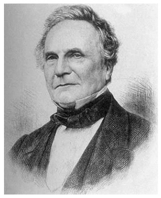
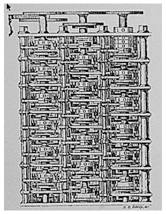
4.1, 4.2 Analysis of Algorithms

Running Time

> As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - Charles Babbage


Charles Babbage (1864)


Analytic Engine (schematic)

Overview

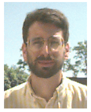
Analysis of algorithms. Framework for comparing algorithms and predicting performance.

Scientific method.
- Observe some feature of the universe.
- Hypothesize a model that is consistent with observation.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate the theory by repeating the previous steps until the hypothesis agrees with the observations.

Universe = computer itself.

Algorithmic Successes

N-body Simulation.
- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: N log N steps, enables new research.


Andrew Appel
PU '81

Discrete Fourier transform.
- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.
- FFT algorithm: N log N steps, enables new technology.


Freidrich Gauss
1805

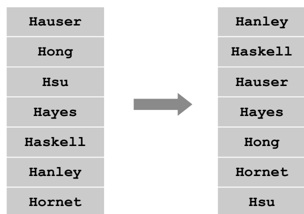
Sorting.
- Rearrange N items in ascending order.
- Fundamental information processing abstraction, enables commerce.


Jon von Neumann
IAS 1945

Case Study: Sorting

Sorting problem. Rearrange N items in ascending order.

Applications. Statistics, databases, data compression, computational biology, computer graphics, scientific computing, ...



Insertion Sort

Insertion Sort

Insertion sort.

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

Insertion Sort: Java Implementation

```java
public class Insertion {
   private static boolean less(double x, double y) {
      return (x < y);
   }

   private static void exch(double[] a, int i, int j) {
      double swap = a[i];
      a[i] = a[j];
      a[j] = swap;
   }
```
sorting helper functions

```java
   public static void sort(double[] a) {
      for (int i = 0; i < a.length; i++) {
         for (int j = i; j > 0; j--) {
            if (less(a[j], a[j-1]))
               exch(a, j, j-1);
            else break;
         }
      }
   }
}
```
insertion sort

Insertion Sort: Observation

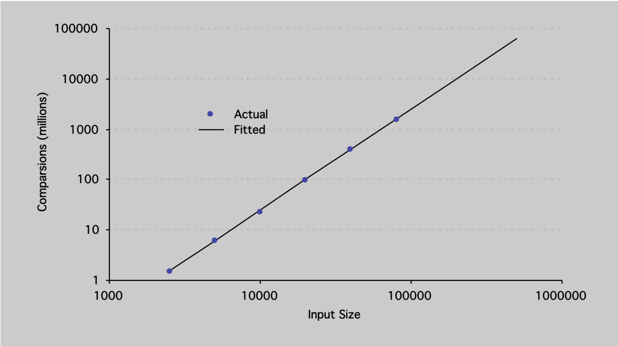
Observe and tabulate running time for various values of N.

- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.
- Timing: Skagen wristwatch.

N	Comparisons	Time
5,000	6.2 million	0.13 seconds
10,000	25 million	0.43 seconds
20,000	99 million	1.5 seconds
40,000	400 million	5.6 seconds
80,000	1600 million	23 seconds

Insertion Sort: Experimental Hypothesis

Data analysis. Plot # comparisons vs. input size on log-log scale.



power law

slope

Regression. Fit line through data points ~ a N^b.

Hypothesis. # comparisons grows quadratically with input size ~ $N^2/4$.

Insertion Sort: Prediction and Verification

Experimental hypothesis. # comparisons ~ $N^2/4$.

Prediction. 400 million comparisons for N = 40,000.

Observations.

N	Comparisons	Time
40,000	401.3 million	5.595 sec
40,000	399.7 million	5.573 sec
40,000	401.6 million	5.648 sec
40,000	400.0 million	5.632 sec

Agrees.

Prediction. 10 billion comparisons for N = 200,000.

Observation.

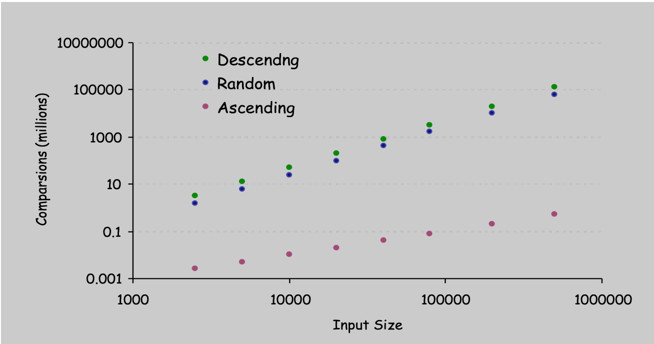
N	Comparisons	Time
200,000	9.997 billion	145 seconds

Agrees.

Insertion Sort: Validation

Number of comparisons depends on input family.

- Descending: $N^2/2$.
- Random: $N^2/4$.
- Ascending: N.



Insertion Sort: Theoretical Hypothesis

Experimental hypothesis.
- Measure running times, plot, and fit curve.
- Model useful for predicting, but not for explaining.

Theoretical hypothesis.
- Analyze algorithm to estimate # comparisons as a function of:
  - number of elements N to sort
  - average or worst case input
- Model useful for predicting and explaining.

Critical difference. Theoretical model is independent of a particular machine or compiler; applies to machines not yet built.

Insertion Sort: Analysis

Worst case. (descending)
- Iteration i requires i comparisons.
- Total = (0 + 1 + 2 + ... + N-1) ~ $N^2 / 2$ compares.

E	F	G	H	I	J	D	C	B	A
						i			

Average case. (random)
- Iteration i requires i/2 comparisons on average.
- Total = (0 + 1 + 2 + ... + N-1) / 2 ~ $N^2 / 4$ compares

A	C	D	F	H	J	E	B	I	G
						i			

Insertion Sort: Theoretical Hypothesis

Theoretical hypothesis.

Analysis	Comparisons	Stddev
Worst	$N^2 / 2$	-
Average	$N^2 / 4$	$1/6\ N^{3/2}$
Best	N	-

Validation. Theory agrees with observations.

N	Actual	Predicted
40,000	401.3 million	400 million
200,000	9.9997 billion	10.000 billion

Insertion Sort: Lesson

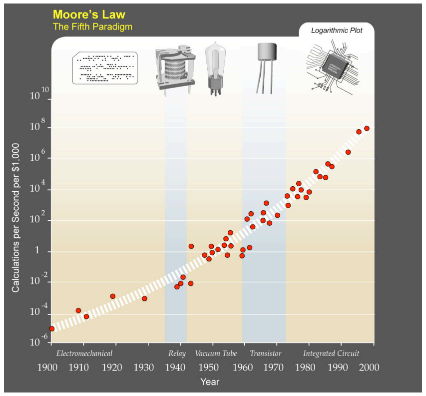
Lesson. Supercomputer can't rescue a bad algorithm.

Computer	Comparisons Per Second	Thousand	Million	Billion
laptop	10^7	instant	1 day	3 centuries
super	10^{12}	instant	1 second	2 weeks

Moore's Law

Moore's law. Transistor density on a chip doubles every 2 years.

Variants. Memory, disk space, bandwidth, computing power per $.



http://en.wikipedia.org/wiki/Moore's_law

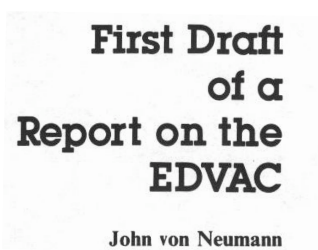
Moore's Law and Algorithms

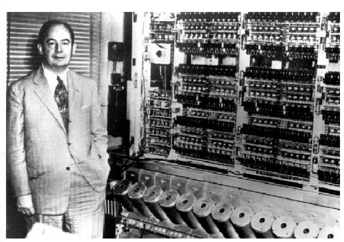
Quadratic algorithms do not scale with technology.
- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

> Software inefficiency can always outpace Moore's Law.
> Moore's Law isn't a match for our bad coding. - Jaron Lanier

Lesson. Need linear algorithm to keep pace with Moore's law.

Mergesort





Mergesort

Mergesort.
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

input

M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E

sort left

E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E

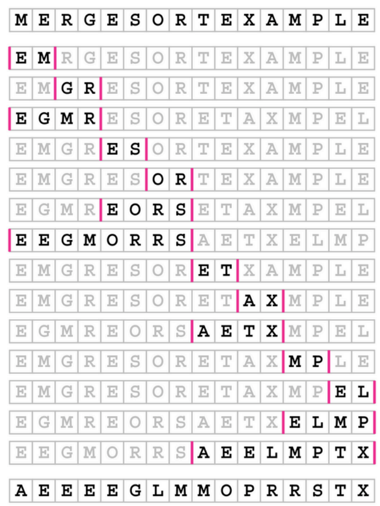
sort right

E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X

merge

A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

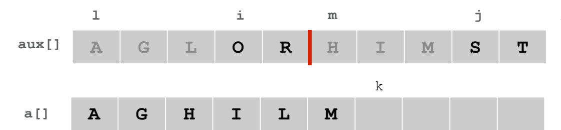
Mergesort: Example



Merging

Merging. Combine two pre-sorted lists into a sorted whole.

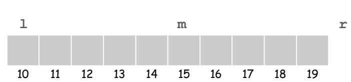
How to merge efficiently? Use an auxiliary array.




```
private static void merge(double[] a, double[] aux, int l, int m, int r) {
   for (int k = l; k < r; k++) aux[k] = a[k];
   int i = l, j = m;
   for (int k = l; k < r; k++) {
      if      (i >= m)                a[k] = aux[j++];
      else if (j >= r)                a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                            a[k] = aux[i++];
   }
}
```

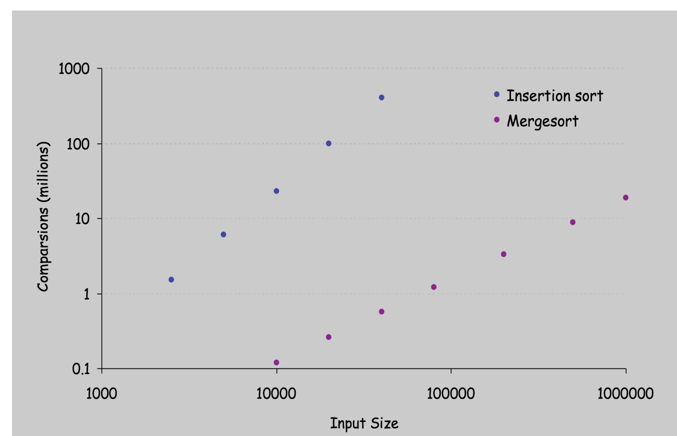

Mergesort: Java Implementation

```
public class Merge {
   private static boolean less(double x, double y)
      // as before

   private static void merge(double[] a, double[] aux, int l, int m, int r) {
      // see previous slide

   private static void sort(double[] a, double[] aux, int l, int r) {
      if (r <= l + 1) return;
      int m = l + (r - l) / 2;
      sort(a, aux, l, m);
      sort(a, aux, m, r);
      merge(a, aux, l, m, r);
   }

   public static void sort(double[] a) {
      double[] aux = new double[a.length];
      sort(a, aux, 0, a.length);
   }
}
```



Mergesort: Preliminary Hypothesis

Experimental hypothesis. Number of comparisons ~ 20N.



Mergesort: Prediction and Verification

Experimental hypothesis. Number of comparisons ~ 20N.

Prediction. 80 million comparisons for N = 4 million.

Observations.

N	Comparisons	Time
4 million	82.7 million	3.13 sec
4 million	82.7 million	3.25 sec
4 million	82.7 million	3.22 sec

Agrees.

Prediction. 400 million comparisons for N = 20 million.

Observations.

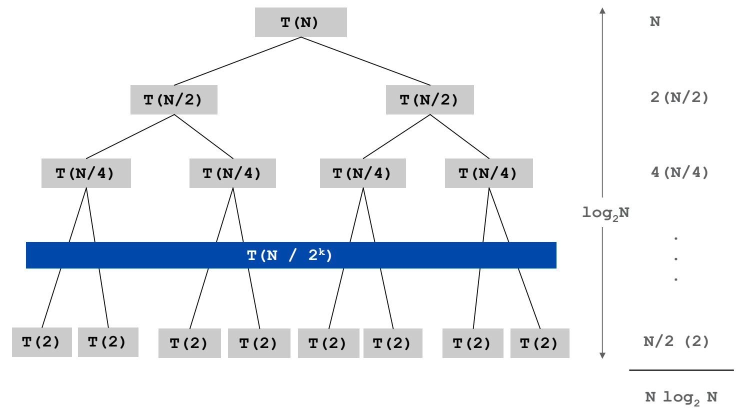
N	Comparisons	Time
20 million	460 million	17.5 sec
50 million	1216 million	45.9 sec

Not quite.

Mergesort: Analysis

Analysis. To mergesort array of size N , mergesort two subarrays of size N/2, and merge them together using ≤ N comparisons.

we assume N is a power of 2



Mergesort: Theoretical Hypothesis

Theoretical hypothesis.

Analysis	Comparisons
Worst	$N \log_2 N$
Average	$N \log_2 N$
Best	$1/2\ N \log_2 N$

Validation. Theory now agrees with observations.

N	Actual	Predicted
10,000	120 thousand	133 thousand
20 million	460 million	485 million
50 million	1,216 million	1,279 million

Mergesort: Lesson

Lesson. Great algorithms can be more powerful than supercomputers.

Computer	Comparisons Per Second	Insertion	Mergesort
laptop	10^7	3 centuries	3 hours
super	10^{12}	2 weeks	instant

N = 1 billion

Scientific Method

Scientific method applies to estimate running time.

- Experimental analysis: not difficult to perform experiments.
- Theoretical analysis: may require advanced mathematics.
- Small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

$\log_2 N$

```
while (N > 1) {
   N = N / 2;
   ...
}
```

N

```
for (int i = 0; i < N; i++)
   ...
```

N^2

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

$N \log_2 N$

```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

2^N

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```


Order of Growth Classifications

Order of growth.

- Estimate running time as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care
- Ex: $6N^3 + 17N^2 + 56 \sim 6N^3$.

Donald Knuth
Turing award '74

Function	Description	When N doubles, running time
1	constant algorithm is independent of input size	does not change
log N	logarithmic algorithm gets slightly slower as N grows	increases by a constant
N	linear algorithm is optimal for processing N inputs	doubles
N log N	linearithmic algorithm scales to huge N	slightly more than doubles
N^2	quadratic algorithm is impractical for large N	quadruples
2^N	exponential algorithm is not usually practical	squares!

Summary

How can I evaluate the performance of my algorithm?

- Computational experiments.
- Theoretical analysis.

What if it's not fast enough?

- Understand why.
- Buy a faster computer.
- Find a better algorithm in a textbook.
- Discover a new algorithm.

Attribute	Better Machine	Better Algorithm
Cost	$$$ or more.	$ or less.
Applicability	Makes "everything" run faster.	Does not apply to some problems.
Improvement	Quantitative improvements.	Dramatic qualitative improvements possible.