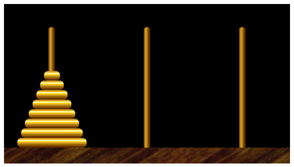
2.3 Recursion



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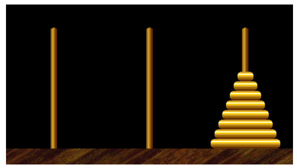


finish

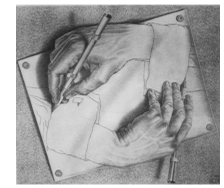
Overview

What is recursion? When one function calls itself directly or indirectly.

Why learn recursion?
- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.
- Mergesort, FFT, gcd.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.



Reproductive Parts
M. C. Escher, 1948

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$$4032 = 2^6 \times 3^2 \times 7^1$$
$$1272 = 2^3 \times 3^1 \times 53^1$$
$$\text{gcd} = 2^3 \times 3^1 = 24$$

Applications.
- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \% \ q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

```
gcd(4032, 1272)  = gcd(1272, 216)
                 = gcd(216, 192)
                 = gcd(192, 24)
                 = gcd(24, 0)
                 = 24.
```

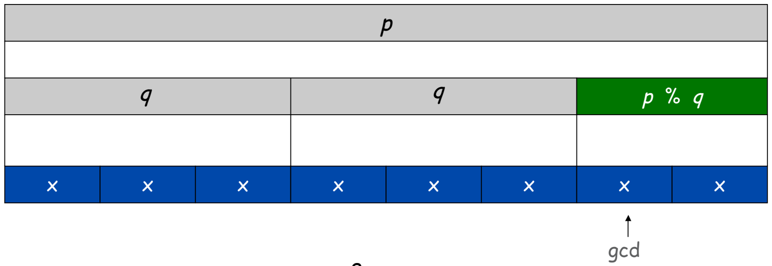
4032 = 3 × 1272 + 216

Gcd. Find largest integer d that evenly divides into p and q.

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \% q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case



p = 8x
q = 3x
gcd(p, q) = x

Gcd. Find largest integer d that evenly divides into p and q.

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \% q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

Java implementation.

```
public static int gcd(int p, int q) {
   if (q == 0) return p;          ← base case
   else return gcd(q, p % q);     ← reduction step
}
```


Recursive Graphics


WEEKEND Arts FINE ARTS LEISURE

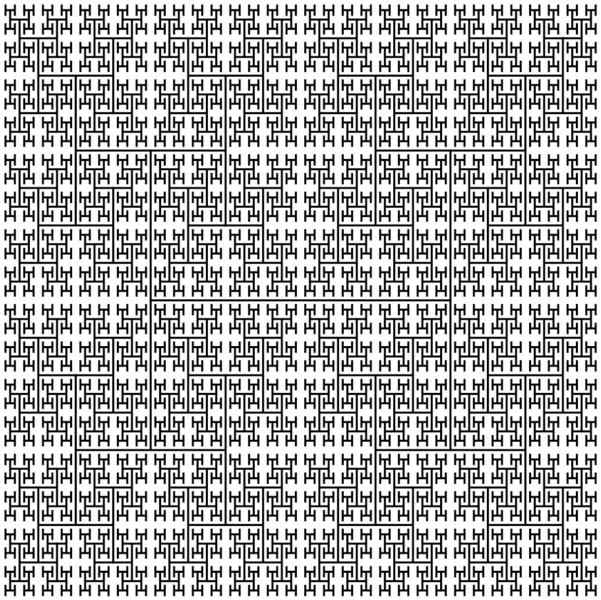
The New York Times

Fruits of Design, Certified Organic

The Gifts to Open Again and Again

Black, White and Read All Over Over

Divine and Devotee Meet Across Hinges

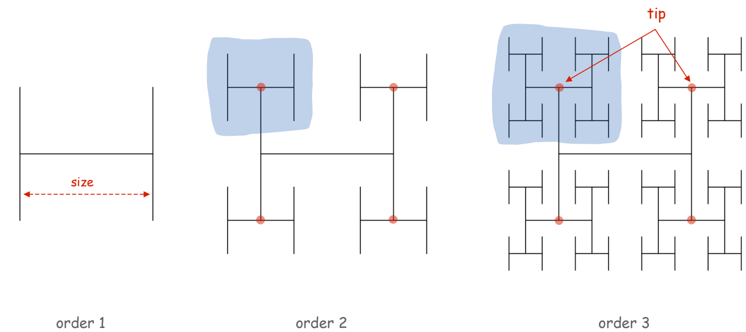


Htree

H-tree of order n.

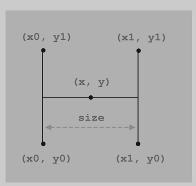
and half the size

- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip.



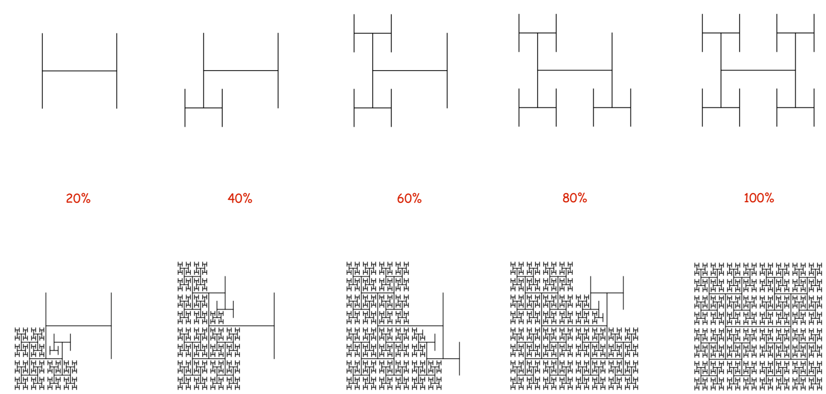
Htree in Java

```
public class Htree {
   public static void draw(int n, double size, double x, double y) {
      if (n == 0) return;
      double x0 = x - size/2, x1 = x + size/2;
      double y0 = y - size/2, y1 = y + size/2;

      StdDraw.line(x0,  y, x1,  y);     ← draw the H, centered on (x, y)
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, size/2, x0, y0);        ← recursively draw 4 half-size Hs
      draw(n-1, size/2, x0, y1);
      draw(n-1, size/2, x1, y0);
      draw(n-1, size/2, x1, y1);
   }

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```



Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.

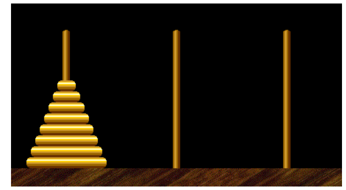


Towers of Hanoi

Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



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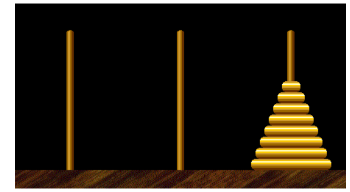
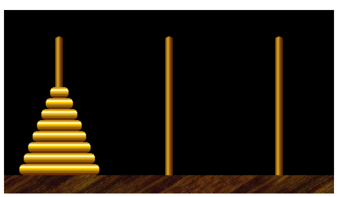


finish

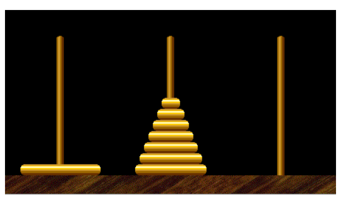
Towers of Hanoi demo

Edouard Lucas (1883)

Towers of Hanoi: Recursive Solution

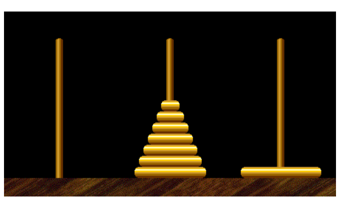


Move n-1 smallest discs right.

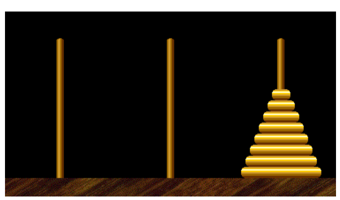


Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.



Towers of Hanoi Legend

Q. Is world going to end (according to legend)?

- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi {

   public static void moves(int n, boolean left) {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

`moves(n, true)` : move discs 1 to n one pole to the left
`moves(n, false)`: move discs 1 to n one pole to the right

Towers of Hanoi: Recursive Solution

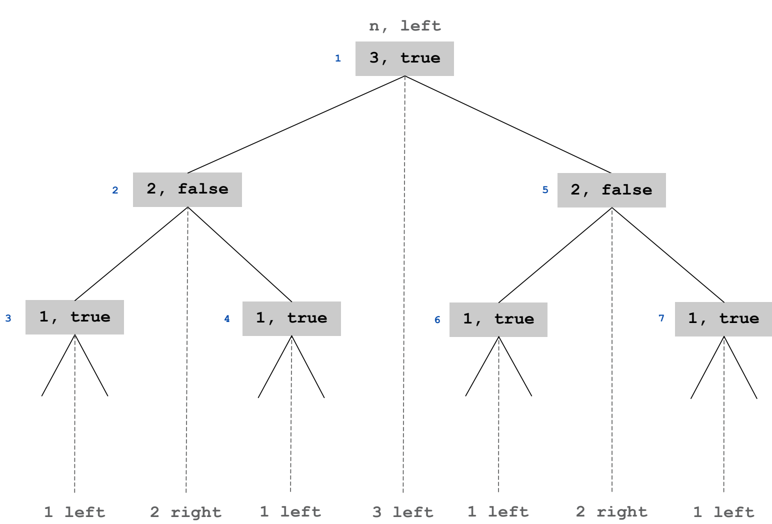
```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions of ruler

Towers of Hanoi: Recursion Tree



Properties of Towers of Hanoi Solution

Remarkable properties of recursive solution.
- Takes $2^n - 1$ steps to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Smallest disc always moves in same direction.

Recursive algorithm yields non-recursive solution!
- Alternate between two moves:
  - move smallest disc to right if n is even (to left if n is odd)
  - make only legal move not involving smallest disc

Recursive algorithm may reveal fate of world.
- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Divide-and-Conquer

Divide-and-conquer paradigm.

- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

> Divide et impera. Veni, vidi, vici. - Julius Caesar

Many important problems succumb to divide-and-conquer.

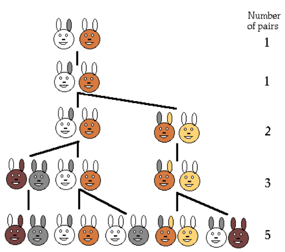
- FFT for signal processing.
- Multigrid methods for solving PDEs.
- Adaptive quadrature for integration.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.
- Midpoint displacement method for fractional Brownian motion.

Pitfalls

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

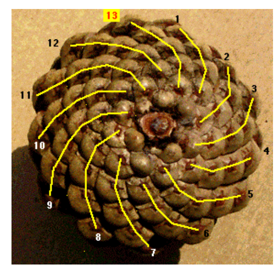
$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

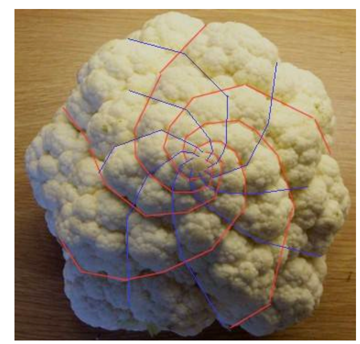


Fibonacci rabbits


L. P. Fibonacci
(1170 - 1250)

Fibonacci Numbers and Nature


pinecone


cauliflower

Possible Pitfalls With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

A natural for recursion?

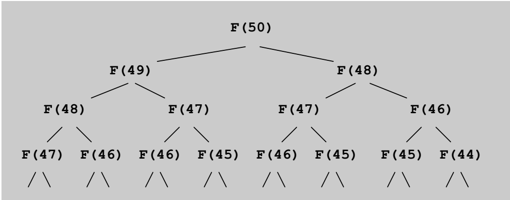
```
public static long F(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

spectacularly inefficient code

Observation. It takes a really long time to compute `F(50)`.

Possible Pitfalls With Recursion

Caveat. Can easily write remarkably inefficient programs.



recursion tree for naïve Fibonacci function

F(50) is called once.
F(49) is called once.
F(48) is called 2 times.
F(47) is called 3 times.
F(46) is called 5 times.
F(45) is called 8 times.
...
F(1) is called 12,586,269,025 times.

↖ F(50)

Binet's formula.

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rfloor$$

↖ ϕ = golden ration ≈ 1.618

Summary

How to write simple recursive programs?

- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Towers of Hanoi by W. A. Schloss.

Why learn recursion?

- New mode of thinking.
- Powerful programming tool.

Divide-and-conquer. Elegant solution to many important problems.