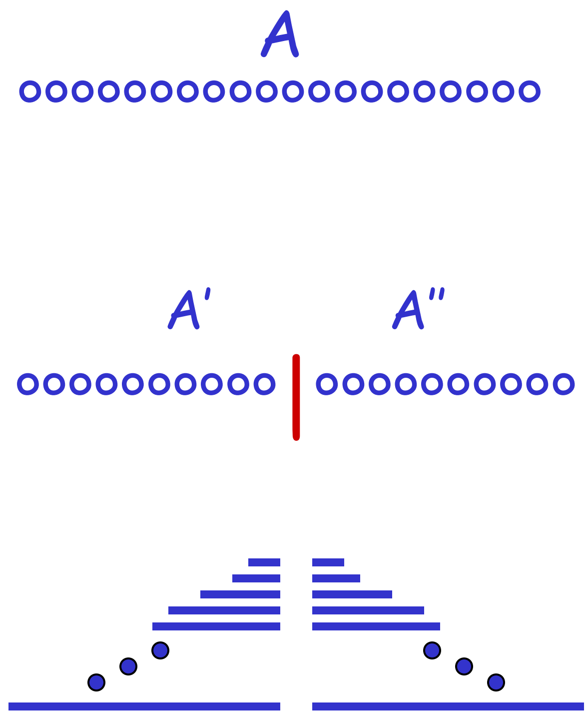

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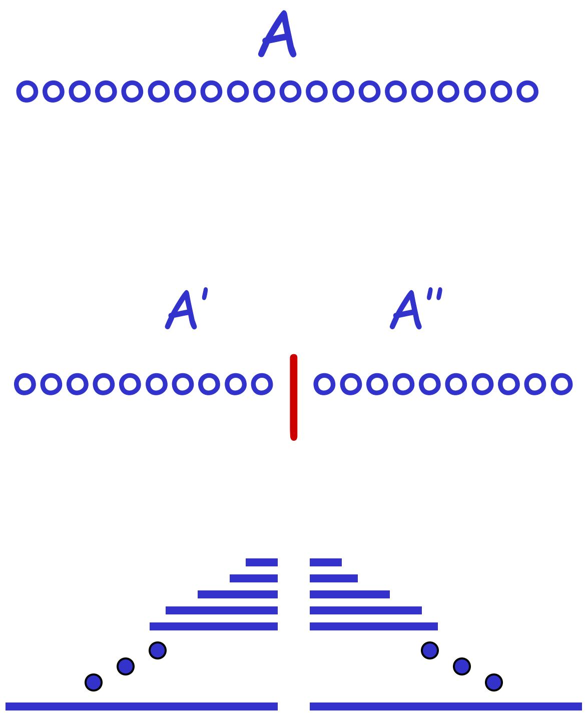

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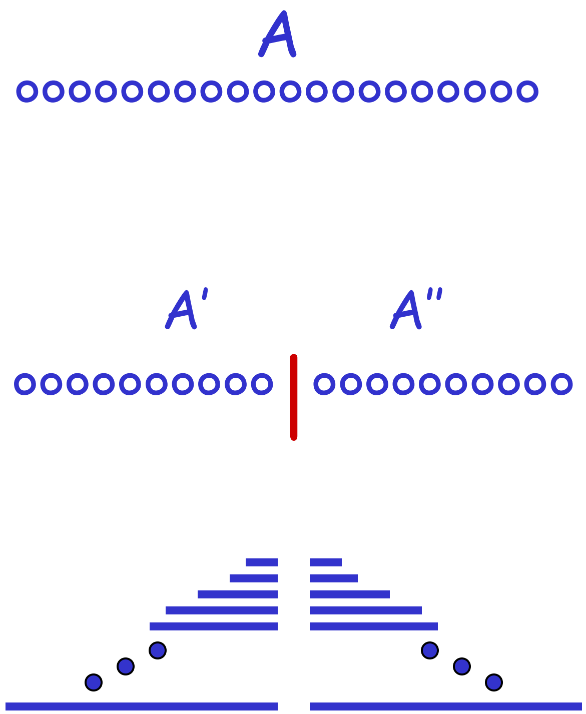

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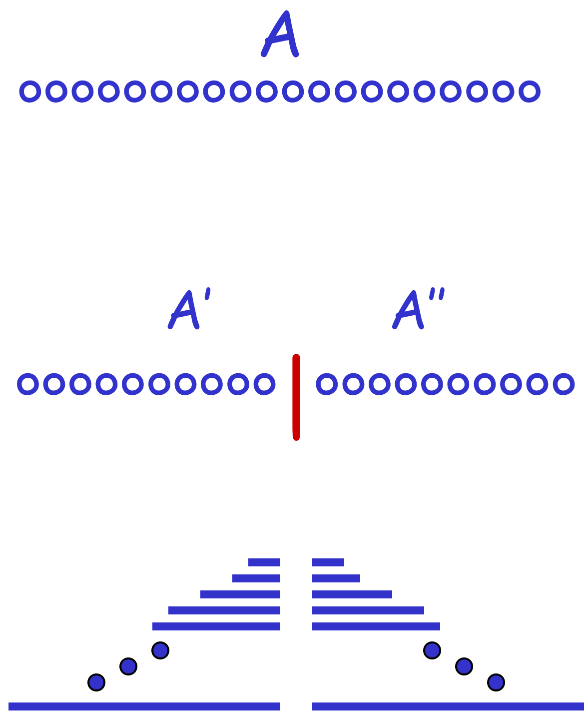
Query answering:

either return (suffix-sum)+(prefix-sum)
or use one of the recursive structures



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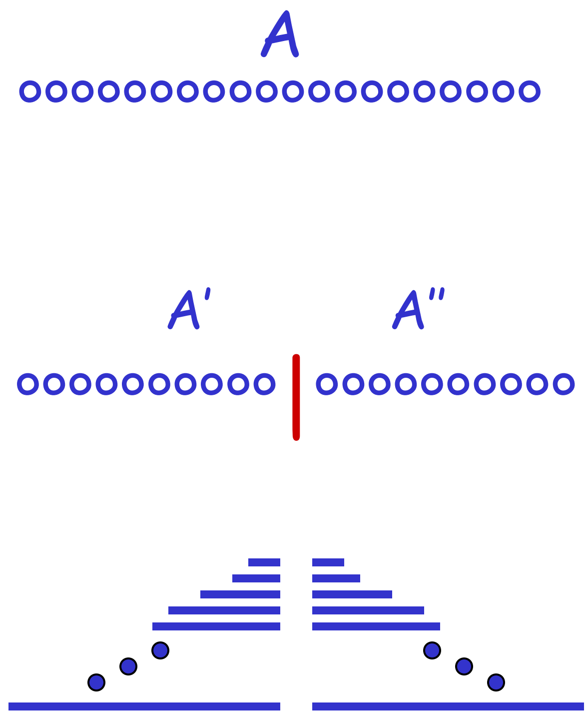
$$S_1(n) \leq n + \frac{n}{(n/2)} S_1(n/2)$$



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$$S_1(n) \leq n + \frac{n}{(n/2)} S_1(\, n/2 \,)$$

$$\Rightarrow \; S_1(n) \leq n \cdot (n/2)^* = n \log_2 n$$

$S_3(n) = ?$

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"3-op-structure"

case $n \leq 4$:   trivial

case $n \geq 5$:   use recursive construction

A

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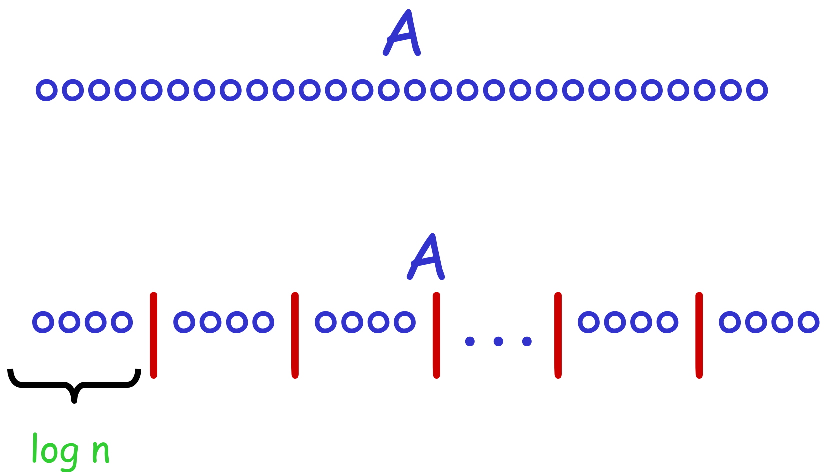
partition A-sequence into
n/log n subsequences of length
≤ log n each



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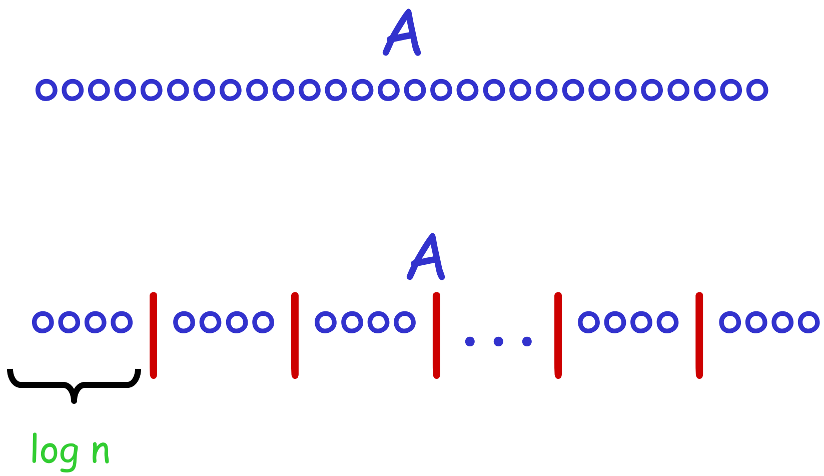

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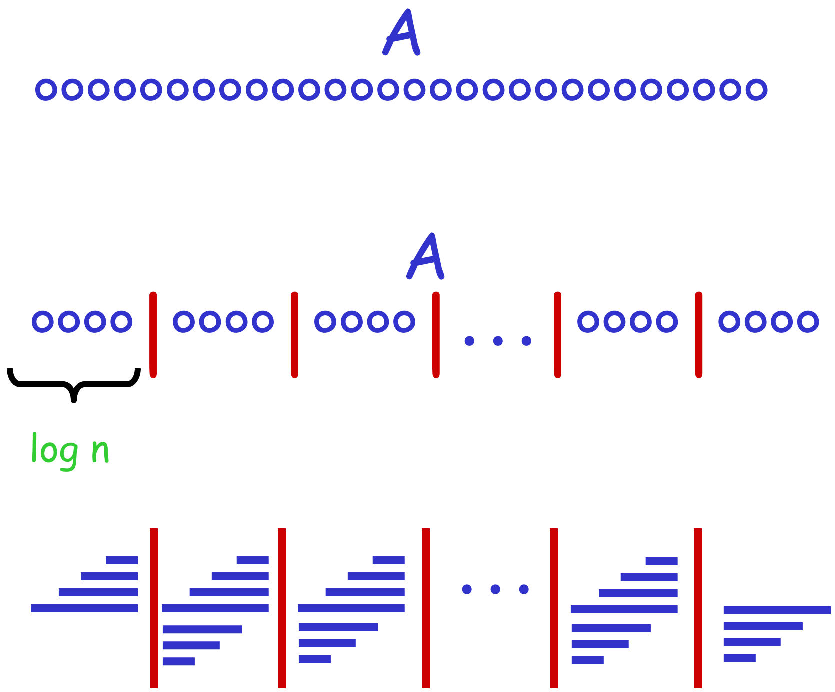

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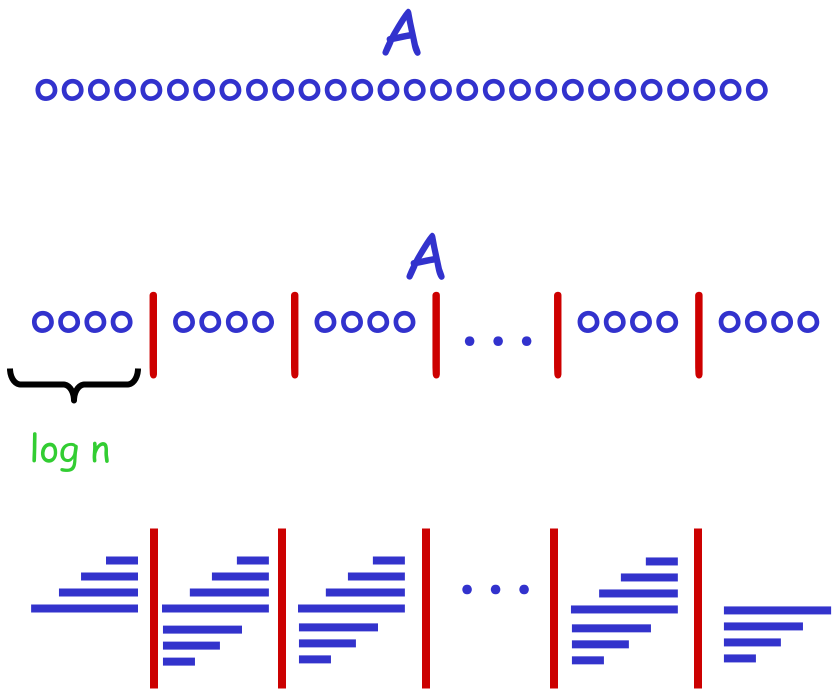

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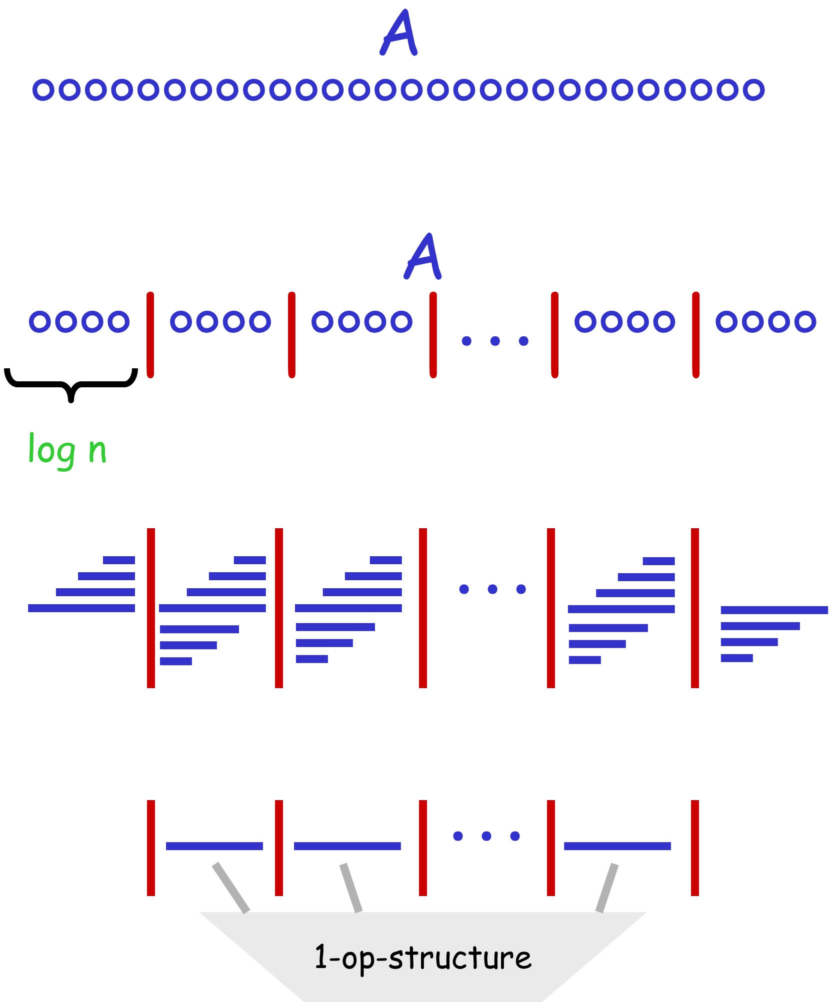

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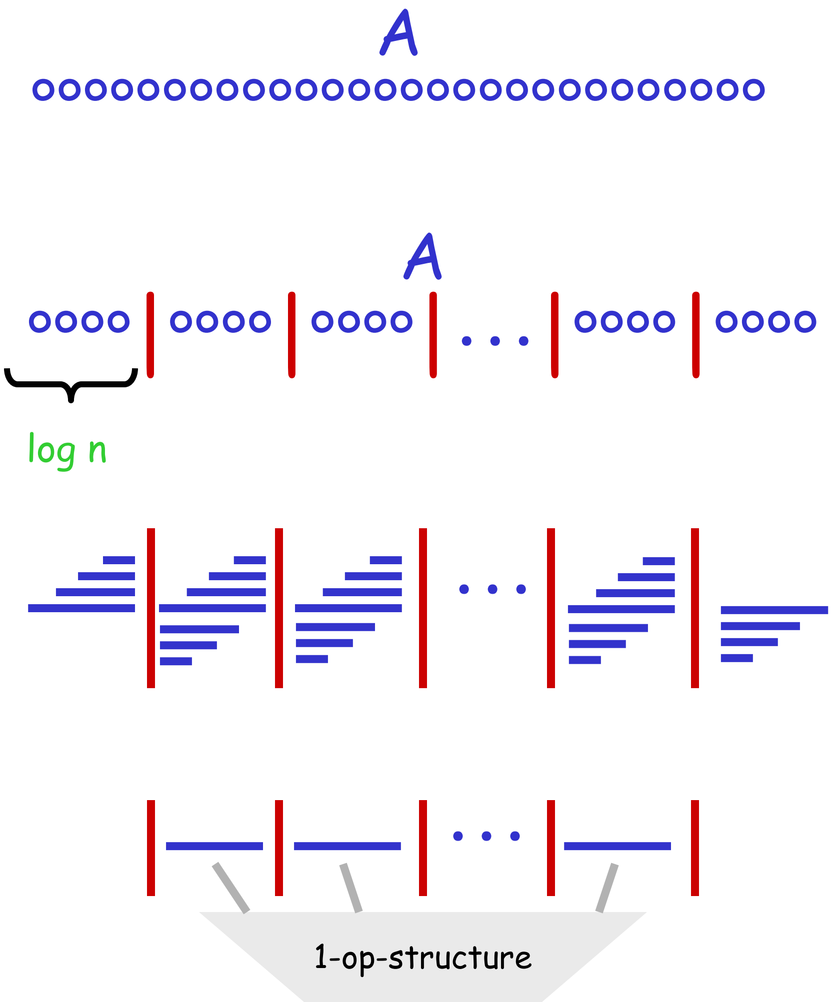

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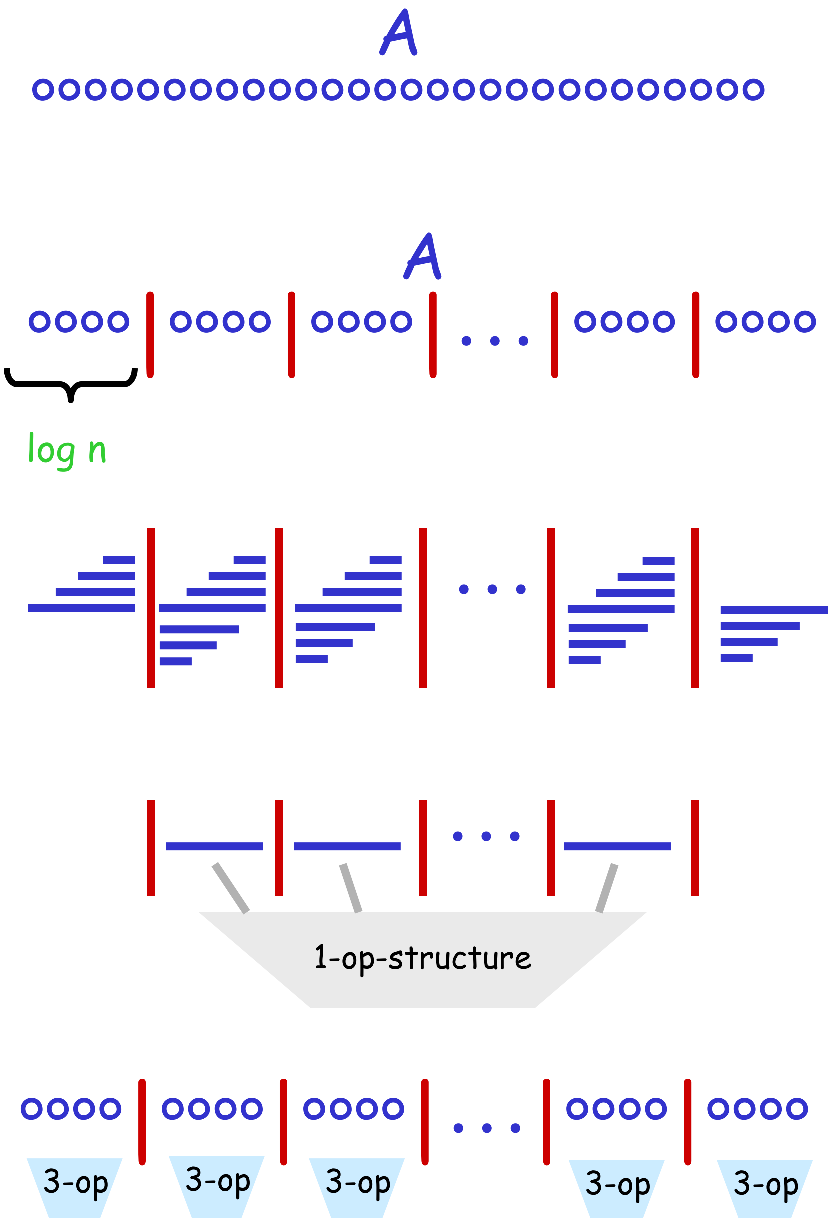

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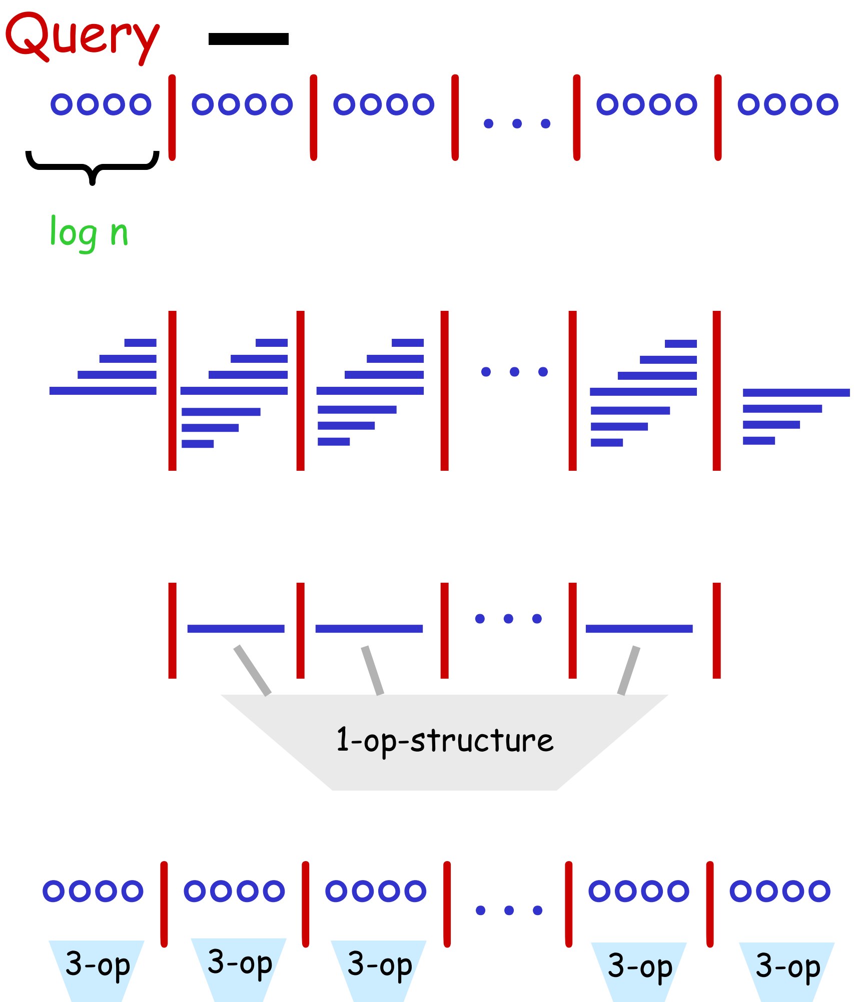

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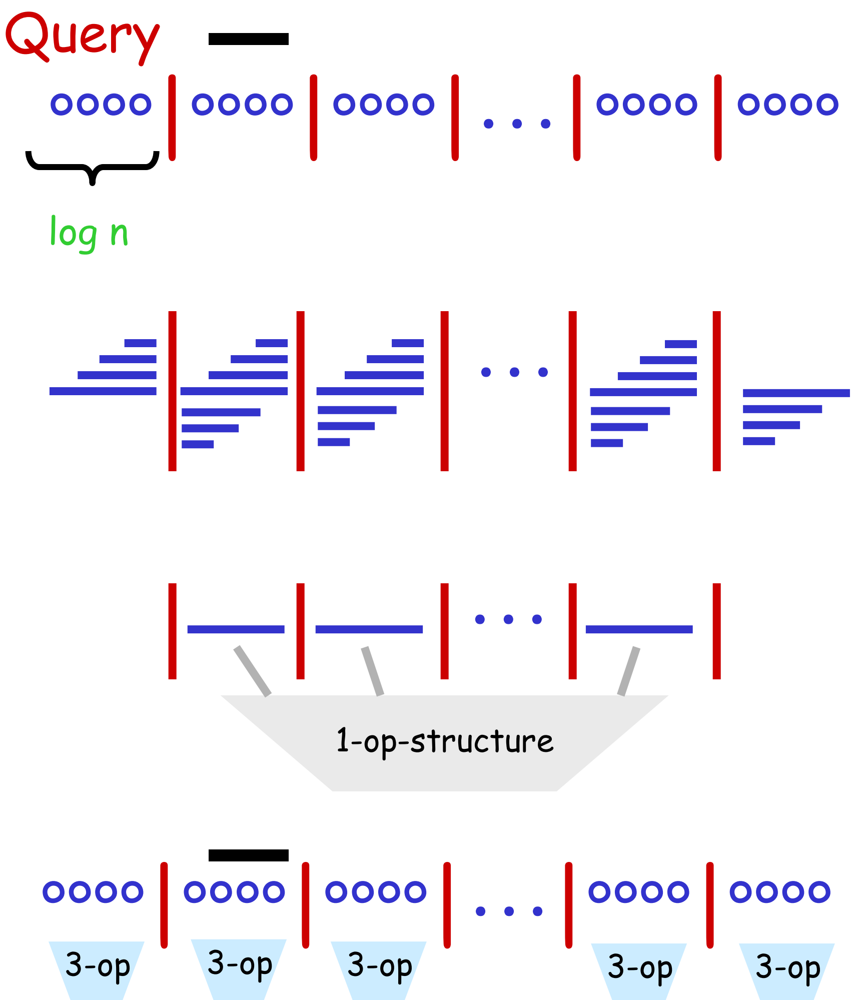
either use one of the recursive 3-op-structures

or return (suffix-sum)+(answer from 1-op-structure)+(prefix-sum)



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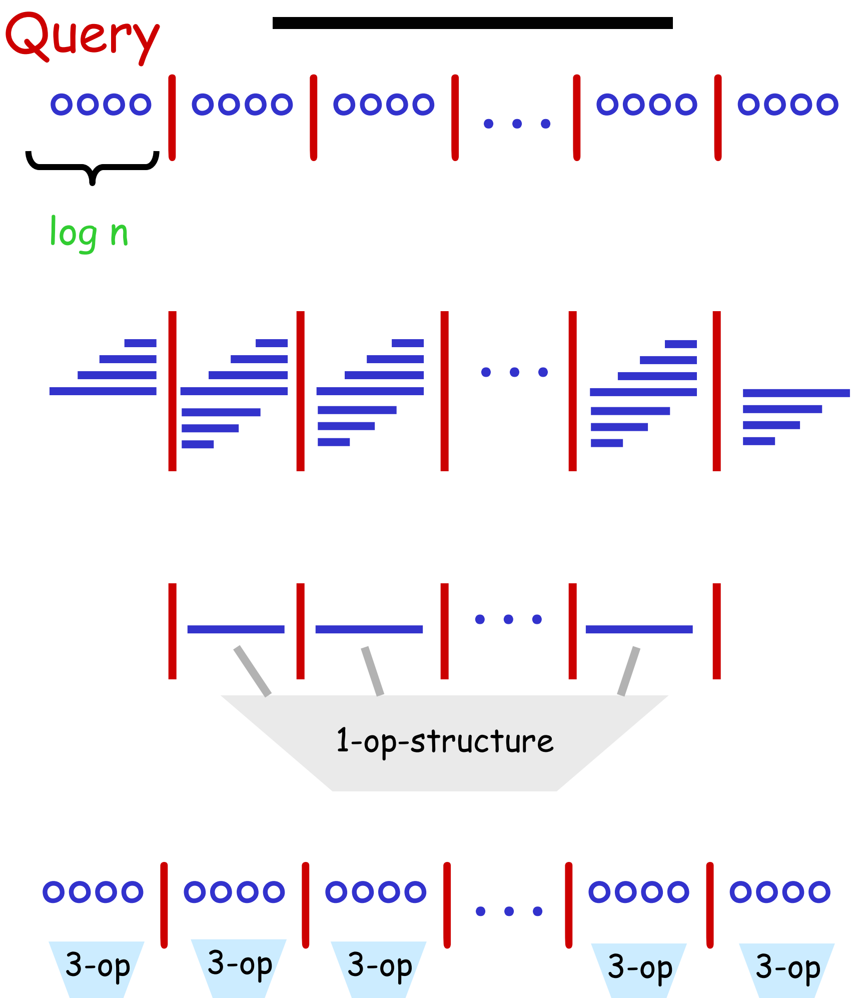

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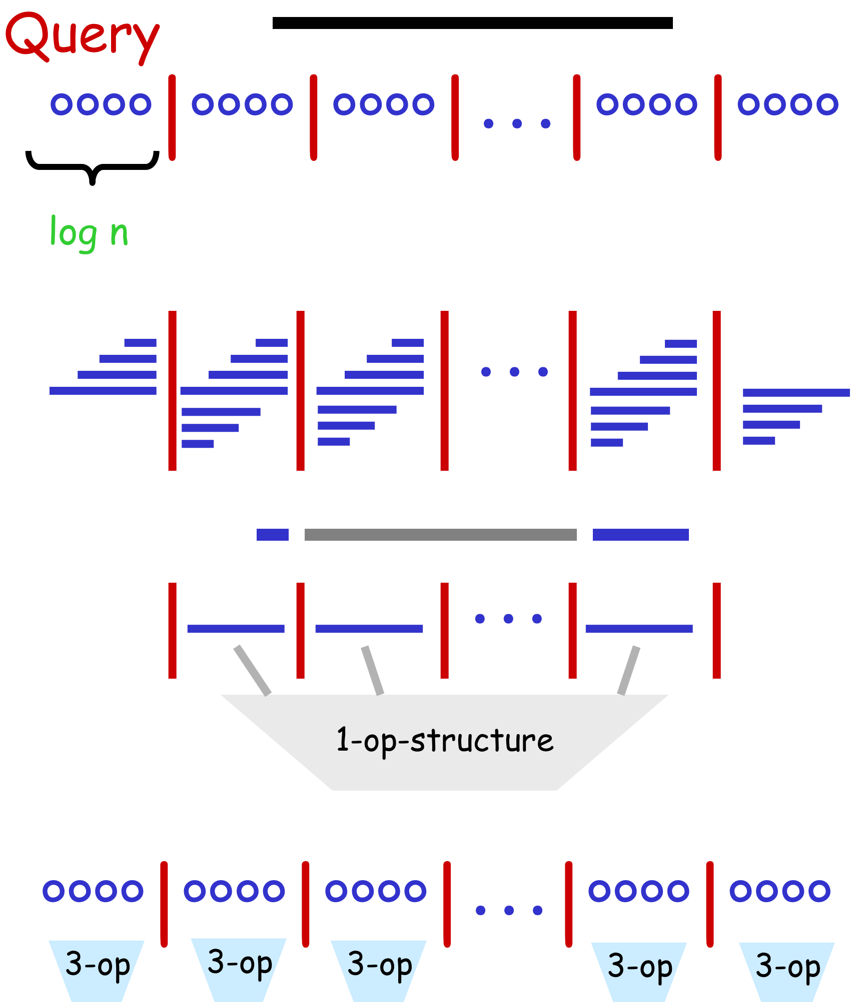

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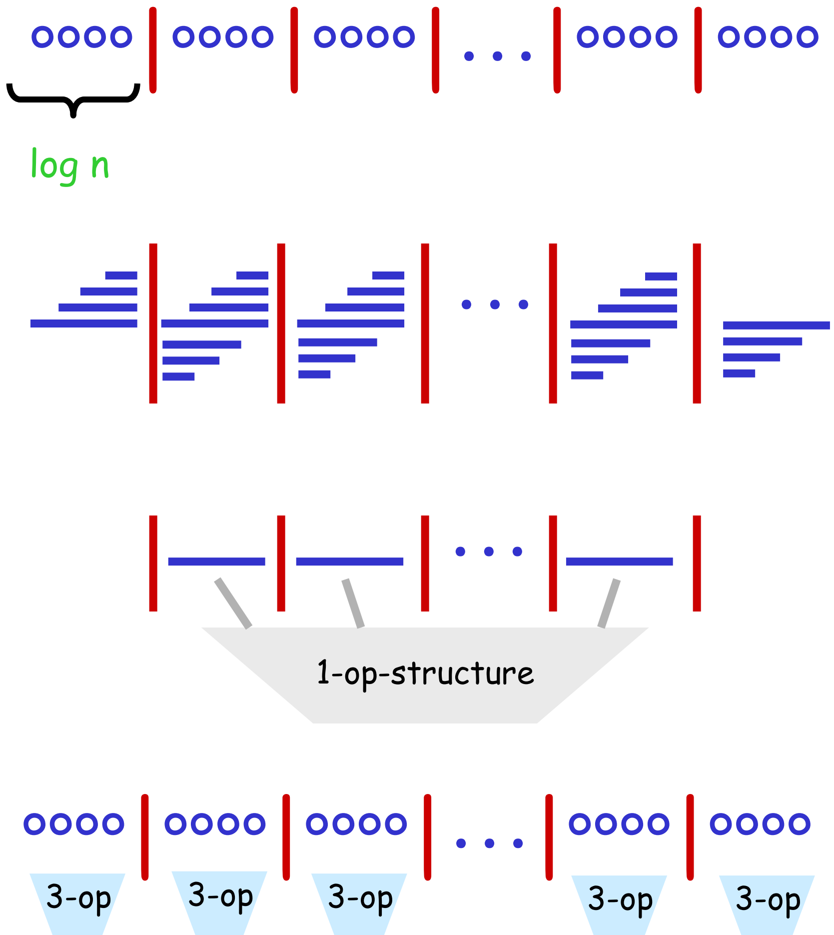

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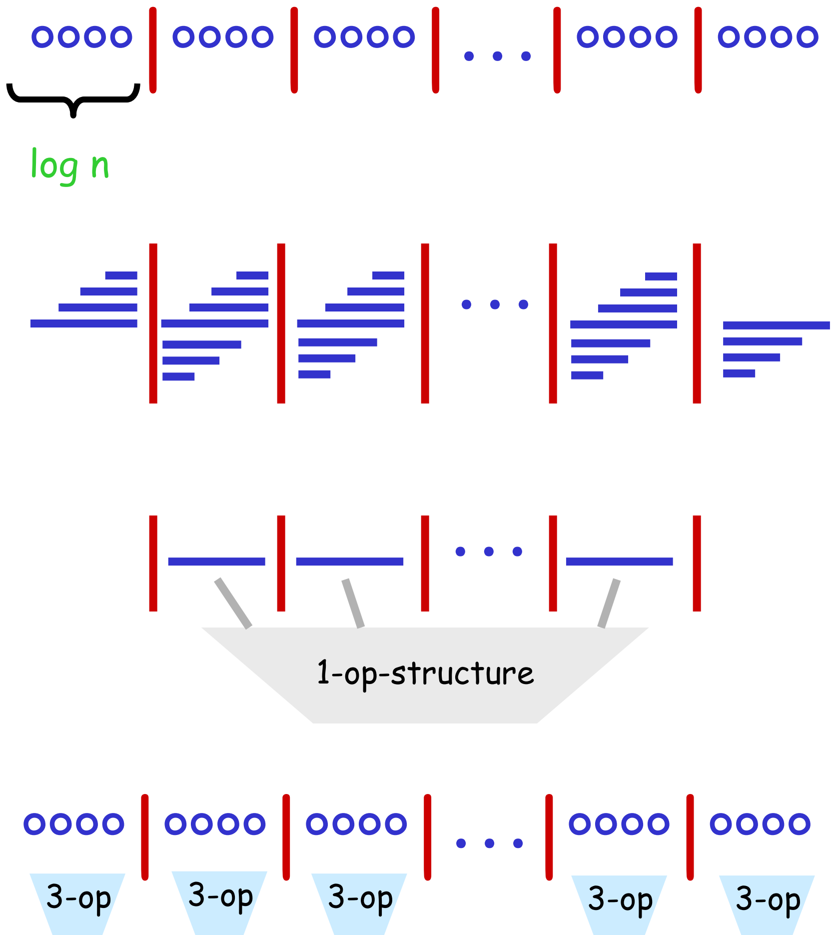
$$S_3(n) \leq 2n + S_1\left(\frac{n}{\log n}\right) + \frac{n}{\log n} \cdot S_3(\log n)$$



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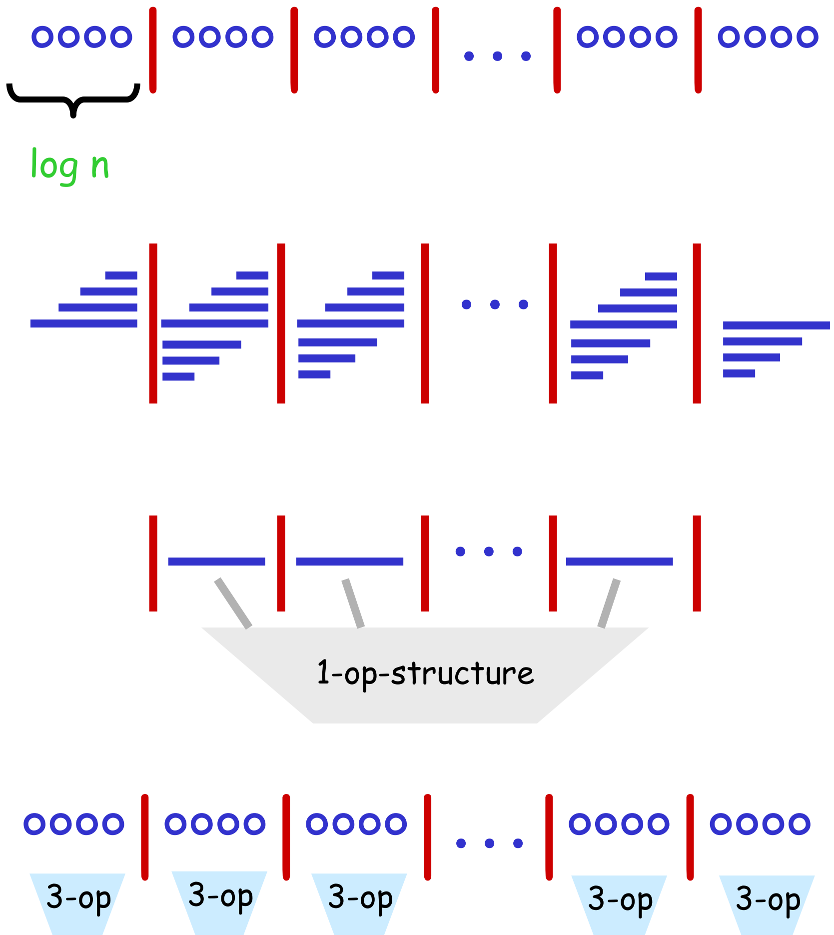

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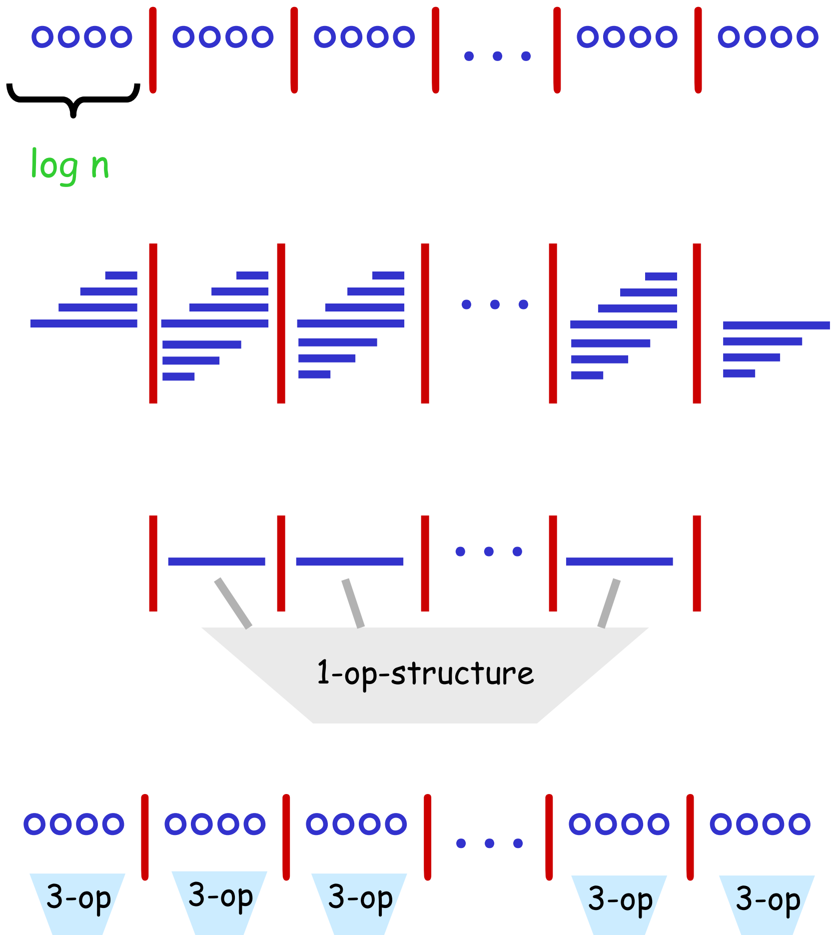
$$S_3(n) \le 2n + \underbrace{S_1\left(\frac{n}{\log n}\right)}_{\le n} + \frac{n}{\log n} \cdot S_3(\log n) \le 3n + \frac{n}{\log n} \cdot S_3(\log n)$$



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$$\Rightarrow \; S_3(n) \le 3n \log^* n$$

$S_5(n) = ?$ $S_7(n) = ?$ $S_9(n) = ?$

$S_{2k+1}(n) = ?$

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Assume: $S_{2k-1}(n) \leq (2k-1) \cdot n \cdot f(n)$

realized by (2k-1)-op-structure

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Assume: $S_{2k-1}(n) \leq (2k-1)\cdot n\cdot f(n)$

realized by (2k-1)-op-structure

Show: $S_{2k+1}(n) \leq (2k+1)\cdot n\cdot f^*(n)$

"(2k+1)-op-structure"

case $n \leq 2k+2$: trivial

case $n \geq 2k+3$: use recursive construction

A

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$n/f(n)$ subsequences of length
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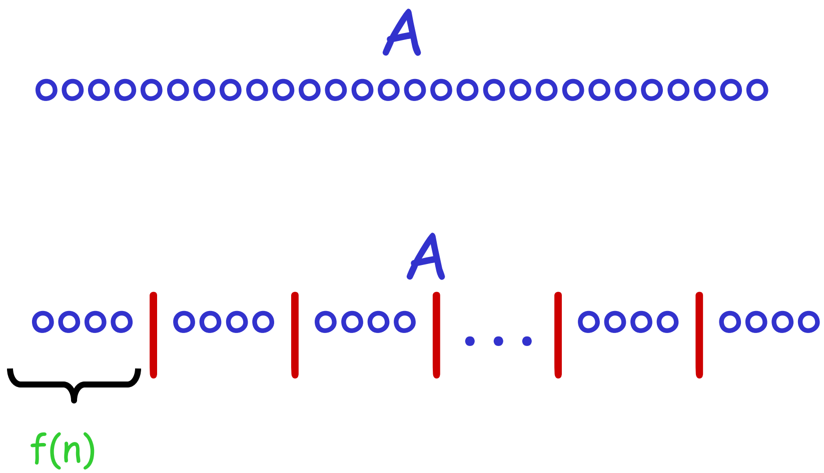

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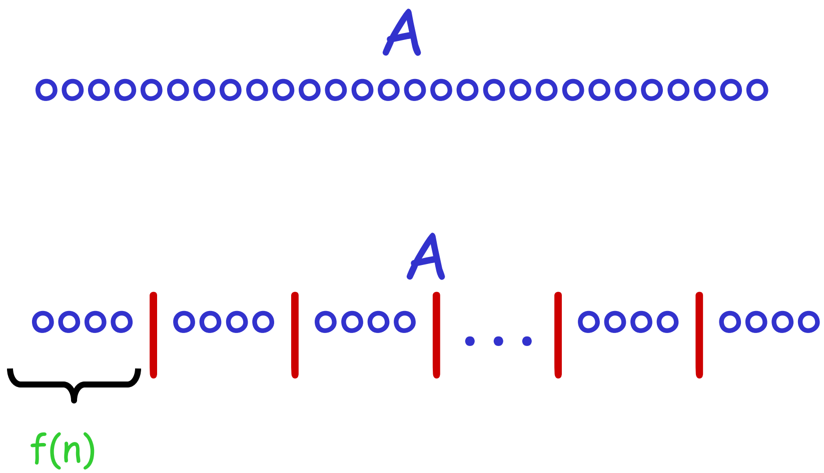

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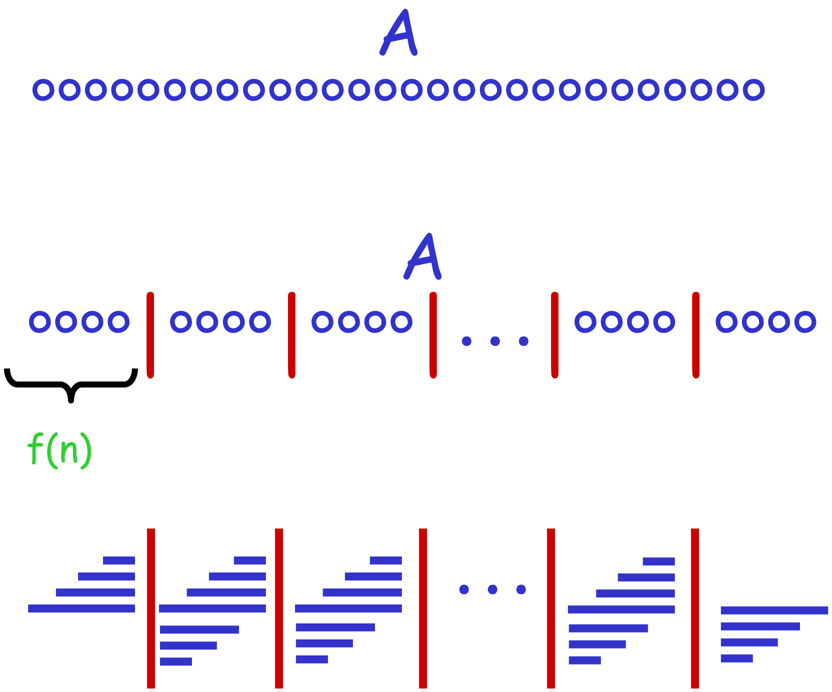

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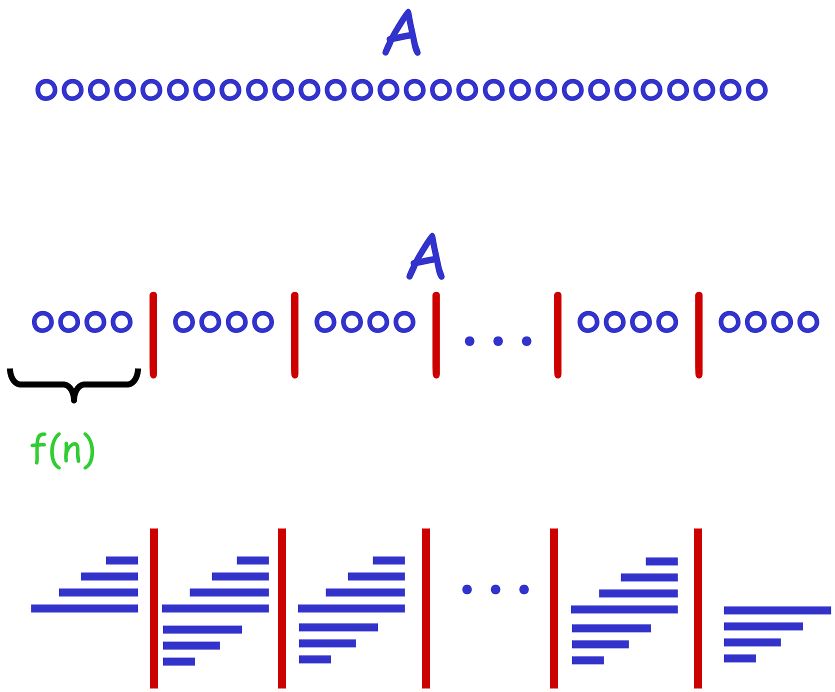

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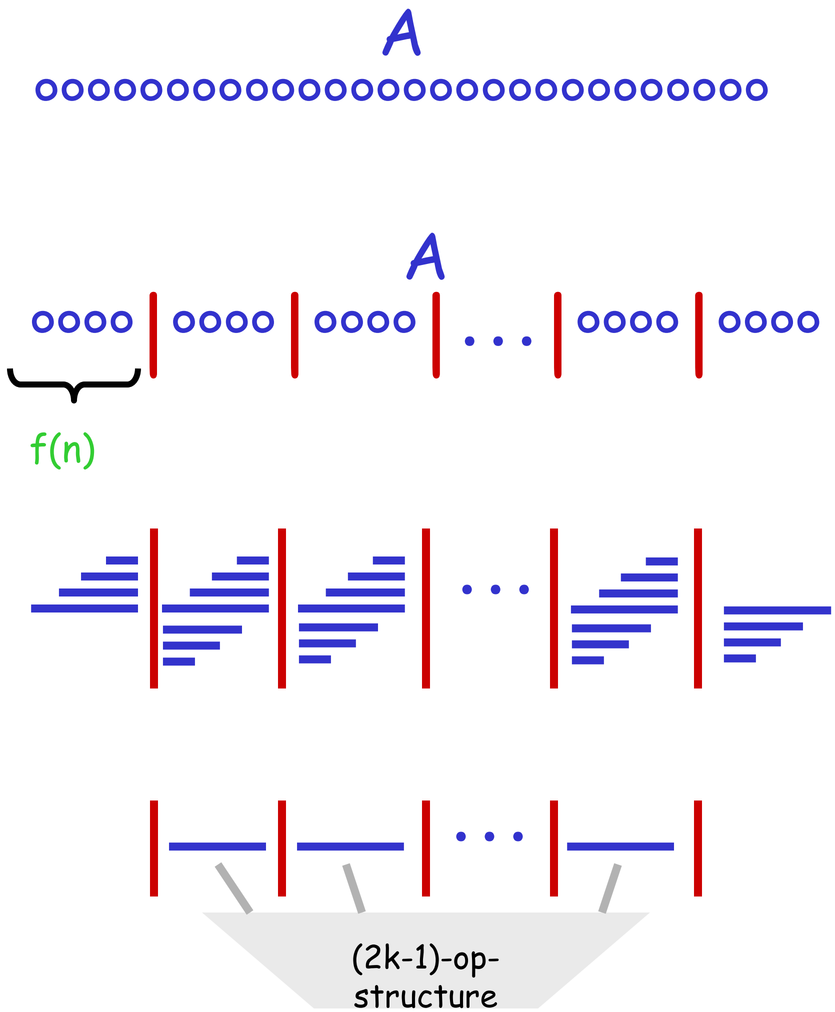

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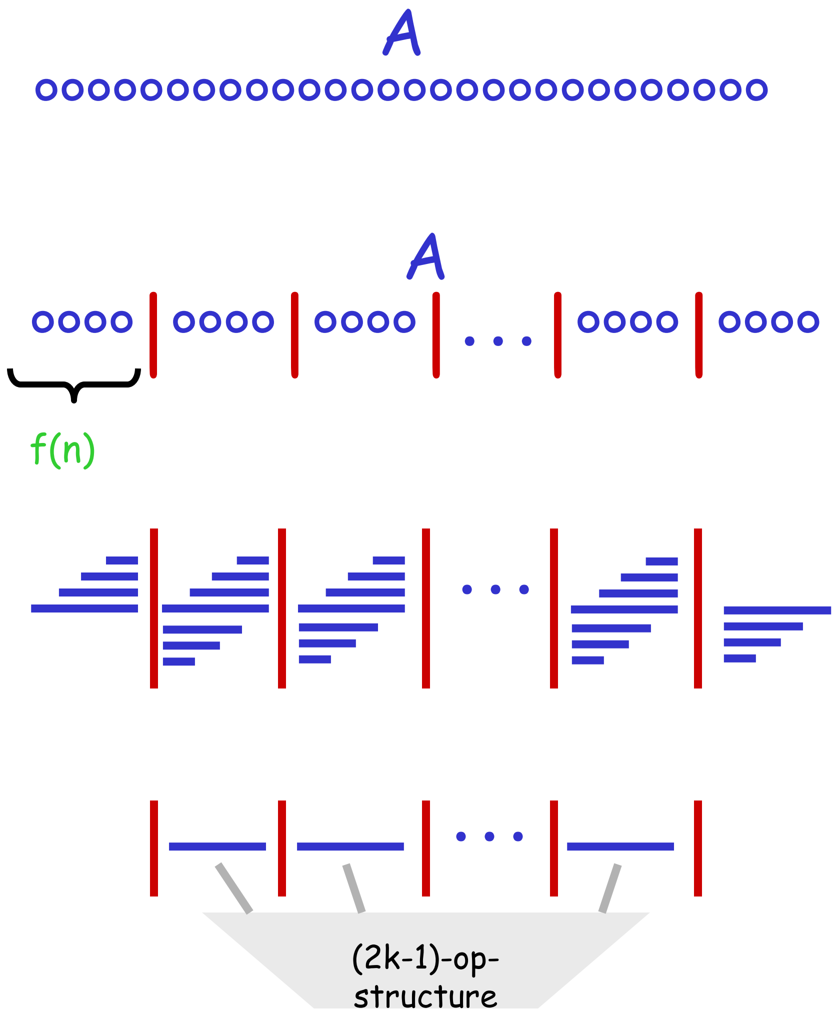

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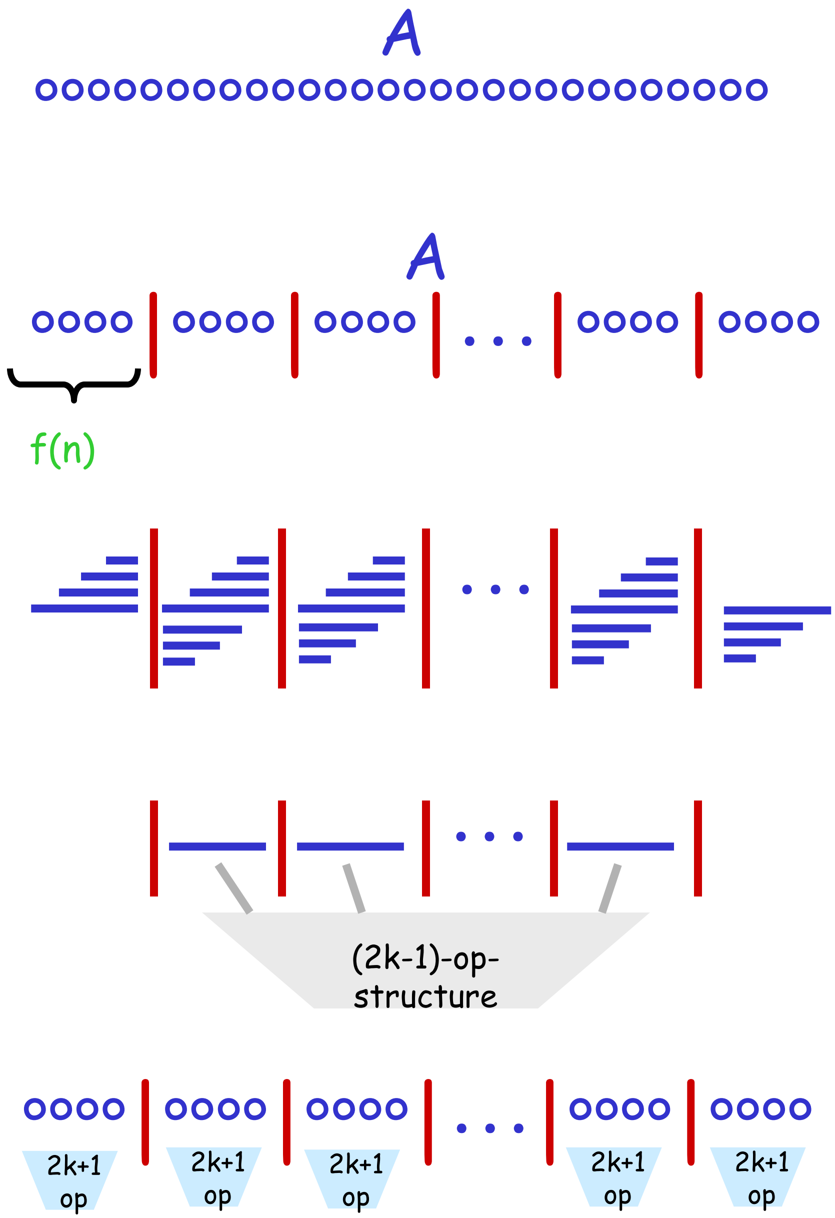
recursively build a (2k+1)-op-structure for each of the n/f(n) subsequences



partition A-sequence into $n/f(n)$ subsequences of length $\leq f(n)$ each

store all prefix- and all suffix-sums within each subsequence

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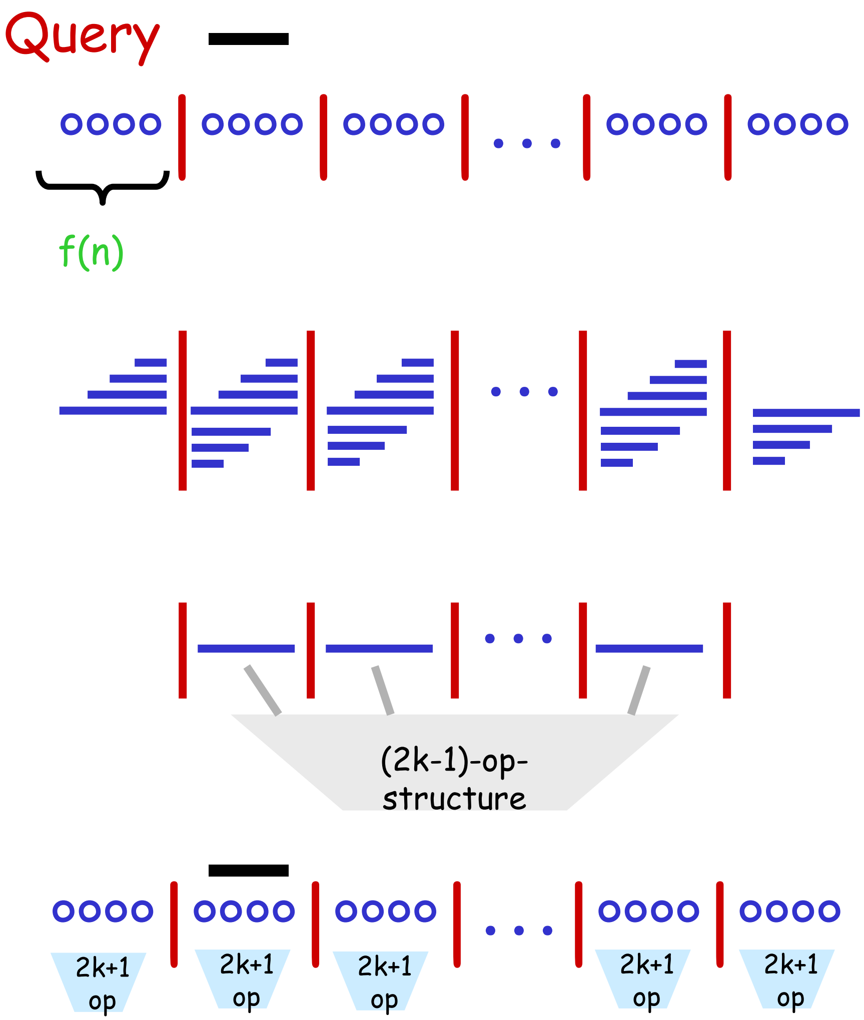

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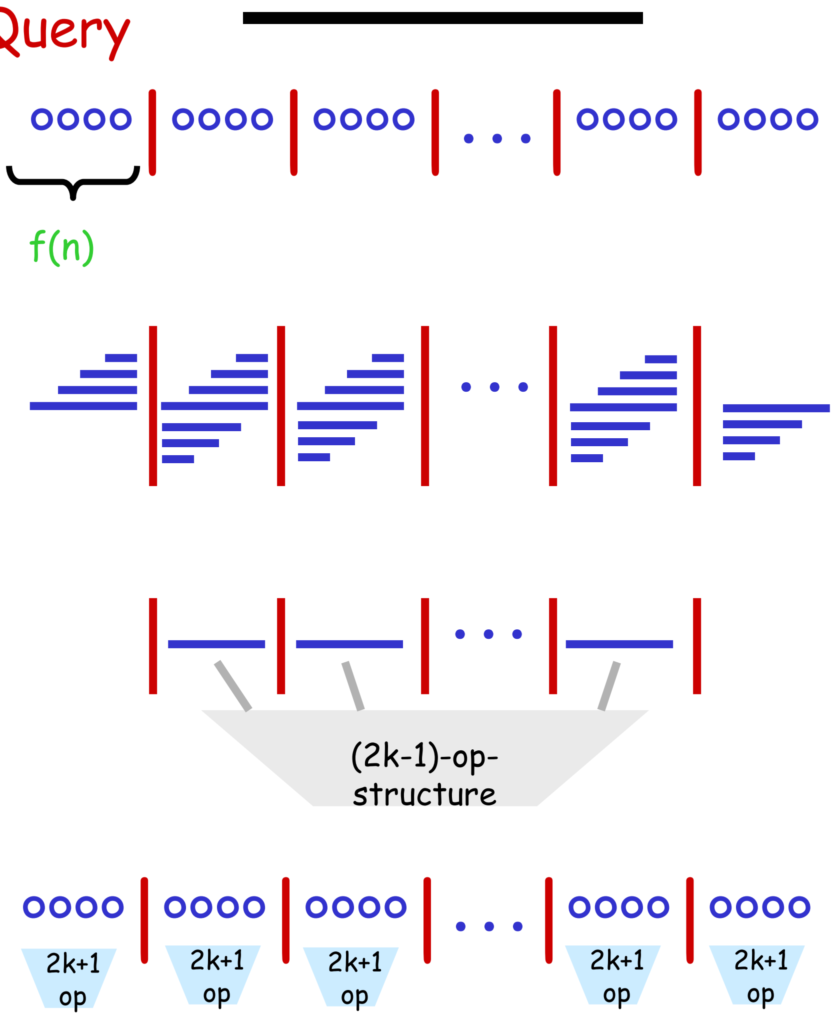
Query



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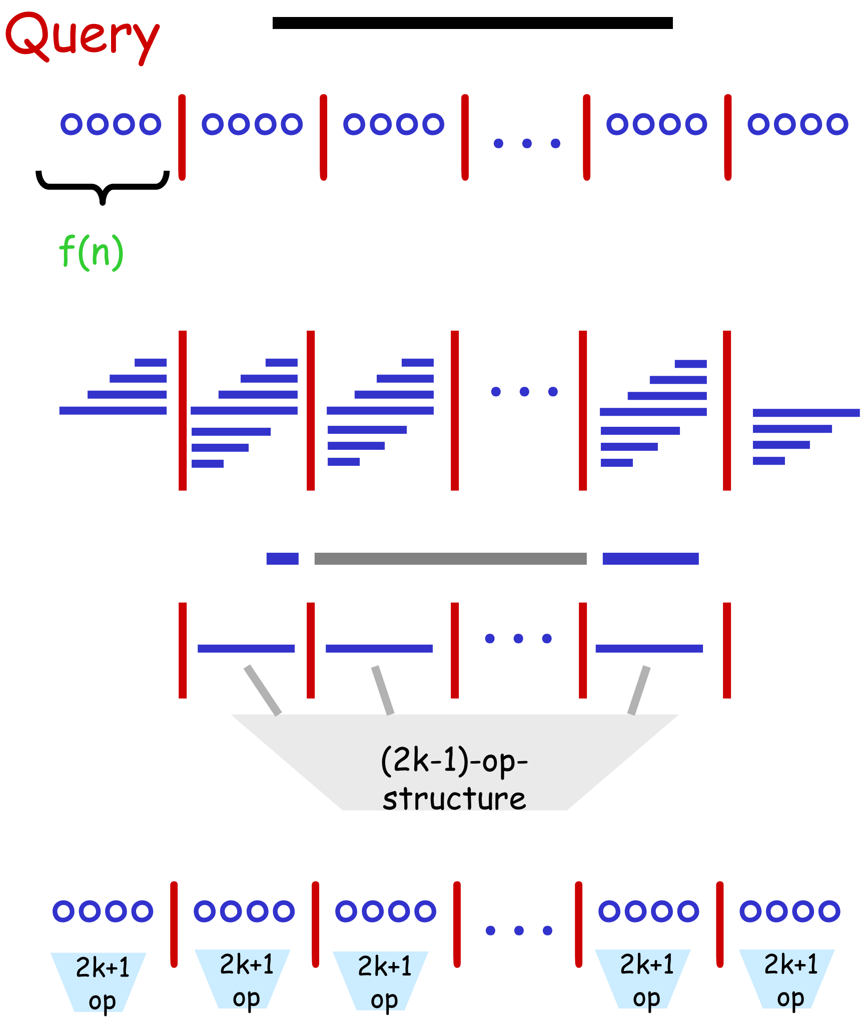

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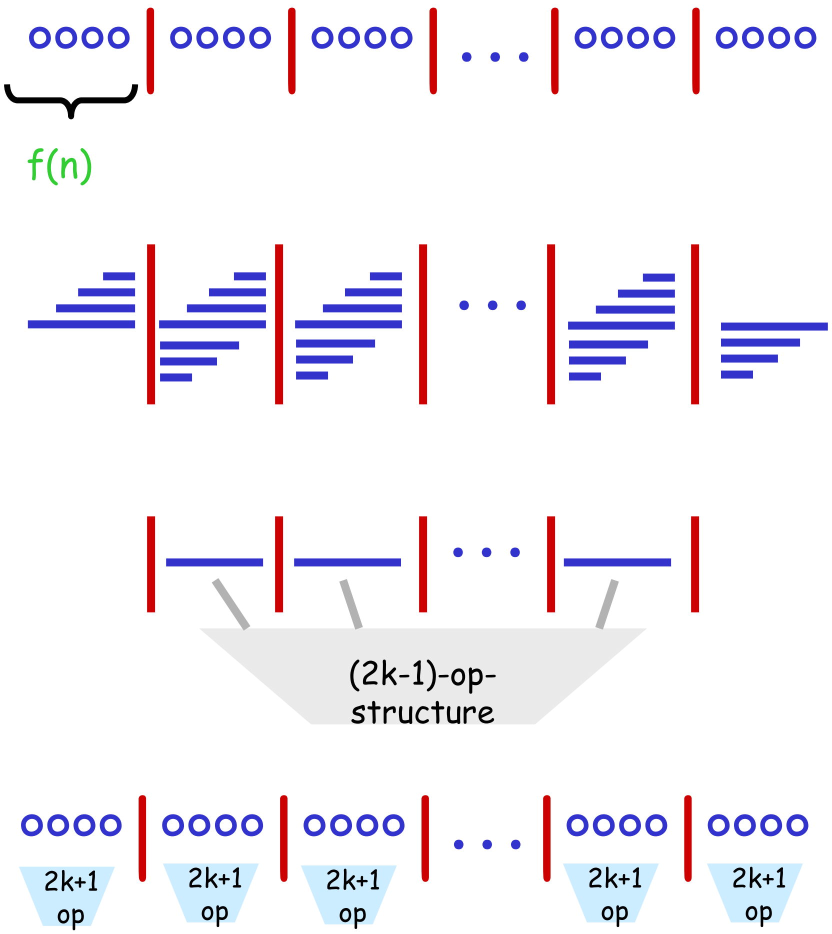

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$$S_{2k+1}(n) \leq 2n + S_{2k-1}\left(\frac{n}{f(n)}\right) + \frac{n}{f(n)} \cdot S_{2k+1}(\,f(n)\,)$$

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$$S_{2k+1}(n) \le (2k+1)\cdot n + \frac{n}{f(n)} \cdot S_{2k+1}(f(n))$$

$$\Rightarrow S_{2k+1}(n) \le (2k+1)n\, f^*(n)$$

$k=1$: $S_1(n) \leq n \log n$

For all $k>1$: $S_{2k-1}(n) \leq (2k-1) \cdot n \cdot f(n)$

$$\Rightarrow \quad S_{2k+1}(n) \leq (2k+1) \cdot n \cdot f^*(n)$$

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For $k = \alpha(n)$: $S_{2\alpha(n)+1} \leq (2\alpha(n)+1) \cdot n \cdot 2$

$= O(\alpha(n) \cdot n)$

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For $O(\alpha(n))$ query cost, space $O(\alpha(n) \cdot n)$ suffices.

Exercise:

For $O(\alpha(n))$ query cost, space $O(n)$ suffices.

Yao; Chazelle, Rosenberg

Union Find with Path Compressions

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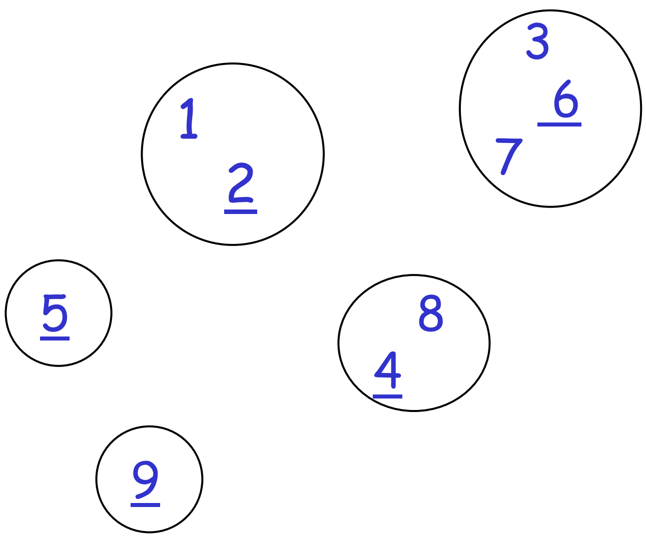
Maintain partition of $S = \{1, 2, \cdots, n\}$

under operations



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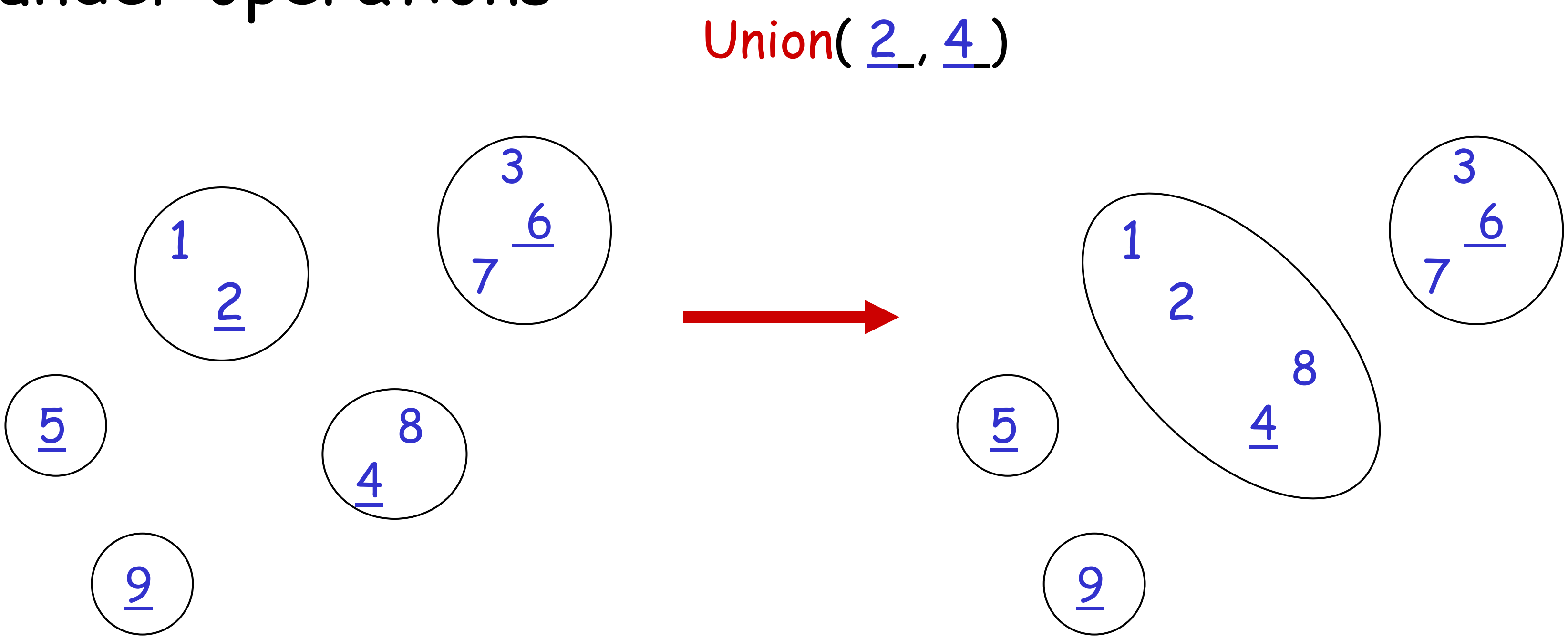

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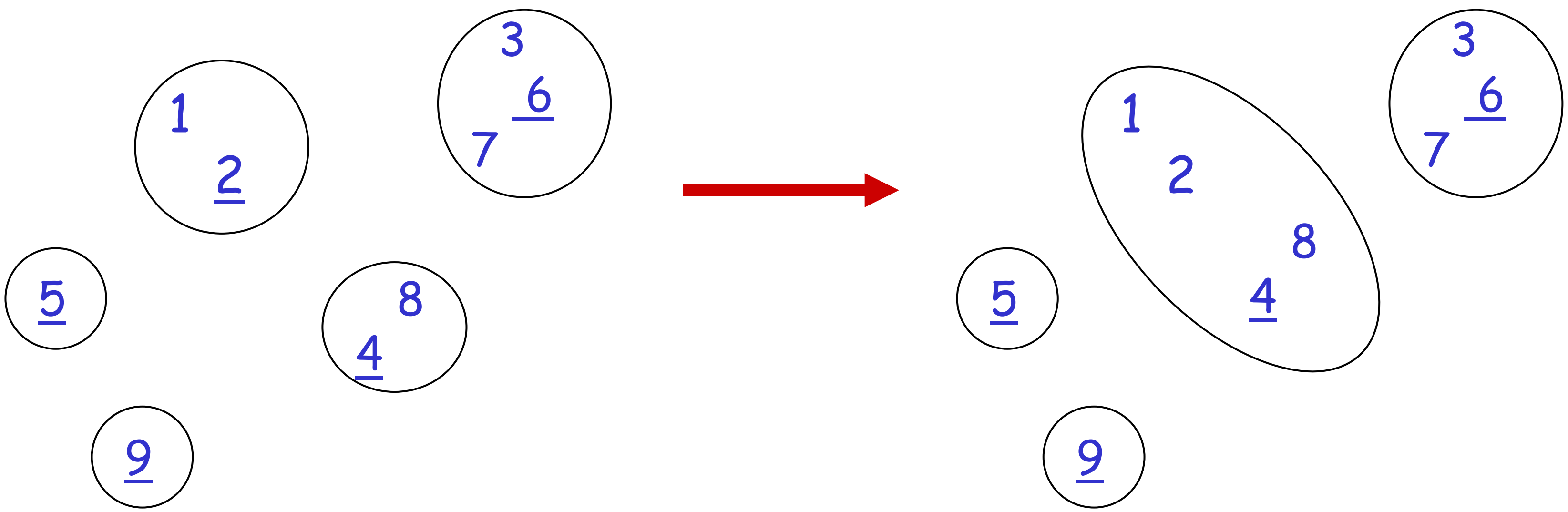
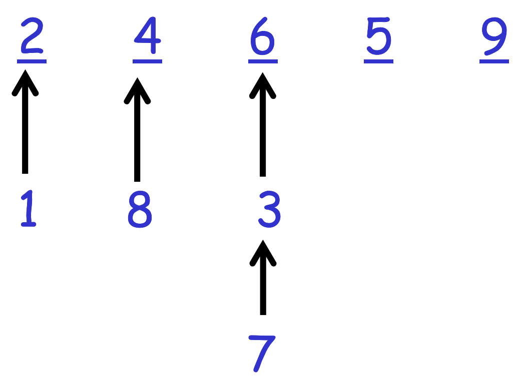
Union(2 , 4)



Find(3) = 6 (representative element)

Impementation

* forest $\mathcal{F}$ of rooted trees with node set S
* one tree for each group in current partition
* root of tree is representative of the group



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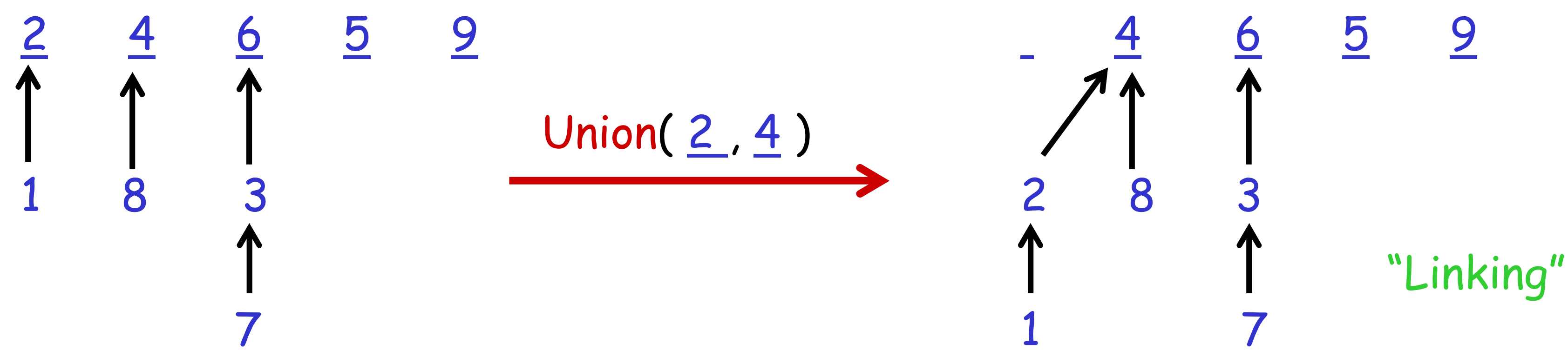

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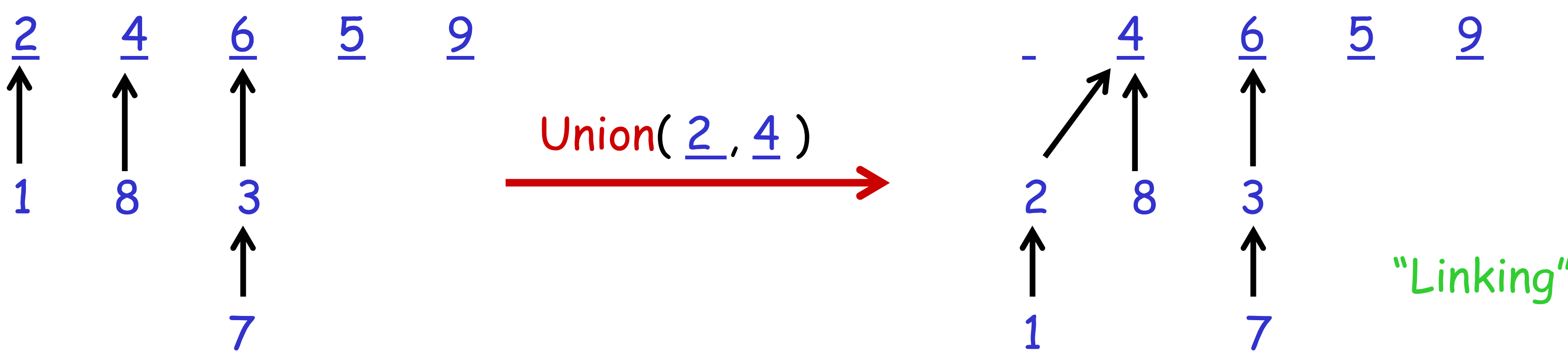
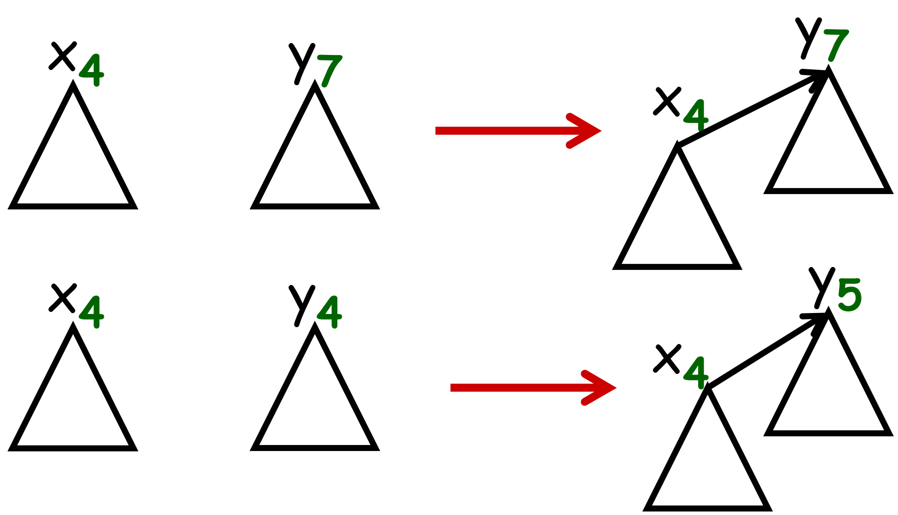


Find(x) follow path from x to root "path follwoing"

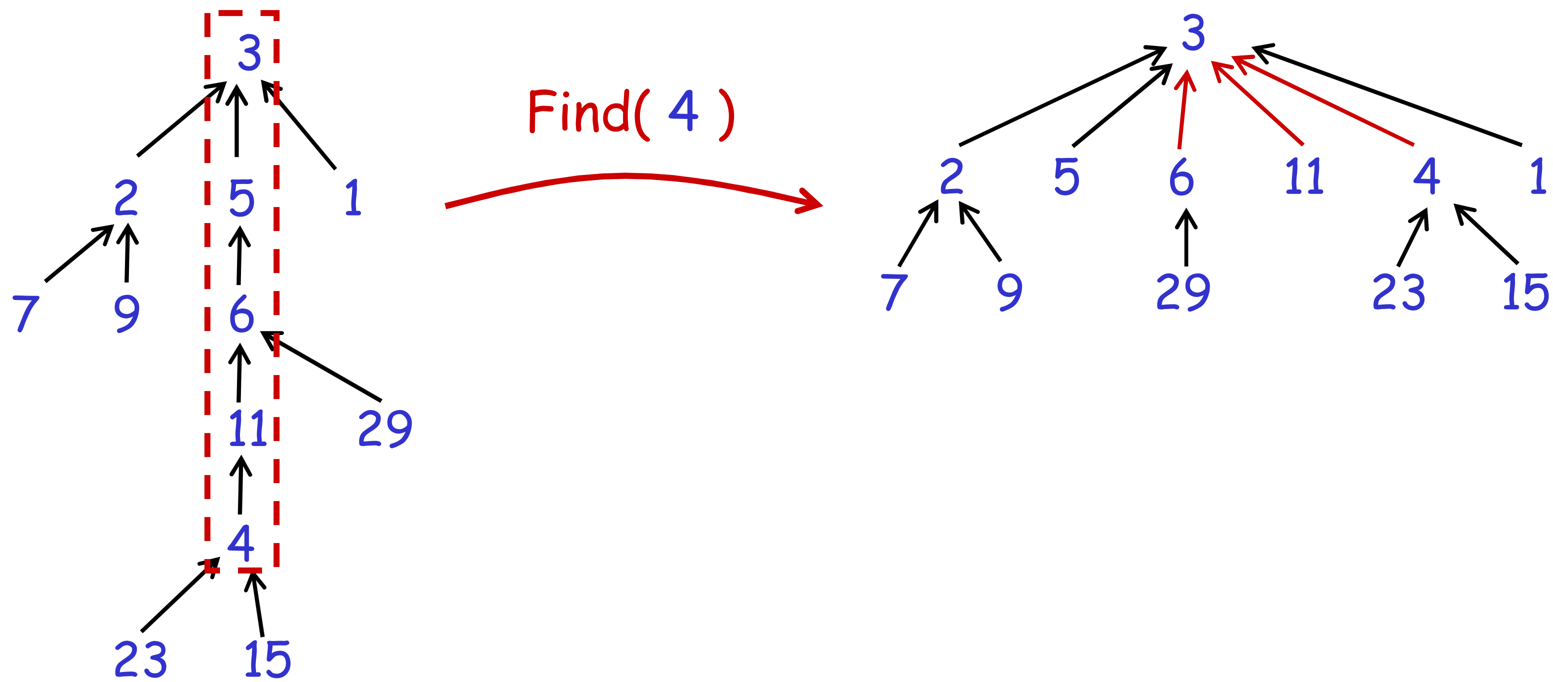
Heuristic 1: "linking by rank"

- each node x carries integer rk(x)
- initially rk(x) = 0
- as soon as x is NOT a root, rk(x) stays unchanged
- for Union(x , y) make node with smaller rank child of the other
  in case of tie, increment one of the ranks



Heuristic 2: Path compression

when performin a Find(x) operation make all nodes in the "findpath" children of the root



sequence of Union and Find operation

Explicit cost model:

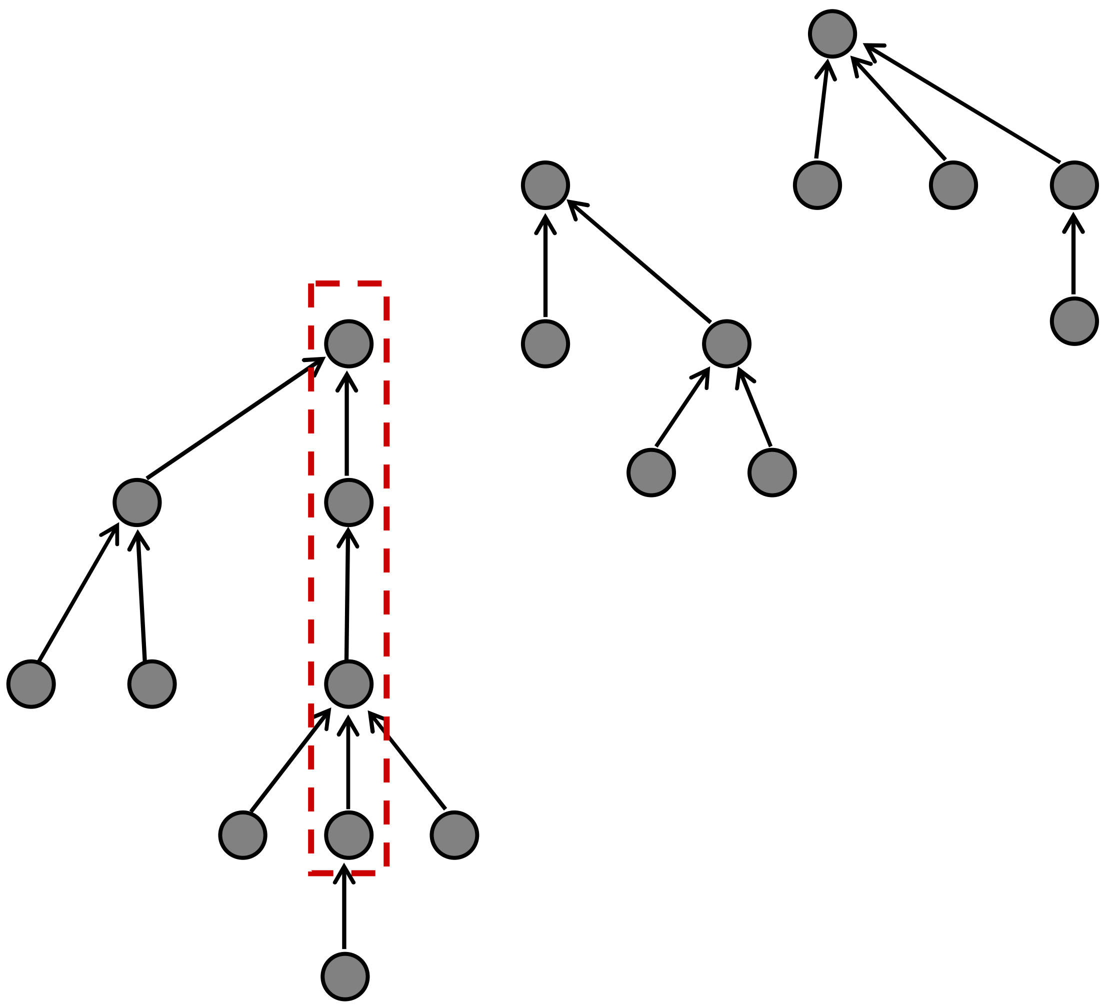
cost(op) = # times some node gets a new parent

Time for Union(x , y) = O(1) = O(cost(Union(x,y)))

Time for Find(x) = O(# of nodes on findpath)

= O(2 + cost(Find(x)))

For analysis assume all Unions are performed first, but Find-paths are only followed (and compressed) to correct node.



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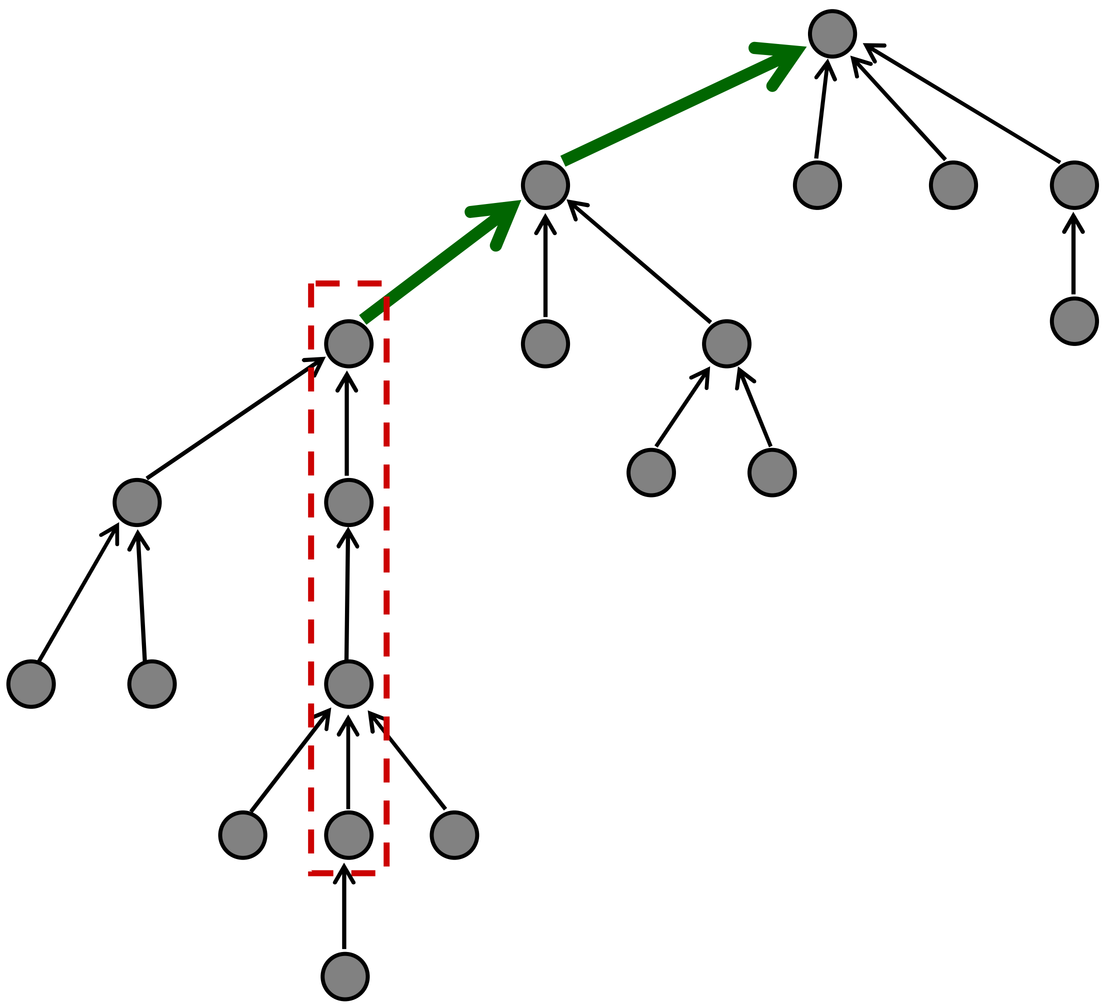
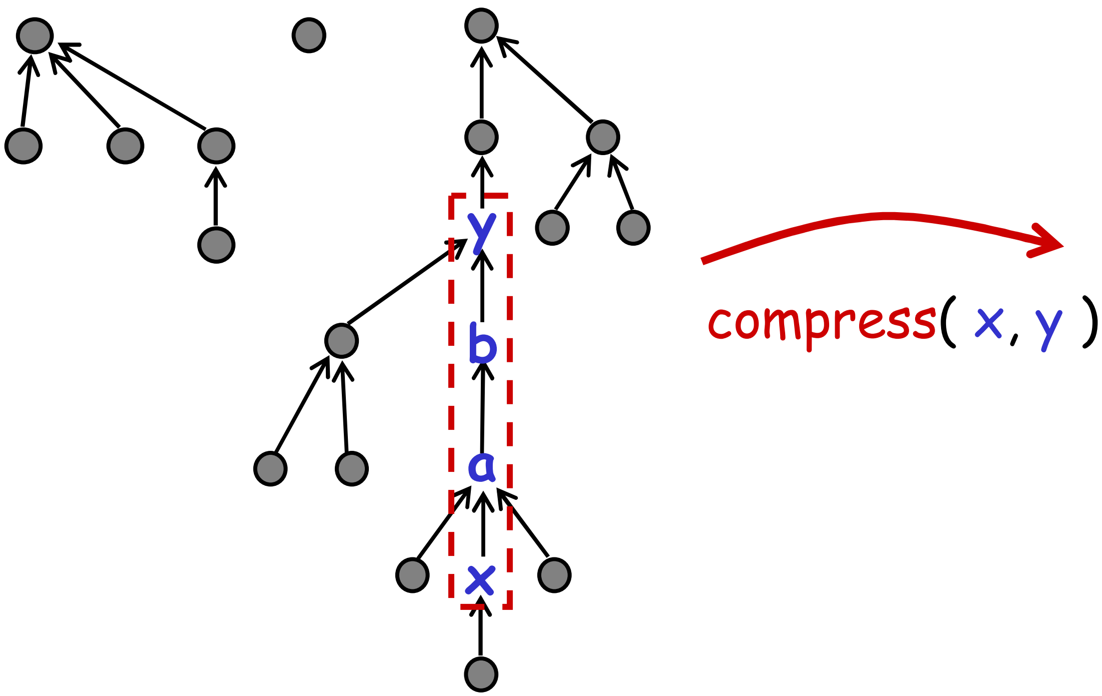


General path compression in forest $\mathcal{F}$



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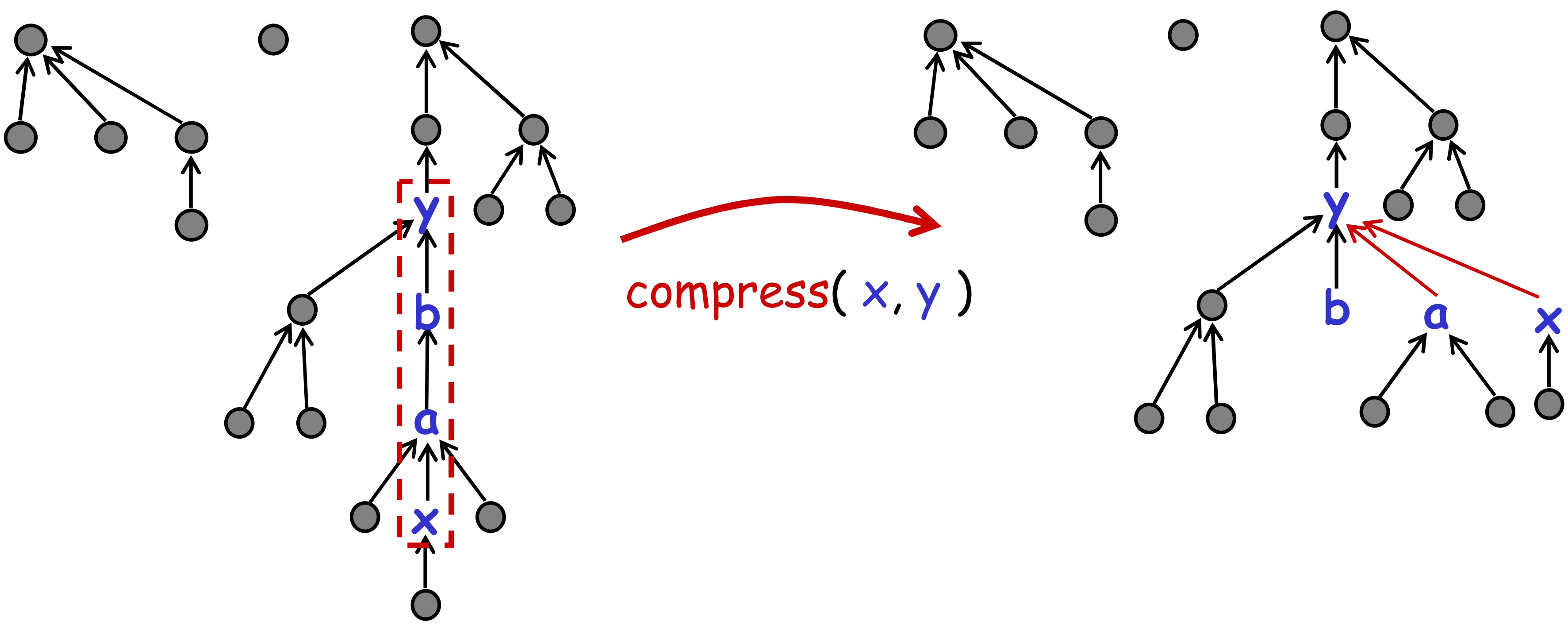

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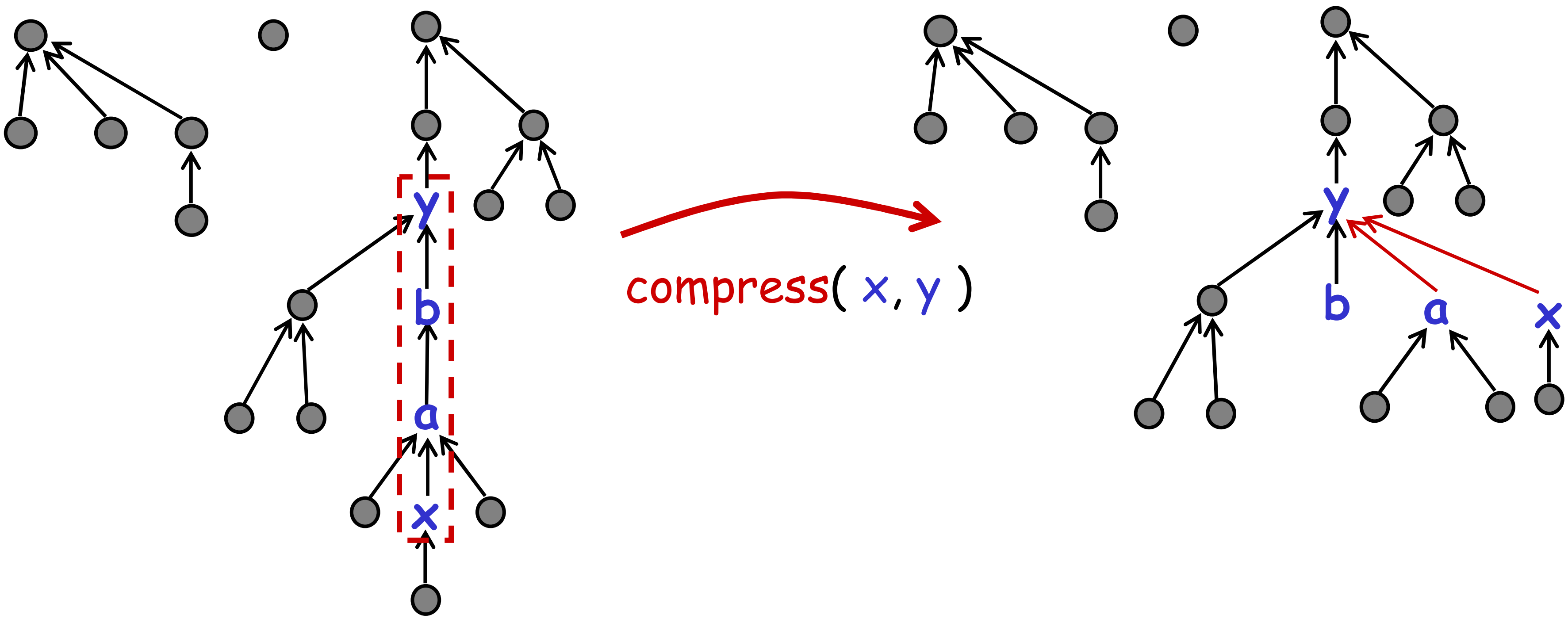

cost(compress(x,y)) = # of nodes that get a new parent

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"rootpath compress"

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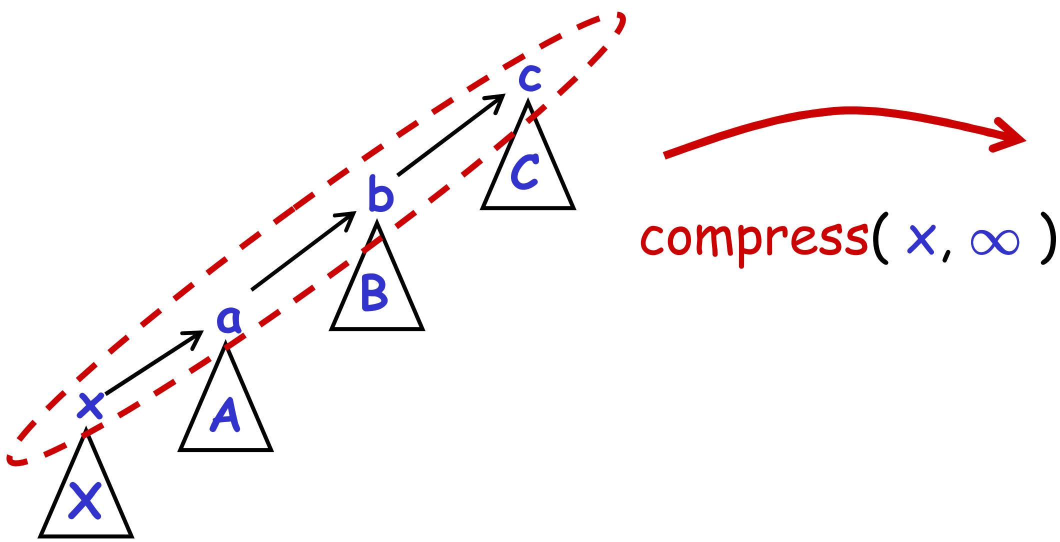

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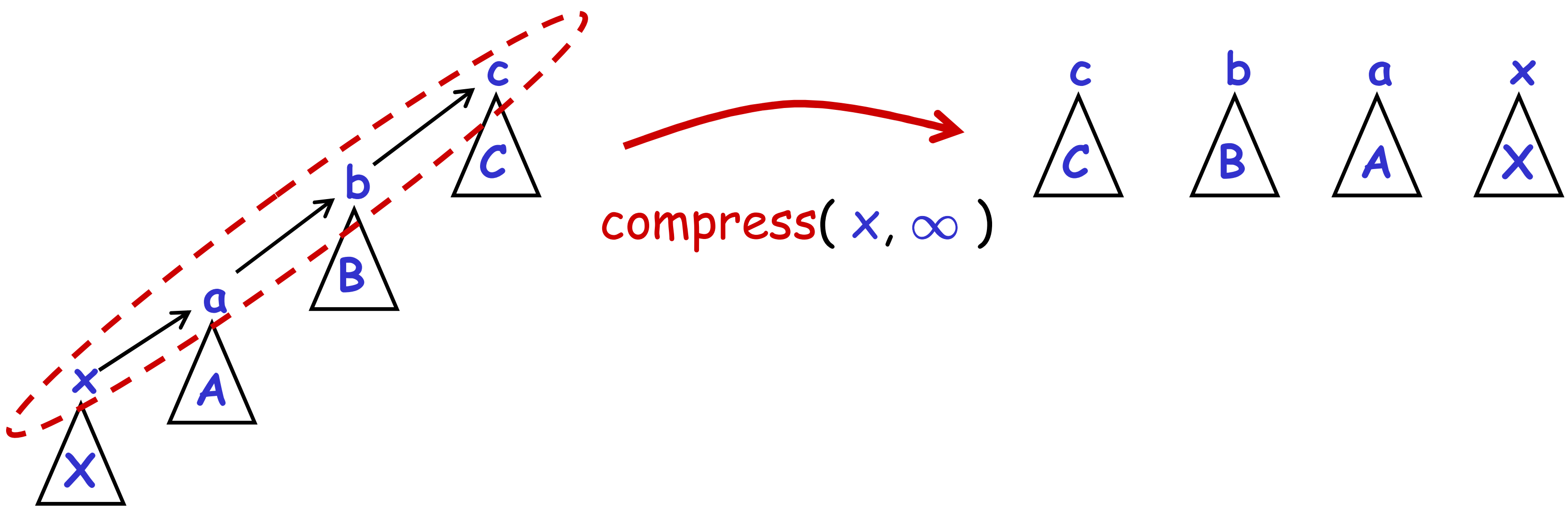

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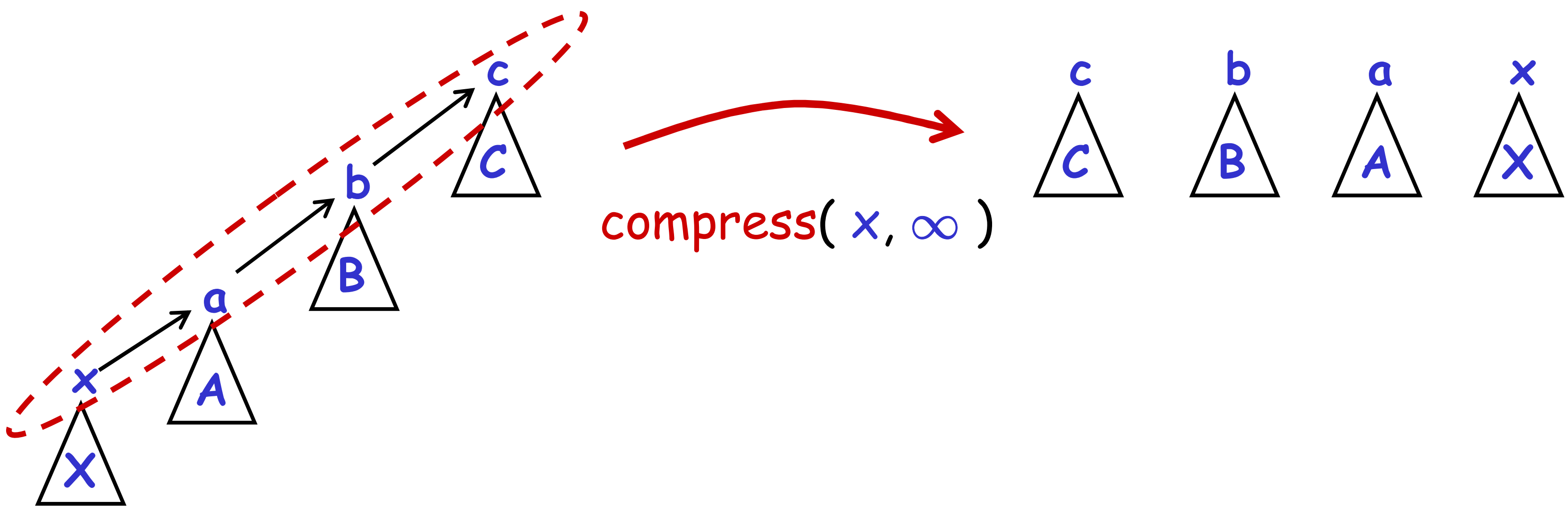

cost(compress(x, ∞)) = # of nodes that get a new parent

= 0

Problem formulation

$\mathcal{F}$ forest on node set X

C sequence of compress operations on $\mathcal{F}$

$|C|$ = # of true compress operations in C

(rootpath compresses excluded)

$\text{cost}(C) = \sum(\text{ cost of individual operations })$

Problem formulation

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C sequence of compress operations on $\mathcal{F}$

$|C|$ = # of true compress operations in C

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cost(C) = $\sum$(cost of individual operations)

How large can cost(C) be at most,
in terms of |X| and |C| ?

Dissection of a forest $\mathcal{F}$ with node set X :

partition of X into "top part" X_t
and "bottom part" X_b

so that top part X_t is "upwards closed",

i.e. $x \in X_t \Rightarrow$ every ancestor of x is in X_t also

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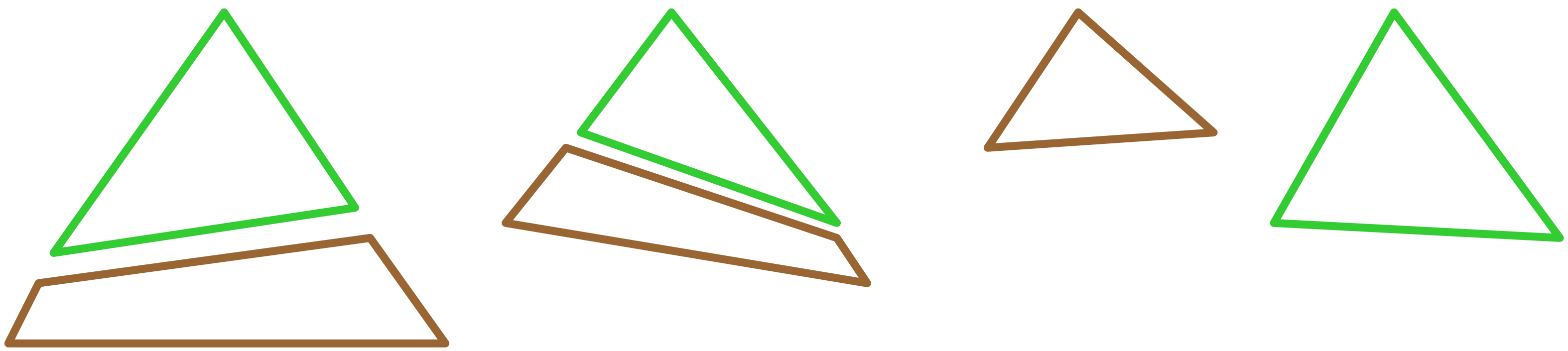

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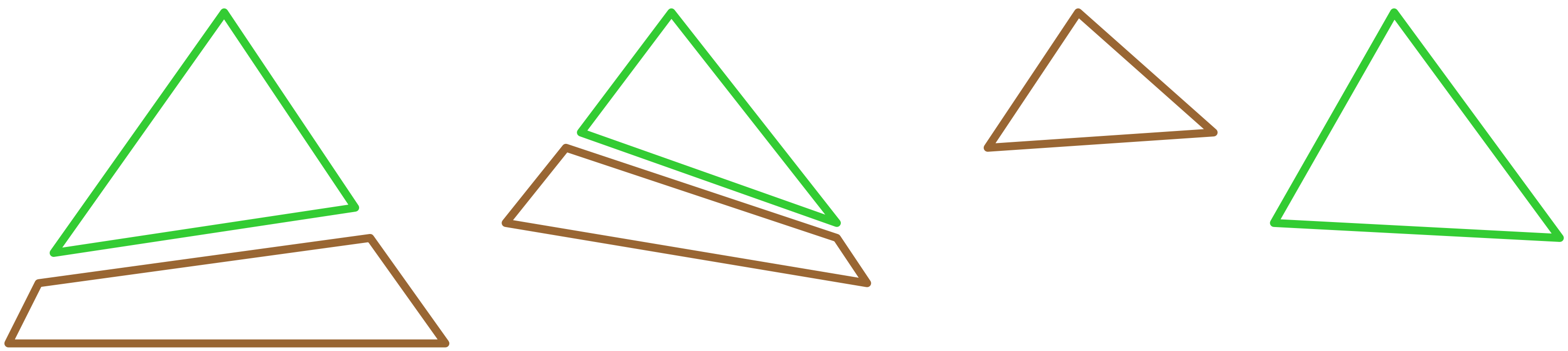

Note: X_t, X_b dissection for $\mathcal{F}$
$\mathcal{F}'$ obtained from $\mathcal{F}$ by
sequence of path compressions

$\Rightarrow$ X_t, X_b is dissection for $\mathcal{F}'$

Main Lemma:

C ... sequence of operations on $\mathcal{F}$ with node set X

X_t, X_b dissection for $\mathcal{F}$ inducing subforests $\mathcal{F}_t$, $\mathcal{F}_b$

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$\Rightarrow$ $\exists$ compression sequences C_b for $\mathcal{F}_b$ and C_t for $\mathcal{F}_t$ with

$$|C_b| + |C_t| \leq |C|$$

and

$$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$

Proof: 1) How to get C_b and C_t from C:

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compression paths from C

case 1: 

into C_t

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compression paths from C

case 1:

into C_t

case 2:

into C_b

case 3:

into C_t

into C_b

Proof:

$$|C_b| + |C_t| \leq |C|$$

compression paths from C

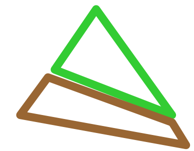
case 1: $x \to y$ $\qquad$ $x \to y$ into C_t

case 2: $x \to y$ $\qquad$ $x \to y$ into C_b

case 3: $x \to x' \to y$ $\qquad$ $x' \to y$ into C_t

$x \to \infty$ into C_b

$$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$



cost(C)	
green node gets new green parent:	accounted by $\text{cost}(C_t)$
brown node gets new brown parent:	accounted by $\text{cost}(C_b)$
brown node gets new green parent: for the first time	accounted by $\lvert X_b \rvert$
brown node gets new green parent: again	accounted by $\lvert C_t \rvert$

$f(m,n)$ ... maximum cost of any compression sequence C with $|C|=m$ in an arbitrary forest with n nodes.

Claim: $f(m,n) \leq (m+n)\cdot\log_2 n$

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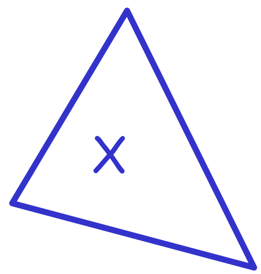
forest $\mathcal{F}$



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C compression sequence $|C|=m$

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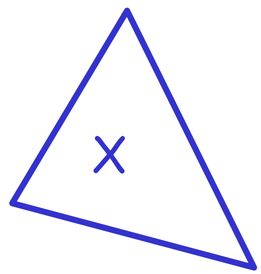

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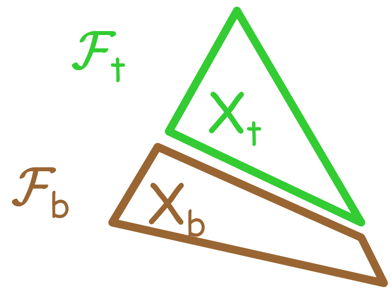

$|X_t|=|X_b|=n/2$

$\mathcal{C}$ compression sequence $|\mathcal{C}|=m$

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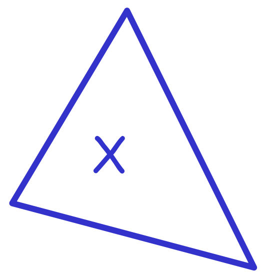
Proof:

forest $\mathcal{F}$



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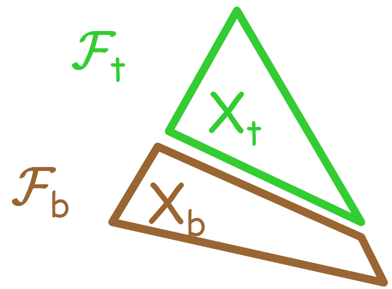

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Main Lemma $\Rightarrow \exists\ C_t, C_b \quad |C_b|+|C_t| \leq |C|$

$m_b + m_t \leq m$

$$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$

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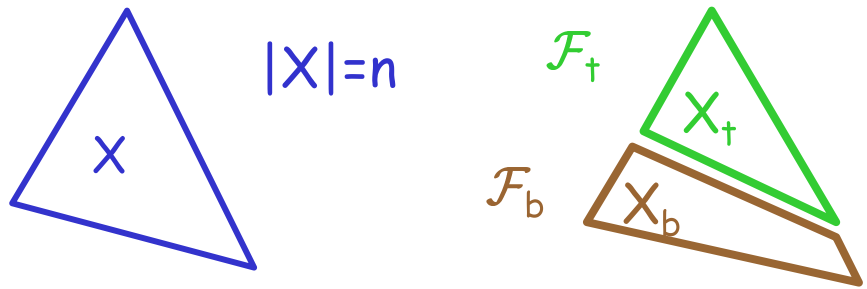

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Proof:

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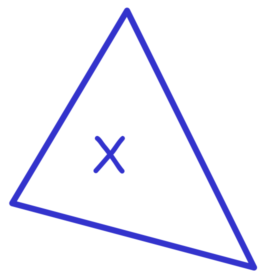

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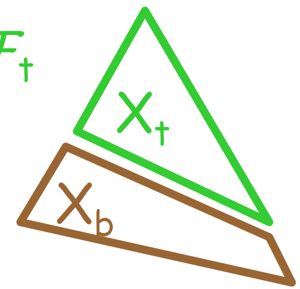
$\mathcal{F}_t$

$\mathcal{F}_b$



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$$\leq (m+n)\cdot\log_2 n$$

Corollary:

Any sequence of m Union, Find operations in a universe of n elements that uses arbitrary linking and path compression takes time at most

$$O(\ (m+n)\cdot \log n)$$

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$$O((m+n) \cdot \log n)$$

By choosing a dissection that is "unbalanced" in relation to m/n one can prove a better bound of

$$O((m+n) \cdot \log_{\lceil m/n \rceil +1} n)$$

Path compression and union by rank

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Def: $\mathcal{F}$ forest, x node in $\mathcal{F}$

$r(x)$ = height of subtree rooted at x

($r(\text{leaf}) = 0$)

$\mathcal{F}$ is a **rank forest**, if

for every node x

for every i with $0 \leq i < r(x)$,

there is a child y_i of x with $r(y_i) = i$.

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Note: Union by rank produces rank forests !

Path compression and union by rank

Def: $\mathcal{F}$ forest, x node in $\mathcal{F}$

$r(x)$ = height of subtree rooted at x

($r(\text{leaf}) = 0$)

$\mathcal{F}$ is a **rank forest**, if

for every node x

for every i with $0 \leq i < r(x)$,

there is a child y_i of x with $r(y_i)=i$.

Note: Union by rank produces rank forests !

Lemma: $r(x)=r \Rightarrow x$ is root of subtree with at least 2^r nodes.

Inheritance Lemma:

$\mathcal{F}$ rank forest with maximum rank r and node set X

$s \in \mathbb{N}$: $X_{>s} = \{ x \in X \mid r(x) > s \}$ $\mathcal{F}_{>s}$

$X_{\leq s} = \{ x \in X \mid r(x) \leq s \}$ $\mathcal{F}_{\leq s}$ induced forests

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i) $X_{\leq s}$, $X_{>s}$ is a dissection for $\mathcal{F}$

ii) $\mathcal{F}_{\leq s}$ is a rank forest with maximum rank $\leq s$

iii) $\mathcal{F}_{>s}$ is a rank forest with maximum rank $\leq r-s-1$

iv) $|X_{>s}| \leq |X| / 2^{s+1}$

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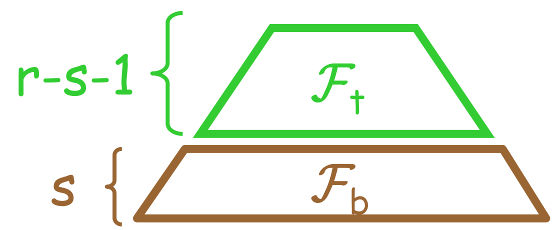

$f(m,n,r)$ = maximum cost of any compression sequence C, with $|C|=m$, in rank forest $\mathcal{F}$ with n nodes and maximum rank r.

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Trivial bounds:

$$f(m,n,r) \leq (r-1)\cdot n$$

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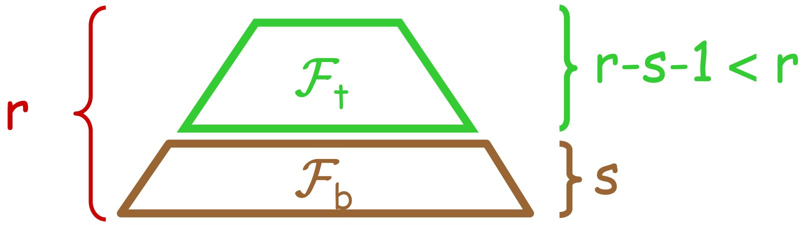
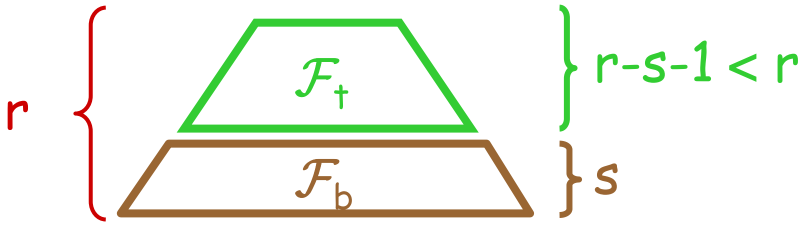


$|X_{>s}| \leq n/2^{s+1}$

$|X_{\leq s}| \leq n$

$f(M,N,R) \leq N \cdot R$



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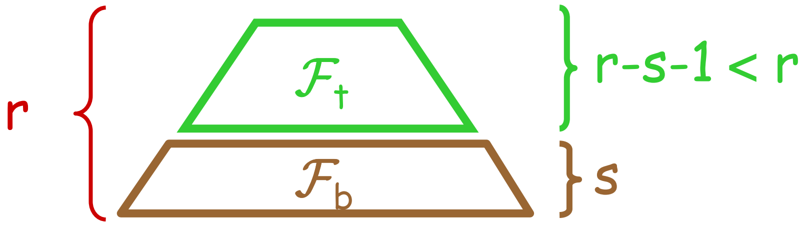

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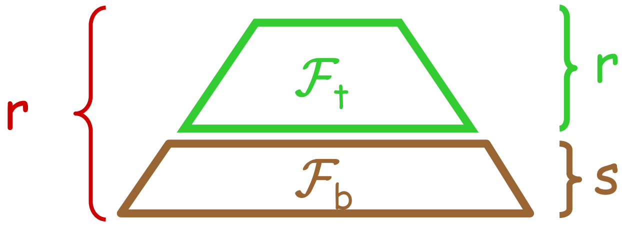

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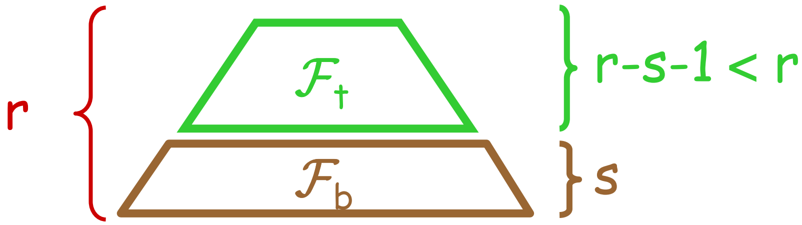
$s = \log r$ $\quad\quad \leq n$



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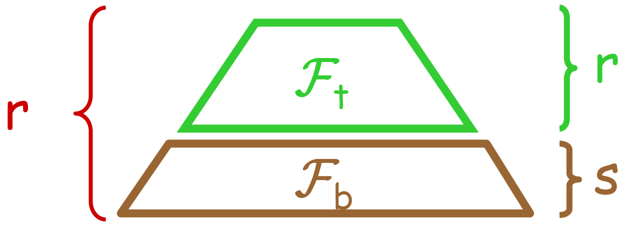

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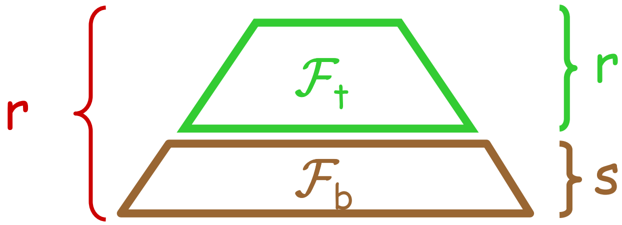

$$r-s-1 < r \qquad |X_{>s}| \le n/2^{s+1}$$

$$s \qquad |X_{\le s}| \le n$$

$$f(M,N,R) \le N \cdot R$$

$$cost(C) \le \underbrace{cost(C_t)}_{\le (n/2^{s+1})\cdot r} + cost(C_b) + \underbrace{|X_b|}_{\le n} + |C_t|$$

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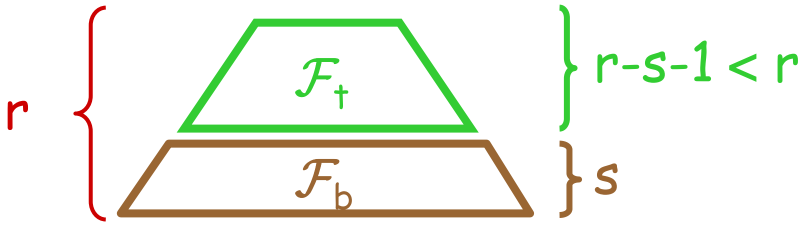
$$cost(C) - |C| \le 2n + (cost(C_b) - |C_b|)$$



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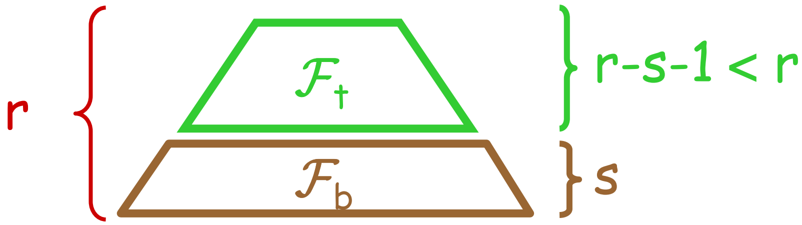

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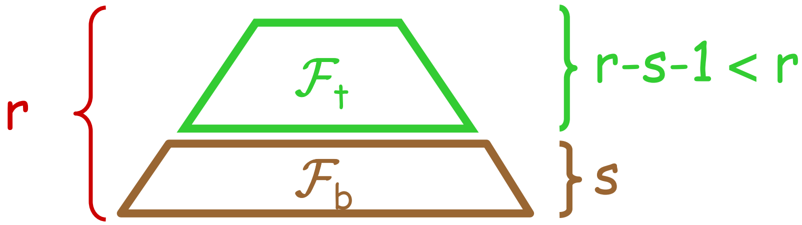
$$(f(m,n,r) - m) \leq 2n + (f(m_b,n,\log r) - m_b)$$



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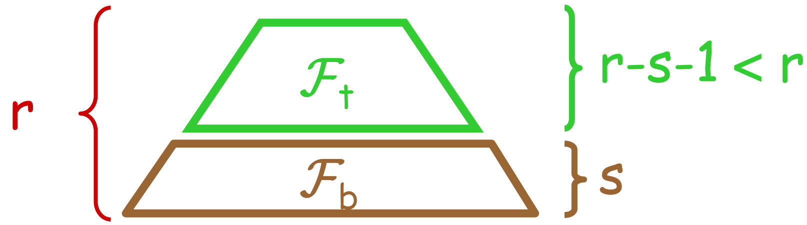

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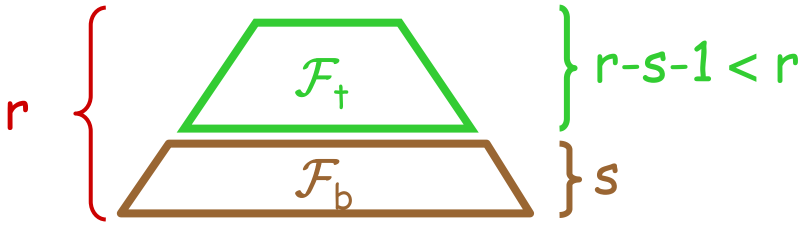

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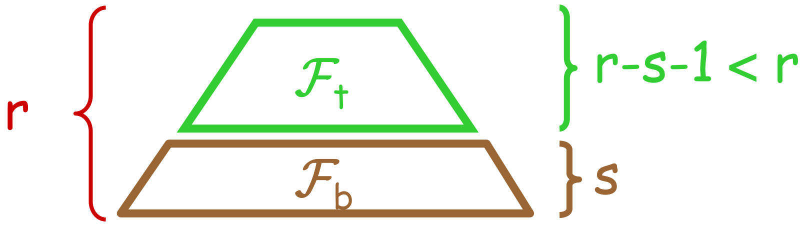

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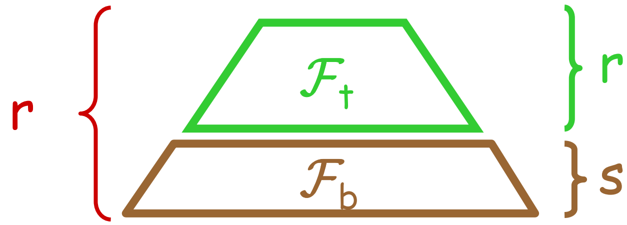

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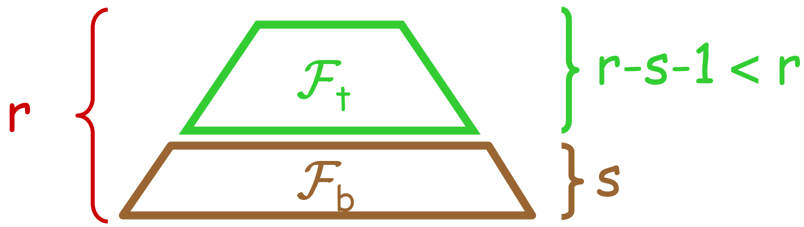

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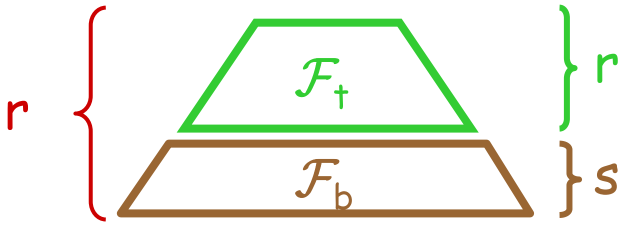
$s = \log \log^* r$

$$\leq |\mathcal{C}_t| + n$$



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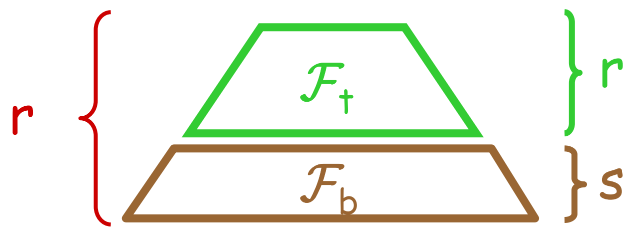

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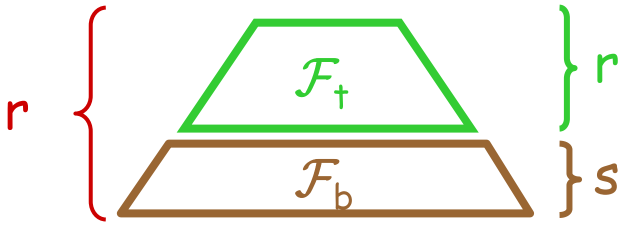

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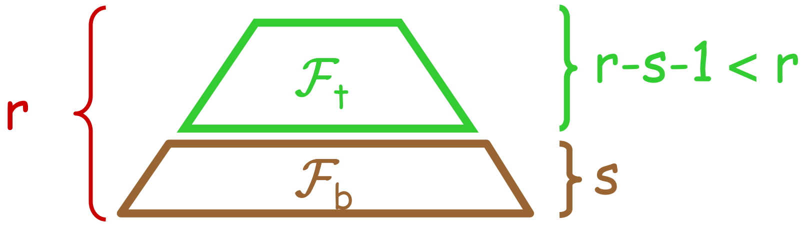

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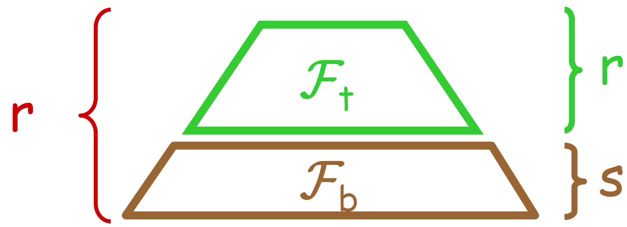

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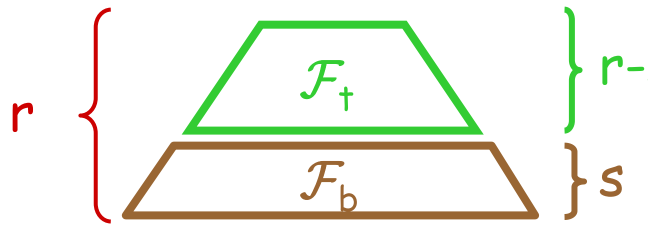

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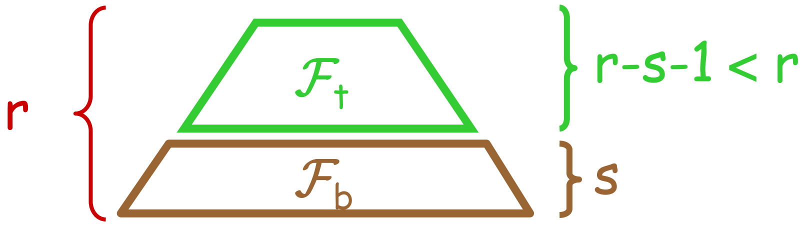

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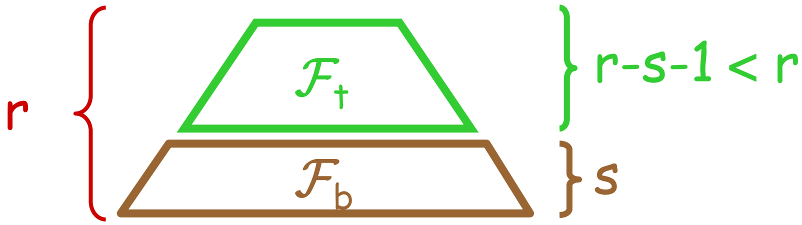

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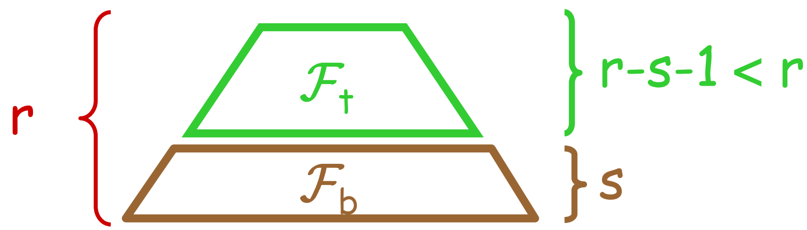
$$f(M,N,R) \leq k \cdot M + 2N \cdot g(R)$$



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$$s = \log g(r)$$

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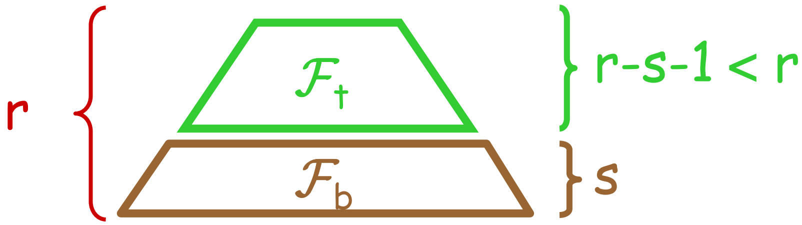

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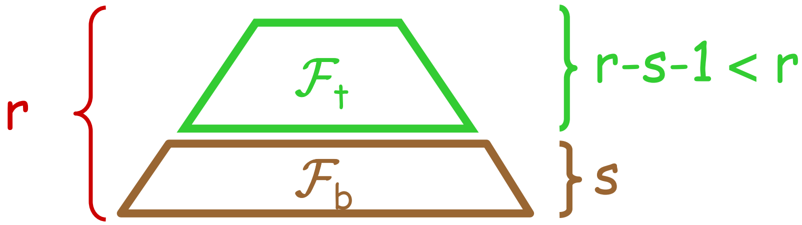

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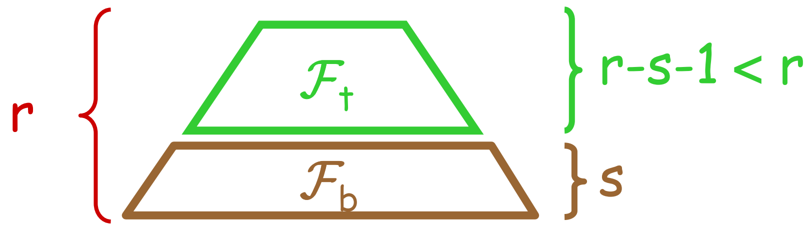

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$$(\,f(m,n,r) - (k+1) \cdot m\,) \leq 2n + (\,f(m_b, n, \log g(r)\,) - (k+1) \cdot m_b\,)$$

$$(\,f(m,n,r) - (k+1) \cdot m\,) \leq 2n \cdot (\log \circ g)^*(r)$$

$$\boxed{f(m,n,r) \leq (k+1) \cdot m + 2n \cdot (\log \circ g)^*(r)}$$

Def.: $g : \mathbb{N} \rightarrow \mathbb{N}$ "nice"

$$g^{\diamond}(r) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 + g^{\diamond}(\lceil \log_2 g(r) \rceil) & \text{if } n > 1 \end{cases}$$

Note: $g^{\diamond} = (\lceil \log_2 \rceil \circ g)^*$

Shifting Lemma:

Assume $k \geq 0$, $g: \mathbb{N} \to \mathbb{N}$, "nice", non-decreasing, $g(r) < r$ for $r > 0$.

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Assume $k \geq 0$, $g: \mathbb{N} \rightarrow \mathbb{N}$, "nice", non-decreasing, $g(r) < r$ for $r > 0$.

If

$$f(m,n,r) \leq k \cdot m + 2 \cdot n \cdot g(r) \qquad \text{for all } m,n,r$$

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then also

$$f(m,n,r) \leq (k+1)\cdot m + 2\cdot n\cdot g^{\diamond}(r) \qquad \text{for all } m,n,r$$

Def: $J_0(r) = \lceil (r-1)/2 \rceil$

$J_k(r) = J_{k-1}^{\diamond}(r)$ for $k>0$

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Note: $r \leq \lfloor \log_2 n \rfloor$ always

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Corollary: $f(m,n,r) \le (\, \alpha(m,n) + 2 \,)m + 2n$

Corollary:

Any sequence of m Union, Find operations in a universe of n elements that uses linking by rank and path compression takes time at most

$$O(\ m \cdot \alpha(m,n) + n\ )$$

Hopcroft – Ullman, Tarjan, van Leeuwen, Kozen, Harfst-Reingold;
Sharir

For $r \leq 65$: $J_1(r) \leq 2$

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Actually:

$f(m,n,r) \le m+2.1n$ for $n<2^{66}$

$f(m,n,r) \le 2m+n$ for $n<2^{2^{24615}}$

Similar proof for $O(m \cdot \alpha(m,n) + n)$ bound also works for

linking by weight and path compression

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linking by rank and generalized path compaction

Heuristic 2': Path compaction

when performing a Find(x) operation make "all" nodes in the "findpath" child of some node further up.

1
↑
2
↑
3
↑
4
↑
5
↑
6
↑
7
↑
8

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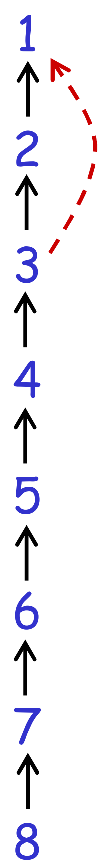

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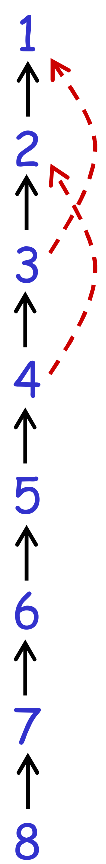

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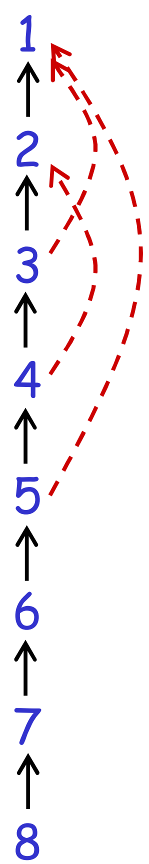

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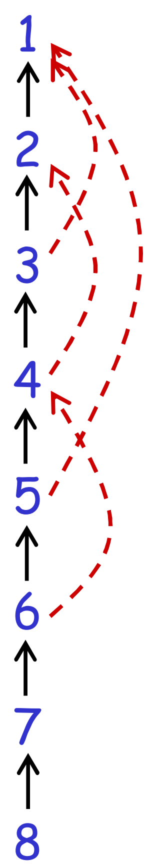

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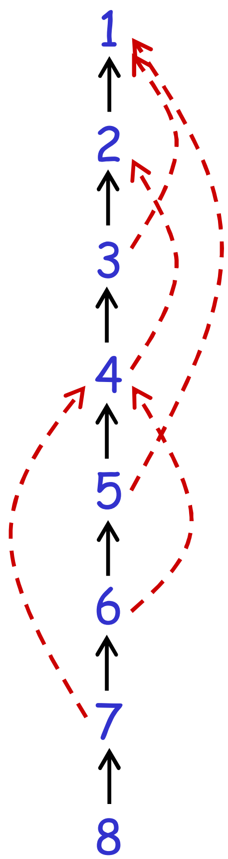

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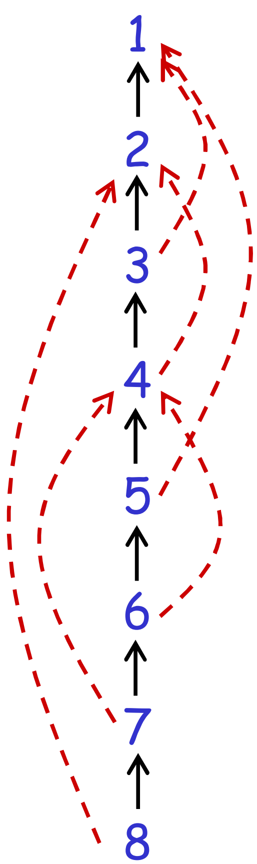

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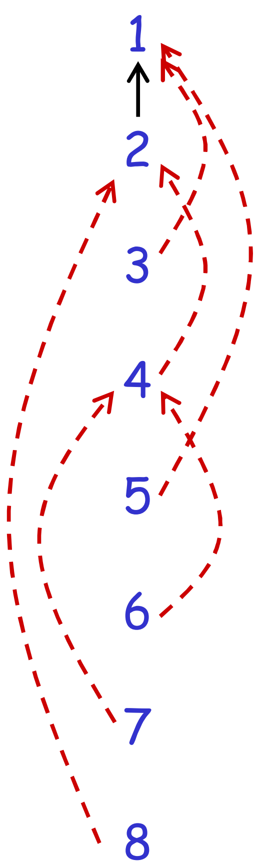

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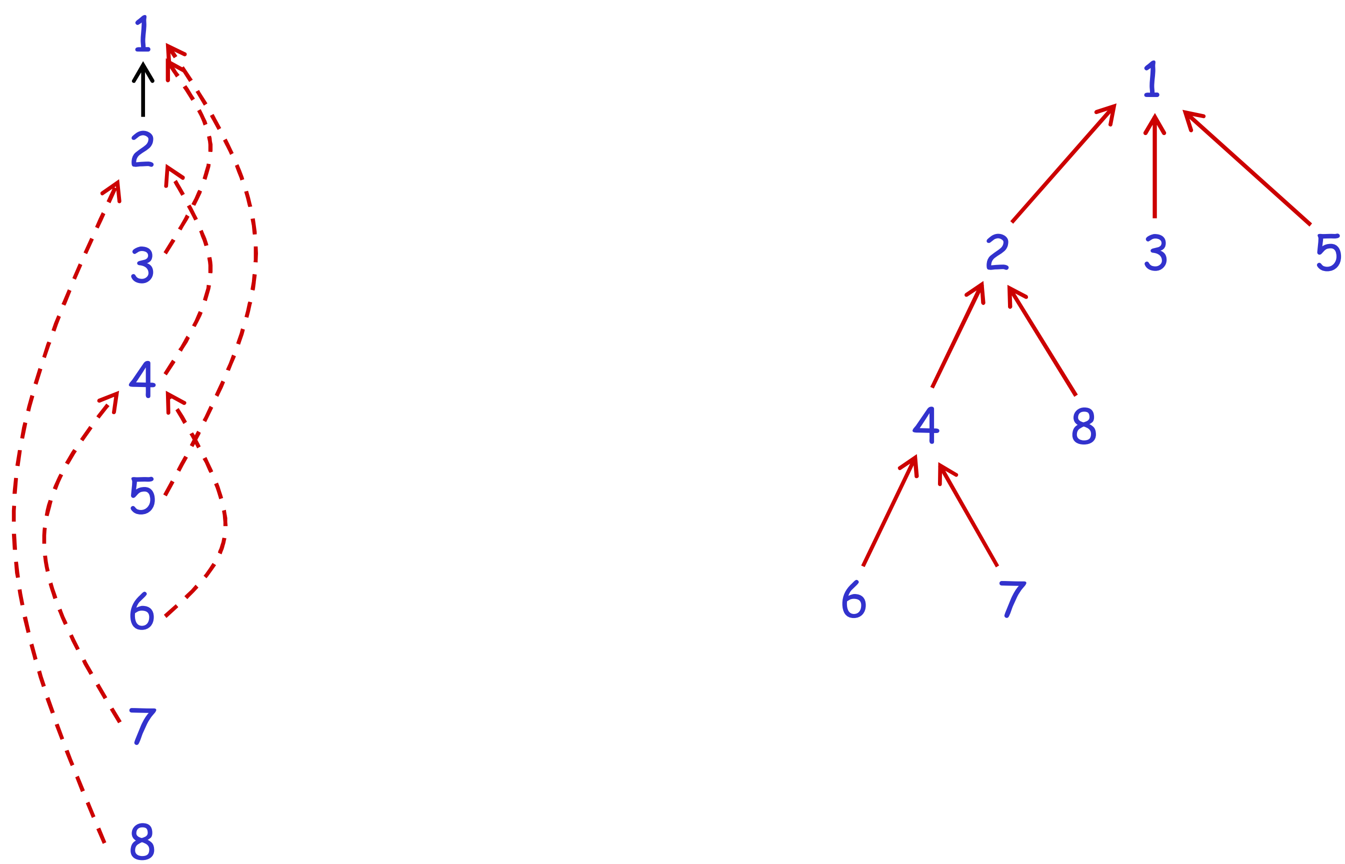

Main Lemma:

$\mathcal{C}$ ... sequence of compress operations on $\mathcal{F}$ with node set X

X_t, X_b dissection for $\mathcal{F}$ inducing subforests $\mathcal{F}_t$, $\mathcal{F}_b$

$\Rightarrow$ $\exists$ compression sequences $\mathcal{C}_b$ for $\mathcal{F}_b$ and $\mathcal{C}_t$ for $\mathcal{F}_t$ with

$$|\mathcal{C}_b| + |\mathcal{C}_t| \leq |\mathcal{C}|$$

and

$$\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{C}_b) + \text{cost}(\mathcal{C}_t) + |X_b| + |\mathcal{C}_t|$$

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Proof:

$$|C_b| + |C_t| \leq |C|$$

compression paths from C

case 1: $x \to y$ $x \to y$ into C_t

case 2: $x \to y$ $x \to y$ into C_b

case 3: $x \to x' \to y$ $x' \to y$ into C_t

$x \to \infty$ into C_b

Proof:

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compaction paths from C

case 1:   into C_t

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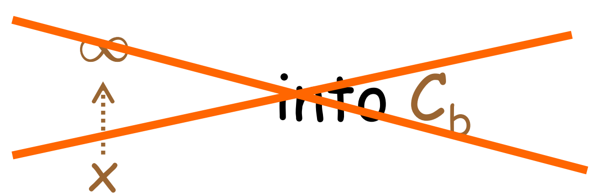
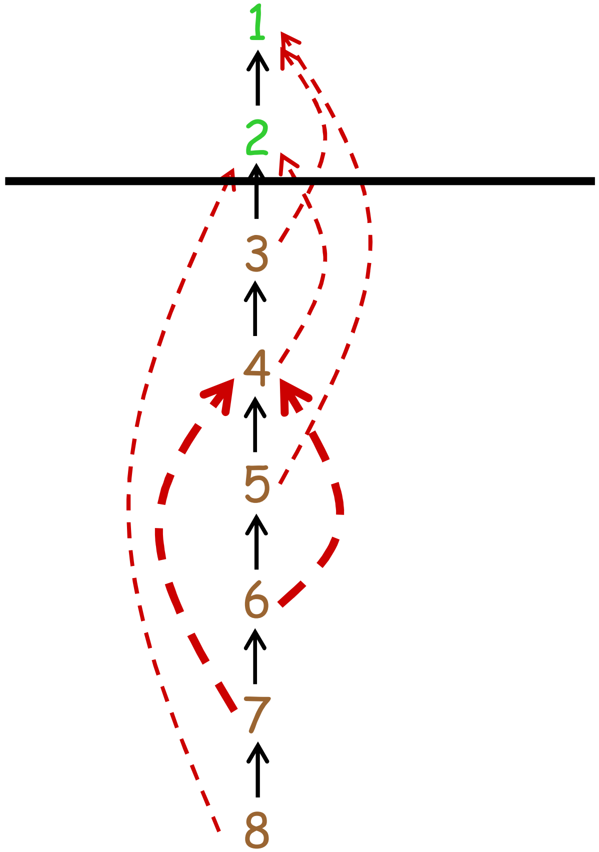
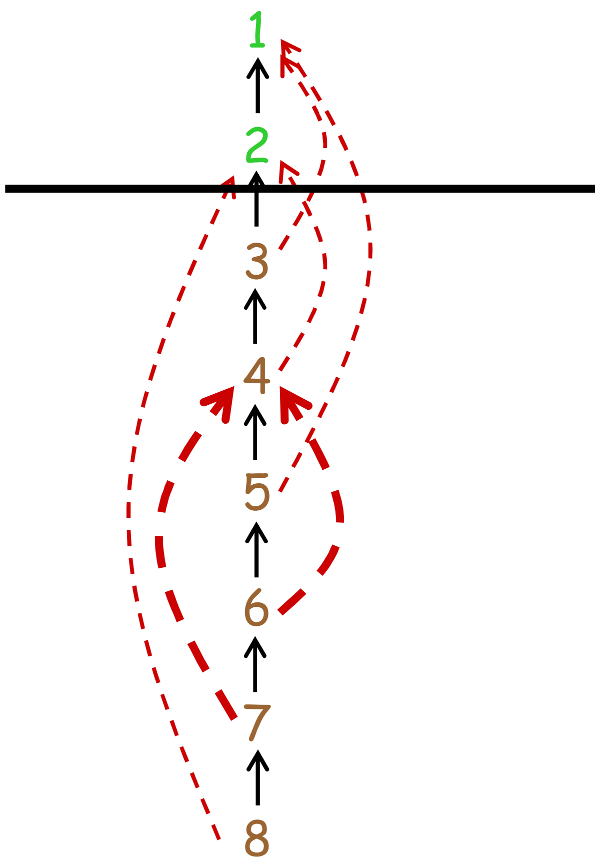
case 3:   into C_t



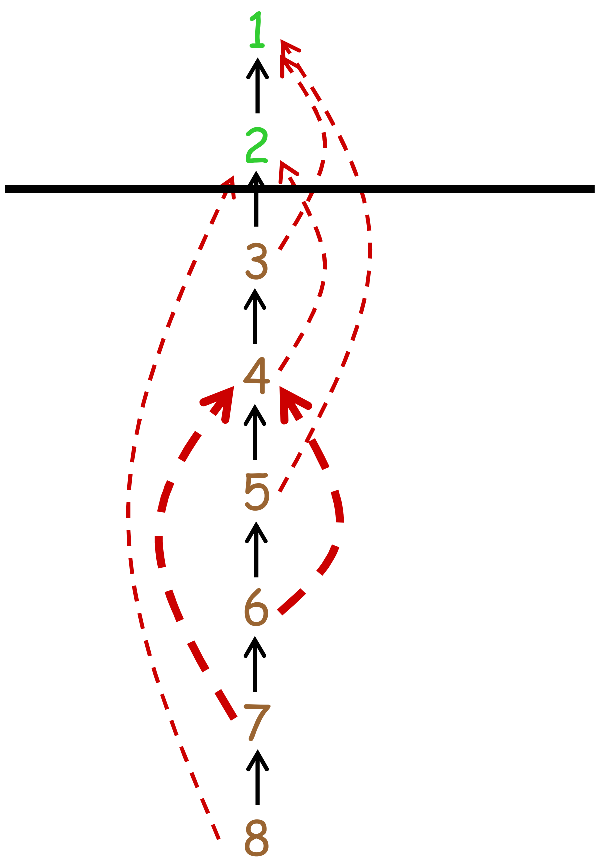
Compaction of path that crosses dissection boundary:





Compaction of path that crosses dissection boundary:

Charge getting a new brown parent to the topmost brown node v that gets a green parent for the first time.



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Charge getting a new brown parent to the topmost brown node v that gets a green parent for the first time.

happens to v at most once;
v can be charged at most height(v)

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$$\text{cost}(C) \le \text{cost}(C_b) + \text{cost}(C_t) + 2 \cdot |X_b| + |C_t|$$

Corollary:

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Open problems:

- path compaction and everything else but linking by rank
- top-down approach for lower bounds