All-Pay-All Auctions

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Abstract

We consider a class of auctions in which bidders receive (asymmetric) shares of the amount of money raised. Examples include corporate takeovers when potential acquirers have "toeholds" and auctions used to raise money for a public good. We show that the optimal selling mechanism in these cases is a simple "all-pay-all" auction in which all bidders pay a weighted sum of all bids. When bidders receive asymmetric shares of the auction's revenue, the all-pay-all auction "levels the playing field" and ensures that the bidder with the highest signal receives the object. In other words, the all-pay-all auction is always efficient. These properties contrast sharply with those of standard auction formats such as the first and second price auction, which are not revenue maximizing and generally not efficient when bidders' shares are asymmetric. Finally, we show that standard auctions may perform particularly poorly when they are used to raise money for a public good.

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1. Introduction

There are many situations in which losing bidders in an auction benefit from driving up the winner's price. Consider, for instance, a wine auction in which several local wine merchants participate. All merchants would prefer their rivals pay a higher price in the auction so that they cannot be as competitive in the aftermarket. Alternatively, when bidders participate in a series of auctions such as the recent European "spectrum auctions," losers benefit from a higher winning bid if they know the winner faces some aggregate budget constraint. In this paper we consider auctions in which each bidder has an incentive to drive up the price because they receive a fixed share of the auction's revenue. In takeover situations, for example, losing bidders who own some of the target's shares ("toeholds") receive payoffs proportional to the sales price (e.g. Singh, 1998). In a "charity auction" (Engers and McManus, 2000), altruistic bidders receive benefits proportional to the amount of money raised. In "knockout auctions" (Graham and Marshall, 1987; McAfee and McMillan, 1992) every member of a bidding ring receives a payment in direct proportion to the winning bid, etc.[1] We analyze such auctions from the point of view of the seller who receives what is left of the revenue after all bidders are paid their shares.

In our model, bidders' shares are not necessarily equal, i.e. we allow for asymmetries, and bidders' information about the object's value may have private and common value elements. Bulow, Huang, and Klemperer (1999) have shown that in the case of pure common values, asymmetries in bidders' shares can have a dramatic effect on equilibrium bidding behavior in an ascending auction (and less so in a sealed-bid auction).[2] For example, a bidder with a larger share has a significantly higher probability of winning, even when she does not have the highest signal. In particular, a bidder with a zero share has zero probability of winning. Moreover, bidders with smaller shares face a stronger winner's curse which suppresses their (and consequently others') bid and leads to low revenues. Bulow, Huang, and Klemperer (1999) suggest that in some cases the seller may benefit from giving free shares to "smaller bidders" to "level the playing field."

In this paper we follow an alternative approach. Instead of changing the distribution of shares to enhance revenues, we propose a different auction format in which all bidders

[1] Other examples include creditors bidding in bankruptcy auctions (Burkart, 1995) and heirs bidding for a family estate (Engelbrecht-Wiggans, 1994).

[2] See also Bikhchandani (1988); Bulow and Klemperer (1999); and Klemperer (1998).

pay a weighted sum of all bids: the *all-pay-all auction*. We show that the all-pay-all rules transform the auction in which bidders receive shares of the revenue into a standard all-pay auction in which bidders receive no shares. As a result, the all-pay-all auction is efficient, i.e. it assigns the object to the highest-signal bidder, even when bidders' shares are asymmetric and both private and common value elements play a role. Moreover, the all-pay-all auction results in the highest possible payoff to the seller and her payoff is independent of the size of her and others' shares.

The all-pay-all rules are akin to those proposed by Cramton, Gibbons, and Klemperer (1987) to dissolve a partnership. In their model, bidder i has a private value, v_i, for an object of which she currently owns a share, r_i. They show that a bidding game in which bidders' payments are based on all bids and owners of larger shares are reimbursed through side-payments, can be used to efficiently dissolve a dissolvable partnership. Not all partnerships are dissolvable because of an "individual rationality" constraint: bidders' expected equilibrium profits in the bidding game cannot be less than their initial "wealth" level $r_i v_i$. Our model differs from theirs in several respects. First, bidders in our model do not face an individual rationality constraint (other than non-negative expected equilibrium payoffs in the auction); all shareholders are willing to sell their share at the highest price. This assumption is common in the literature on takeover bidding. Indeed, the individual rationality constraint is generally not satisfied in the standard first price auction or ascending auction discussed in Singh (1998) and Bulow, Huang, and Klemperer (1999). Second, each bidder's payment in the all-pay-all auction is non-negative and depends on her own and others' shares (and there are no side-payments). Finally, we single out one of the shareholders, the "seller," and ask the question which auction format is revenue-maximizing from her point of view.[3]

We apply the all-pay-all model to the case when an auction is used to raise money for a public good (see also Engers and McManus, 2000). We assume a symmetric situation in which each bidder values \$1 given to the public good at \$$\alpha$. The benefits that bidders receive from the auction's revenue do not diminish its value nor is the sum of bidders' shares necessarily less than one (since one bidder's benefit does not affect another's). In

[3]This issue is not addressed by Cramton, Gibbons, and Klemperer (1987) who treat all shareholders equally and determine circumstances under which an efficient dissolvement is possible without maximizing the payoff to a particular shareholder. A final (technical) extension is that our model includes both private and common values.

fact, when one or more outside parties match the auction's revenue, α can be greater than one. We show that the best way to raise money for the public good is by means of an all-pay-all auction or a simple "lowest price" all-pay auction. In addition, we demonstrate that standard auctions may perform rather poorly in terms of raising money. In the first price auction, for example, *a bidder who is indifferent between keeping $1 for herself or giving it to the public good bids only the object's expected value.* These low bids are caused by a strong "free rider" effect when bidders prefer to lose and receive their share of another's bid rather than win the object. Stated differently, due to the positive externalities of a high bid, all bids are kept low. This free rider effect is even stronger in a second price auction where *bids are always finite even when bidders value $1 given to the public good at more than $1.* Asking for voluntary contributions would therefore be better than conducting a second price (or ascending) auction in this case.

Our derivation of the bidding functions for the first and second price auctions follows Engelbracht-Wiggans (1994) who considers the special case of two bidders. In a very interesting paper, Engers and McManus (2000) consider "charity auctions" in which altruistic bidders receive benefits proportional to the amount of money raised. They also extend Engelbrecht-Wiggans' analysis to the case of an arbitrary number of bidders. In addition, they prove that the second price auction raises more money than a first price auction under fairly general conditions.[4] Proposition 5 can be seen as a generalization of their result to allow for the possibility that the auction's revenue is matched by one or more outside parties. We show that auctions in which only the winner pays, e.g. a k^{th} price auction for $k = 1, 2, \ldots$, all yield the same revenue when $\alpha = 0$ or $\alpha = 1$ (see Proposition 6). We conjecture that a k^{th} price auction raises more (less) money than a $(k+1)^{th}$ price auction when $\alpha > 1$ ($\alpha < 1$).

Finally, our work is related to that of Jehiel, Moldovanu, and Stacchetti (1996). They consider auctions in which the winning bidder imposes an individual-specific negative externality on losing bidders. In this case, it may be in a bidder's best interest not to participate in the auction (Jehiel and Moldovanu, 1996). Our approach differs in two ways. First, we consider the case when the winning bidder imposes a positive externality on others. More importantly, however, externalities are exogenously specified in Jehiel,

[4] Engers and McManus (2000) also provide useful comparisons of alternative auction formats in the limit as the number of bidders tends to infinity.

Moldovanu, and Stacchetti (1996) while we allow them to be endogenously determined, i.e. their magnitudes increase with the auction's revenue.

This paper is organized as follows. The next section presents a simple example of two bidders with equal shares whose values for the object are private, independent, and uniformly distributed. Section 3 generalizes the example and shows that the all-pay-all auction is efficient and results in the highest possible payoff to the seller, even when bidders' shares are unequal. In section 4 we apply our results to the case where an auction is used to raise money for a public good. Section 5 concludes. All proofs can be found in the Appendix.

2. An Example

The owners of three adjacent parcels of land, "Left," "Middle," and "Right," face the following dilemma. Left contains oil but its owner, L, needs to acquire all three parcels to build a pipeline. Right, however, is owned by a real-estate developer, R, who needs the other parcels to plan a residential area. Finally, Middle's owner, M, has no use for her own land nor for the others'. Since M is the only one not interested in obtaining more land, she is chosen to design an auction in which L and R can bid for all three parcels (sold as one unit). The proceeds of the auction are split among the owners in proportion to the size of their land: L and R each receive an α share while M receives the remaining $(1-2\alpha)$ part, where $\alpha < \frac{1}{2}$. Which auction format is profit maximizing from M's point of view?

In many situations, the answer to this optimality question is simply "it doesn't matter, all auction formats yield the same revenue." One of the assumptions underlying this revenue equivalence result, however, is that the different formats assign the same (zero) expected payoff to a bidder with the lowest possible valuation. To show that this assumption is not valid in the above example, we compare a standard first and second price auction. In what follows we refer to M as the "seller" and to L and R as "bidders."

Suppose owning all three parcels is worth v_i to bidder $i = L, R$, where v_i is private information and independent of v_j, $j \neq i$. For simplicity, we take the v_i to be uniformly distributed on $[0, 1]$. It is straightforward to verify that the optimal bids in the first price auction are given by $B_1(v) = v/(2-\alpha)$, see Engelbrecht-Wiggans (1994).[5] Since the lowest-value bidder loses for sure, her expected payoff is simply the expected value of the other's bid: $\pi_1(0) = \alpha/(4-2\alpha)$. In the second-price (or ascending) auction optimal bids are given by $B_2(v) = (v+\alpha)/(1+\alpha)$, see Singh (1998).[6] The payoff to the lowest-value bidder now follows from the observation that her bid $B_2(0) = \alpha/(1+\alpha)$ is the second-highest bid for sure, so her expected payoff is $\pi_2(0) = \alpha^2/(1+\alpha)$. Note that $\pi_1(0) > \pi_2(0)$ for all $0 < \alpha < \frac{1}{2}$, i.e. bidders are better off when a first price auction is used than when a second price auction is used.

An auction that is better for the bidders is often worse for the seller. Comparing the revenues of the first and second price auctions shows this is also true in the above example. The expected revenue raised by the second price auction, $R_2 = (1+3\alpha)/(3+3\alpha)$, exceeds that of a first price auction, $R_1 = 2/(6-3\alpha)$, for all $0 < \alpha < \frac{1}{2}$. Can we conclude that the second price auction is revenue maximizing in this case (ignoring reserve prices and/or entry fees)? To show that the second price auction is not optimal in general, consider the first price all-pay auction. In this case, both the winning and the losing bidder have to pay their bid. It is straightforward to verify that the optimal bids are $B_{AP}(v) = \frac{1}{2}v^2/(1-\alpha)$ and that the resulting revenue is $R_{AP} = 1/(3-3\alpha)$.[7] Note that R_{AP} is less than R_2 for $\alpha < \frac{1}{3}$ but that R_{AP} *exceeds* R_2 for $\frac{1}{3} < \alpha < \frac{1}{2}$.

What is needed is a mechanism that yields a zero expected payoff for the lowest-value bidder. To accomplish this, a payment has to be extracted from the losing bidder as in a standard first price all-pay auction. However, to ensure that the expected payoff of the lowest-value bidder is zero, this payment has to be based on both the winner's and the loser's bid.[8] To this end, consider the following rules:

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[5] Consider a bidder with value v who bids as if she has value w and who faces a rival that bids according to $B_1(\cdot)$. The bidder's probability of winning is w and her expected payment if she wins is $B_1(w)$. Hence, her expected payoff is: $\pi^e(B_1(w)|v) = (v - B_1(w))w + \alpha(B_1(w)w + B_1((1+w)/2)(1-w))$. The first-order condition for profit maximization yields: $\partial_w \pi^e(B_1(w)|v) = v - w = 0$, so it is optimal for a bidder with value v to bid $B_1(v)$.

[6] Suppose the rival bids according to this solution and a bidder with value v bids $B_2(w)$. Her probability of winning is w and her expected payment if she wins is $B_2(w/2)$. Hence, her expected payoff is: $\pi^e(B_2(w)|v) = (v - B_2(w/2))w + \alpha(B_2(w/2)w + B_2(w)(1-w))$. The first-order condition for profit maximization yields: $\partial_w \pi^e(B_2(w)|v) = v - w = 0$, so it is optimal for a bidder with value v to bid $B_2(v)$.

[7] A bidder with value v who bids as if she has value w and who faces an opponent that bids according to $B_{AP}(\cdot)$ has expected payoffs $\pi^e(B_{AP}(w)|v) = vw - (1-\alpha)B_{AP}(w) + \alpha/(6-6\alpha)$, where the final term is α times the rival's expected bid. The first order condition for profit maximization dictates that $w = v$.

[8] Indeed, the expected payoff of the lowest-value bidder in a standard first price all-pay auction (where the loser's payment is proportional to the loser's bid) is $\pi_{AP}(0) = \alpha/(6-6\alpha)$. Note that $\pi_{AP}(0) > \pi_2(0)$

1. The highest bidder obtains the land.

2. Both bidders pay $\frac{1-\alpha}{1-2\alpha}$ times their own bid plus $\frac{\alpha}{1-2\alpha}$ times the other's bid.

These rules may seem to violate the participation constraint that requires non-negative expected equilibrium payoffs for the bidders. However, this is not the case as we now show. First, a bidder can ensure a zero payoff by bidding zero: she then pays $\frac{\alpha}{1-2\alpha}$ times the other's bid and receives $\alpha(\frac{1-\alpha}{1-2\alpha}+\frac{\alpha}{1-2\alpha})$ times the other's bid. More generally, let $B(\cdot)$ denote the optimal bidding function given the above rules, then a bidder's expected payoffs can be written as:

$$\begin{aligned}\pi^e(b|v) &= vB^{(-1)}(b) - \frac{1-\alpha}{1-2\alpha}\,b - \frac{\alpha}{1-2\alpha}\,E(B)\\ &+ \alpha\left(\frac{1-\alpha}{1-2\alpha}\,b + \frac{\alpha}{1-2\alpha}\,E(B) + \frac{1-\alpha}{1-2\alpha}\,E(B) + \frac{\alpha}{1-2\alpha}\,b\right), \qquad (2.1)\end{aligned}$$

where $E(B)=\int_0^1 B(y)\,\mathrm{d}y$ is the rival's expected bid. Combining the terms on the right side of (2.1) yields

$$\pi^e(b|v) \;=\; vB^{(-1)}(b) - b. \qquad (2.2)$$

In other words, bidders' expected payoffs under the above rules are the same as those of a standard first price all-pay auction! A bidder's *net* payment, which includes the payments she makes in the auction and her share of the auction's revenue, is therefore equal to her bid. The optimal bidding function for the all-pay auction is given by $B(v)=\frac{1}{2}v^2$, and equilibrium payoffs are $\pi^e(B(v)|v)=\frac{1}{2}v^2$. Hence, bidders' expected equilibrium payoffs are non-negative and zero for the lowest-value bidder.

The auction's revenue follows by adding bidders' expected payments:

$$R_{APA} = 2\,\frac{1-\alpha}{1-2\alpha}\,\frac{1}{6} + 2\,\frac{\alpha}{1-2\alpha}\,\frac{1}{6} = \frac{1}{3-6\alpha}. \qquad (2.3)$$

Note that the all-pay-all auction yields higher revenues than the other auctions considered so far and that its revenue tends to infinity when α tends to $\frac{1}{2}$.[9] The payments that accrue to the seller, M, are $(1-2\alpha)R_{APA}=\frac{1}{3}$, *independent of the size of the parcels!* Figure 1

for $0<\alpha<\frac{1}{3}$ and $\pi_{AP}(0)<\pi_2(0)$ for $\frac{1}{3}<\alpha<\frac{1}{2}$.

[9]In this limit, the all-pay-all auction turns into a standard second price all-pay auction, or "war-of-attrition," in which the highest bidder wins and both bidders pay the loser's bid. When $\alpha=\frac{1}{2}$, the loser can afford to bid an arbitrary high amount, b, since she pays b but also receives $\frac{1}{2}(b+b)$.

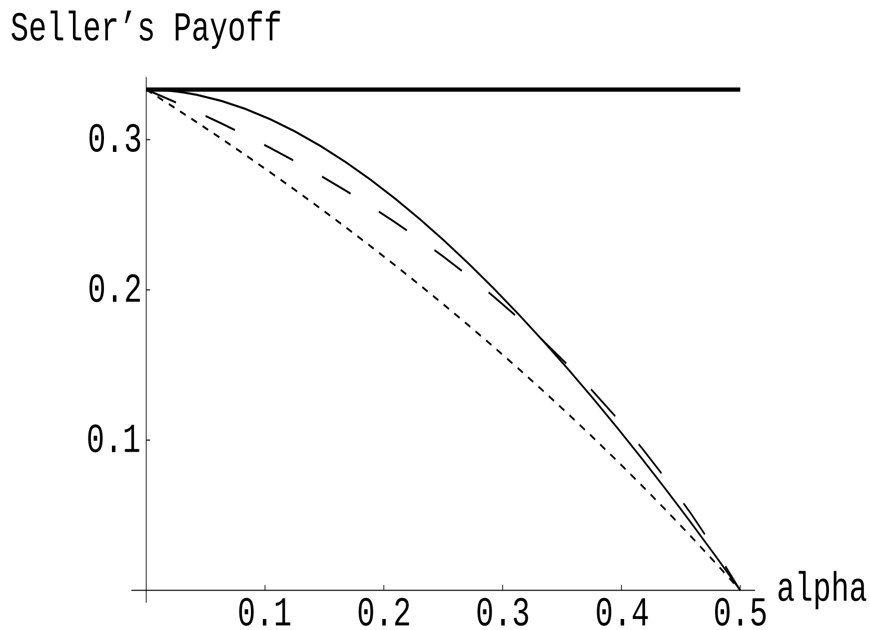

Figure 1: *The seller's payoff in an all-pay-all auction (thick solid line), an all-pay auction (long dashes), a first price auction (short dashes), and a second price auction (thin solid line) for $\alpha \leq 1/2$.*

shows the seller's profit in the different auction formats for $0 \leq \alpha \leq \frac{1}{2}$. The all-pay-all auction is the only format that gives M a non-zero profit as $\alpha \to \frac{1}{2}$, keeping M's payoff and L and R's combined payoffs at $\frac{1}{3}$.

3. The General Model

Consider $n \geq 2$ bidders who compete for a single object. Each bidder $i = 1, \cdots, n$ receives a signal, s_i, about the object's value. Bidders' signals are independent and distributed according to the (commonly known) distribution function $F(\cdot)$.[10] Without loss of generality, we can normalize the signals such that they have support $[0, 1]$, i.e. $F(0) = 0$ and $F(1) = 1$. Bidder i's value for the object, v_i, will in general depend on all n

[10]Our results readily extend to the case of affiliated signals (Milgrom and Weber, 1982).

signals. Following Milgrom and Weber (1982) we assume that $v_i = u(s_i, \{s_j\}_{j \neq i})$ for all i, where $u(\cdot, \cdot)$ is increasing in its first argument and non-decreasing in its second argument. Note that each bidder's value is a symmetric function of others' signals. For later use we define $v(x, y) = E(v_i | s_i = x, y_i = y)$ where $y_i = \max_{j \neq i} s_j$ is the maximum signal of all bidders different from i. Since the joint distribution of bidders signals is symmetric, the definition of $v(x, y)$ is independent of i.[11]

Let x_i denote the payment that bidder $i = 1, \cdots, n$ makes in the auction and let α_i denotes bidder i's share in the auction's revenue $R = \sum_{i=1}^n x_i$. We will not put any restrictions on the α_i except that $0 \leq \alpha_i < 1$ and $\sum_{i=1}^n \alpha_i < 1$. The remainder $\alpha_0 = 1 - \sum_{i=1}^n \alpha_i$ is the fraction of the revenue kept by the seller. Note that bidder i's net payment, which incorporates the α_i share she receives from the auction's revenue, is given by $x_i - \alpha_i \sum_{j=1}^n x_j$. The auction format determines the link between bids and payments in the auction. For instance, in a first price auction only the highest bidder makes a payment equal to her bid, while in an all-pay auction all bidders make payments equal to their bids. Consider the following rules:[12]

1. The object is assigned to the highest bidder.

2. Each bidder pays her own bid plus α_i/α_0 times the sum of all bids.

In other words, all bidders pay a non-negative fraction of all bids: the *all-pay-all auction*. As in the example of the previous section, a zero bid results in a zero payoff under the above rules. To see this, let $B_{-i} = \sum_{j \neq i} b_j$ denote the sum of all others' bids. A zero bid leads to a net payment of

$$\frac{\alpha_i}{\alpha_0} E(B_{-i}) - \alpha_i \Big(E(B_{-i}) + \sum_{j=1}^n \frac{\alpha_j}{\alpha_0} E(B_{-i}) \Big) \; = \; \frac{\alpha_i}{\alpha_0} E(B_{-i}) - \alpha_i \left(1 + \frac{1 - \alpha_0}{\alpha_0} \right) E(B_{-i}) \; = \; 0.$$

Proposition 1 shows that the optimal bids in the all-pay-all auction, in which bidders receive part of the revenue, are simply those of a standard all-pay auction in which bidders receive no share of the revenue.

[11]The case of independent private values corresponds to $v_i = s_i$, $i = 1, \cdots, n$, which yields $v(x, y) = x$. Similarly, a common value results when $v_i = \sum_j v_j$, in which case $v(x, y) = x + y + (n-2)E(z | z \leq y)$.

[12]As noted in the Introduction, Cramton, Gibbons, and Klemperer (1987) propose similar rules to dissolve a partnership.

Proposition 1. *The symmetric Nash equilibrium of the all-pay-all auction is given by*

$$B(s) \;=\; \int_0^s v(y,y)\,\mathrm{d}F(y)^{n-1}. \tag{3.1}$$

Bidders' expected equilibrium payoffs satisfy $\pi^*(s) \geq 0$ for all $s \in [0,1]$ with $\pi^*(0) = 0$.

Note that the optimal bidding function is independent of bidders' shares and increasing in the signal. *Hence, the all-pay-all auction awards the object to the bidder with the highest signal even when bidders' shares are asymmetric.*

This property contrasts sharply with those of standard auction formats. Bulow, Huang, and Klemperer (1999), for instance, consider the case of two bidders and a pure common value (which is nested in our model), and show that small asymmetries in bidders' shares can have a dramatic effect on equilibrium behavior in an ascending auction. In particular, the bidder with a (slightly) larger share has a significantly higher chance of winning even when her signal is not the highest. (A similar phenomenon occurs in the first price sealed-bid auction, although the effect of share asymmetries is less pronounced in this case.) Efficiency is not an issue in a pure common value setup but these results suggest that the outcome of a standard auction will not be efficient when private values are introduced (as in our model). In contrast, the outcome of the all-pay-all auction is independent of bidders' shares and is always efficient.

To determine the seller's payoff from the auction, note that the expected bid that follows from (3.1) is given by $E(B) = E(v(Y_2, Y_2))/n$ where Y_2 is the second-highest of all signals.[13] The *ex ante* expected payoff to the seller, $\pi_0 = \alpha_0 \sum_{i=1}^{n} E(x_i)$, can thus be

[13] Using (3.1) the expected bid $E(B) = \int_0^1 B(s)\,\mathrm{d}F(s)$ can be written as

$$E(B) \;=\; \int_0^1 \int_0^s v(z,z)\,\mathrm{d}F(z)^{n-1}\mathrm{d}F(s) \;=\; \int_0^1 \int_z^1 v(z,z)\,\mathrm{d}F(s)\,\mathrm{d}F(z)^{n-1},$$

where we changed the order of integration. The term on the far right side equals

$$E(B) \;=\; (n-1) \int_0^1 v(z,z)\,f(z)\,(1-F(z))\,F(z)^{n-2}\,\mathrm{d}z \;=\; E(v(Y_2,Y_2))/n,$$

since the density of the second-highest order statistic is: $f_{Y_2}(z) = n(n-1)f(z)(1-F(z))F(z)^{n-1}$, see Mood, Graybill, and Boes (1963).

worked out as

$$\pi_0 \;=\; \alpha_0 \sum_{i=1}^{n} \{E(B) + (\alpha_i/\alpha_0)\, n\, E(B)\} \;=\; E(v(Y_2, Y_2)). \tag{3.2}$$

Since the lowest type has zero expected equilibrium payoffs the revenue raised by the all-pay-all auction, $R_{APA} = E(v(Y_2, Y_2))/\alpha_0$, is the maximum possible revenue when the seller does not exclude (low-signal) bidders, i.e. when no reserve price or entry fee is used.

Proposition 2. *In the absence of reserve prices and/or entry fees, the all-pay-all auction is efficient and yields the highest possible payoff to the seller:* $\pi_0 = E(v(Y_2, Y_2))$*, independent of the sizes of the bidders' and seller's shares. The payoff to the winning bidder is* $E(v(Y_1, Y_2) - v(Y_2, Y_2))$.

To compare the revenue from an all-pay-all auction with that of a standard auction, recall that the lowest-signal bidder has strictly positive payoffs in a first or second price auction. These formats therefore yield lower payoffs to the seller (who earns nothing when her share tends to 0). Moreover, for the case of pure common values, Bulow, Huang, and Klemperer (1999) show that (small) asymmetries in bidders' shares substantially reduce the revenue from an ascending auction (and less so from a sealed bid first price auction). They point out that in this case the target firm may benefit from "leveling the playing field" by selling a cheap toehold to a "white knight." In contrast, an all-pay-all auction automatically levels the playing field and maximizes revenue independent of the distribution of shares.

In the special case of private values, i.e. when $v(x, y) = x$, it is straightforward to include a reserve price in the above analysis to obtain an "optimal" all-pay-all auction. It is well known that with independent, symmetric signals the seller can maximize her payoff by choosing a cut-off signal, or "screening level," s^*, below which it does not pay to bid. The optimal screening level satisfies

$$s^* \;=\; \frac{1 - F(s^*)}{f(s^*)}. \tag{3.3}$$

for all auctions in which the object is assigned to the highest bidder (including the all-pay auction) and is independent of the number of bidders (Riley and Samuelson, 1981; Myerson, 1981). Since the all-pay-all rules transform the auction with shares into a standard all-pay auction without shares, we can use (3.3) to determine the optimal all-pay-all auction. To ensure a zero expected payoff for the bidder with value s^* (i.e. the lowest type that finds it worthwhile to bid), the seller can simply announce a reserve price $r^* = s^* F(s^*)^{n-1}$.[14] The introduction of this optimal reserve price increases the auction's revenue but lowers its efficiency, since the object may go unsold when all bidders' signals are less than s^*.[15]

4. How (Not) To Raise Money For A Public Good

In this section we compare the revenue properties of alternative auction formats in the symmetric case when all bidders receive a common benefit equal to α times the revenue. Engers and McManus (2000) studied such auctions under the pretense that bidders receive a "warm-glow" or altruistic utility proportional to the auction's proceeds. Consider, for instance, an auction that is used to raise money for a charity whose work is valued by the bidders. In contrast to Engers and McManus, we discuss cases where standard auctions, in which only the winner pays, are dominated by other mechanisms such as lotteries, voluntary contributions, and especially the all-pay-all auction.

We will drop the restriction that $\alpha < 1/n$. Consider, for instance, the case when the auction's revenue is used to finance a public good and all bidders value \$1 contributed to the public good at \$$\alpha$. A natural assumption would be that α lies between 0 and 1 (and is independent of the number of bidders). Suppose, however, that the auction's revenue is matched by a third party. Such a matching is equivalent to a doubling of α, which could thus be as high as 2.[16] To allow for the possibility of revenue matching by one or more parties, we will not impose an upper bound on α. Consequently, bidders' shares do

[14] Bidders' optimal strategies now become

$$B(s) \;=\; r^* + \int_{s^*}^{s} y \, \mathrm{d}F(y)^{n-1},$$

when $s \geq s^*$ and zero otherwise. Bidders who bid zero have zero expected payoffs in the all-pay-all auction, while bidders with signals $s \geq s^*$ have expected equilibrium payoffs $\pi^*(s) = \int_{s^*}^{s} (s - z) \, \mathrm{d}F(z)^{n-1}$.

[15] Total expected surplus with a reserve price, $r^* = s^* F(s^*)^{n-1}$, is given by $E(Y_1 | Y_1 \geq s^*)\,(1 - F_{Y_1}(s^*))$ and expected revenue is $R = E(Y_2 | Y_2 \geq s^*)\,(1 - F_{Y_2}(s^*)) + n r^* (1 - F(s^*))$.

[16] More generally, a third party's contribution proportional to the auction revenue is equivalent to a rescaling of α.

not necessarily add up to 1, which is natural since the benefit one bidder receives from the public good does not affect another's. In addition, the auction's revenue can be fully allocated to the public good (i.e. bidders' benefits from the auction's revenue do not diminish its value).

First, consider the case $\alpha < 1/n$. The all-pay-all rules then become: 1) the highest bidder receives the object, and 2) all bidders pay their own bid plus $\alpha/(1 - n\alpha)$ times the sum of all bids. Proposition 2 guarantees that (in the absence of reserve prices and entry fees) the all-pay-all auction is revenue maximizing. The amount of money raised is $R_{APA} = E(v(Y_2, Y_2))/(1 - n\alpha)$, which diverges to infinity as $\alpha \to 1/n$.[17] Of course, this divergence occurs because of the assumption that bidders' benefits from the public good are *linear* in the amount of money raised. A more realistic model would incorporate some decreasing marginal utility for the public good, which would result in a finite revenue. In this section we keep the linear-payoff assumption, however, to show how much worse other auction formats can be in terms of raising money.

We start with the all-pay auction in which each bidder pays her own bid. In this case, the expected benefit a bidder receives from others' bids is independent of her own bid. Hence the benefits bidders receive from the public good merely reduces the cost of their bid by a factor $(1 - \alpha)$. Optimal bids, therefore, are inflated by $1/(1 - \alpha)$ and so is the auction's revenue: $R_{AP} = E(v(Y_2, Y_2))/(1 - \alpha)$. Notice that the all-pay revenue is finite when $\alpha = 1/n$ but diverges when $\alpha = 1$, i.e. when bidders value \$1 given to the public good at \$1. This result seems intuitive and one would expect the same to occur in a first and second price auction. This, however, is not the case as we now show.

Proposition 3. *The amount of money raised by a first price auction is increasing in* α *with* $R_1(\alpha = 0) = E(v(Y_2, Y_2))$ *and* $R_1(\alpha = 1) = E(v(Y_1, Y_1))$*.*

So even when bidders are indifferent between keeping \$1 for themselves or giving it to the public good, they only bid a finite amount. More precisely, when $\alpha = 1$, a bidder with signal s bids the object's expected value assuming the highest of the others' signals is also s. To see this, consider bidder 1 with signal s_1 and let y_1 denote the highest of the others' signals as before. When $s_1 \geq y_1$, bidder 1's expected payoff when she bids

[17]When $\alpha \geq 1/n$, a simple "lowest price all-pay auction," in which the highest bidder wins and everyone pays the lowest price, will result in an infinite revenue.

$v(s_1, s_1)$ is $E(v(s_1, y_1)|y_1 \leq s_1)$. In fact, this is her payoff for all bids with which she wins. When she lowers her bid too far and loses, however, bidder 1's expected payoff becomes $E(v(y_1, y_1)|y_1 \leq s_1)$. Since $v(\cdot, \cdot)$ is increasing in its first argument this expected payoff is less than $E(v(s_1, y_1)|y_1 \leq s_1)$. In other words, when $s_1 > y_1$ bidder 1 never gains but may lose when choosing a bid different from $v(s_1, s_1)$. Similarly, when $s_1 \leq y_1$, bidder 1's expected payoff when she bids $v(s_1, s_1)$ is $E(v(y_1, y_1)|y_1 \geq s_1)$. This payoff is the same for all bids with which bidder 1 loses. A bid that would lead her to win the auction yields a lower expected payoff of $E(v(s_1, y_1)|y_1 \geq s_1)$.

The first price auction leads to relatively low bids because of a strong "free rider" problem; with a low value, a bidder is better off losing and receiving (part of) the winning bid. This free rider effect can be even stronger in the second price auction.[18]

Proposition 4. *The amount of money raised by a second-price auction is increasing in* α *with* $R_2(\alpha = 0) = E(v(Y_2, Y_2))$, $R_2(\alpha = 1) = E(v(Y_1, Y_1))$, *and* $R_2(\alpha = \infty) = v(1, 1)$.

In other words, a bidder who values \$1 given to the public good at *more* than \$1, bids at most $v(1, 1)$ in the auction (which is the maximum possible revenue). Recall that $\alpha > 1$ can occur when one or more outside parties match the revenue raised in the auction. The difference with a first price auction is that the highest bidder does not benefit from increasing her bid, which further suppresses bids. *Clearly, when* $\alpha > 1$, *it would be better not to conduct a second price auction but just ask for voluntary contributions!*

Recall that in the example of section 2 the second price auction resulted in a higher revenue than the first price auction. Bulow, Huang, and Klemperer (1999) show that this holds more generally when $n = 2$ and $\alpha < 1/2$. Engers and McManus (2000) prove that the second price auction does better when $\alpha < 1$. The following proposition also considers the case $\alpha > 1$.[19]

Proposition 5. *The second-price auction raises less (more) money than a first price auction when* $\alpha > 1$ *(*$\alpha < 1$*).*

[18] The positive externalities present in auctions contrast with the negative externalities that occur in "raffles," i.e. a bidder who buys more lottery tickets lowers everyone else's chance of winning (Morgan, 1997).

[19] The assumption of a common α is crucial for the result of Proposition 5. When there are small asymmetries between bidders, the first price auction raises more revenue than a second price auction even when $\alpha < 1$, see Proposition 6 in Bulow, Huang, and Klemperer (1999).

Of course, both the first and second price auctions raise less money than an all-pay-all auction when $\alpha < 1/n$ or a simple lowest price all-pay auction when $\alpha \geq 1/n$.

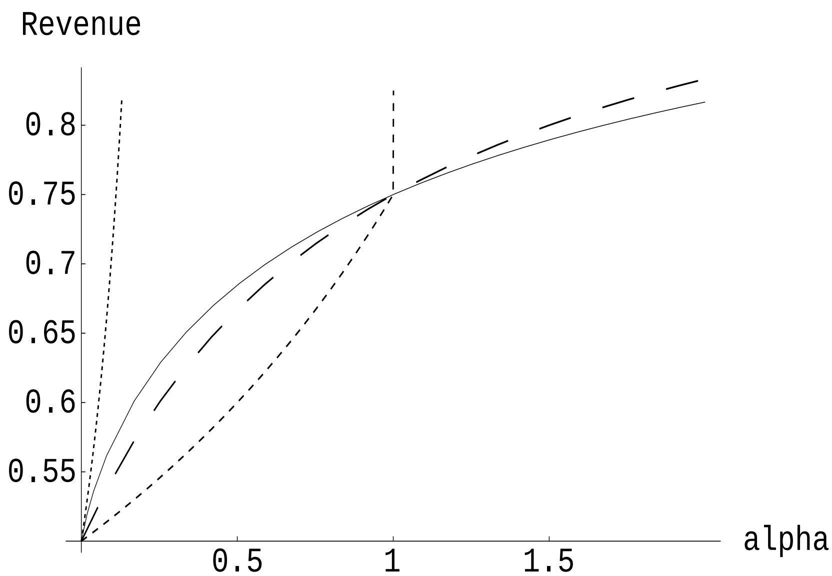

Figure 2: *The revenue from an all-pay-all auction (dotted line), a first price auction (short dashes), second price auction (long dashes), and a third price auction (solid line) for* $0 \leq \alpha \leq 2$.

The above propositions show that the revenue from a first and second price auction are the same for $\alpha = 0$, which is the standard revenue equivalence result, and $\alpha = 1$. This latter result is no coincidence.

Proposition 6. *Any auction in which only the highest bidder makes a payment, e.g. the k^{th} price auction for $k = 1, \cdots, n$, yields revenue $R = E(v(Y_2, Y_2))$ for $\alpha = 0$ and $R = E(v(Y_1, Y_1))$ for $\alpha = 1$.*

When $\alpha > 1$, it seems likely that a $(k+1)^{th}$ price auction will result in less revenue than a k^{th} price auction since one more high bidder does not benefit from increasing her bid. For $\alpha < 1$ we would expect the opposite ranking (as in Proposition 5). This claim is verified for the case of three bidders and uniform private values in Figure 2, which shows revenues in a first, second, and third price auction by α.[20] The solid line represents the revenue from a third price auction, which is higher than that of a second price auction (long dashes) or a first price auction (short dashes) when $0 < \alpha < 1$. Note that the revenue from a first price auction diverges when $\alpha > 1$ since the winner benefits from her own bid. In contrast, the maximum revenue of the k^{th} price auction for $k > 1$ is $R_k(\alpha = \infty) = v(1, 1)$. These results lead us to conjecture:

Conjecture. *Let $k = 1, \cdots, n-1$. The $(k+1)^{th}$ price auction results in more (less) revenue than the k^{th} price auction when $\alpha < 1$ ($\alpha > 1$).*

5. Conclusions

In this paper we have considered auctions in which bidders receive shares of the auction's revenue. We propose a new auction format, the *all-pay-all auction*, in which all bidders pay a weighted sum of all bids. The all-pay-all auction ensures efficiency, i.e. the object is assigned to the highest-value bidder, and maximizes the seller's revenue. This contrasts with the properties of standard auction formats in which a bidder's payment depends only on her own bid. Standard auctions generally are not efficient when bidders' shares are asymmetric. In addition, they result in a strictly positive payoff for the lowest type and are thus not revenue maximizing from the seller's point of view.

The inferiority of standard auctions is most dramatic when they are used to raise money for a public good. A bidder's "share" in this context is the benefit she derives from the public good, i.e. a share of α means that a bidder values \$1 given to the public good at \$$\alpha$. In this case, the sum of bidders' shares is not necessarily less than one since one bidder's benefit from the public good does not affect another's. In fact, when matching of the auction's revenue by one or more outside parties is possible, an individual bidder's share α need not be less than one.

[20] The bidding functions for the first and second price auction are derived in the proofs of Propositions 3 and 4 respectively. The revenue of a first price auction is $R_1 = 3/(6 - 2\alpha)$ for $\alpha \leq 1$ and the revenue of a second price auction is $R_2 = (1 + 2\alpha)/(2 + 2\alpha)$ for all $\alpha \geq 0$. The bidding function for the third price auction is

$$B_3(x) \;=\; 2 - \frac{2}{1-\alpha} + \frac{2x}{1-\alpha} + \frac{\alpha}{2(1-\alpha)} \left(1 + \sqrt{1 + 8/\alpha}\right) (1-x)^{\frac{1}{2}(\sqrt{1+8/\alpha}-1)}$$

and the corresponding revenue is $R_3 = (3\,\alpha\, B_3(0) - \frac{1}{2})/(3\,\alpha - 1)$ for all $\alpha \geq 0$.

The all-pay-all auction raises more money for the public good than any other auction and its revenue diverges when α tends to $1/n$. In other words, if it were known that the marginal value of the public good is α one need only attract more than $1/\alpha$ bidders to raise an arbitrary amount. (Of course, this divergence is due to the assumption that the marginal value of the public good is constant.) In contrast, all standard auctions yield finite revenues for $\alpha \leq 1$. Even when bidders are indifferent between keeping \$1 or giving it to the public good, they only bid a finite amount. The low bids are caused by the positive externalities one bidder's high bid exerts on others. Moreover, in a k^{th} price auction for $k > 1$, optimal bids remain finite even when the constant marginal value of the public good *exceeds* \$1. The latter formats further suppress bids because the high bidders do not benefit from an increase in their own bids. Clearly, in many of these situations it would be better to ask for voluntary contributions or conduct a "raffle" (where a higher investment by one bidder imposes a negative externality on others). Ideally, however, one should employ an all-pay-all auction.

A. Appendix

Proof of Proposition 1. Suppose bidders $j \neq i$ use the strategy $B(\cdot)$ and bidder i's signal is $s_i = x$. Bidder i's expected payoff in the all-pay-all auction when she bids as if her signal is y is

$$\begin{aligned}\pi_i^e(B(y)|x) \;&=\; \int_0^y v(x,z)\,\mathrm{d}F(z)^{n-1} - \Big(1+\frac{\alpha_i}{\alpha_0}\Big)\,B(y) - \frac{\alpha_i}{\alpha_0}\,(n-1)\,E(B)\\ &\quad+\,\alpha_i\,\Big(\Big(1+\frac{\alpha_i}{\alpha_0}\Big)\,B(y) + \frac{\alpha_i}{\alpha_0}\,(n-1)\,E(B)\Big)\\ &\quad+\,\alpha_i\,\Big(\sum_{j\neq i}\{E(B) + \frac{\alpha_j}{\alpha_0}\,(n-1)\,E(B) + \frac{\alpha_j}{\alpha_0}\,B(y)\}\Big),\end{aligned}$$

where $E(B)$ is the expected value of another's bid. Combining the $B(y)$ and $E(B)$ terms on the right side, the expected payoff simplifies to

$$\pi_i^e(B(y)|x) \;=\; \int_0^y v(x,z)\,\mathrm{d}F(z)^{n-1} - B(y).$$

Using (3.1) we have

$$\pi_i^e(B(y)|x) \;=\; \int_0^y (v(x,z) - v(z,z))\,\mathrm{d}F(z)^{n-1}. \tag{A.1}$$

Since v is increasing in its first argument, the integrand is positive when $z < x$ and negative when $z > x$, so bidder i's expected payoff is maximized when $y = x$. Note from (A.1) that a bidder's equilibrium payoff when she has signal s is $\pi^*(s) = \int_0^s (v(s,z) - v(z,z))\,\mathrm{d}F(z)^{n-1}$, which is non-negative for all $s \in [0,1]$ and zero when $s = 0$. Hence, the participation constraint, which requires non-negative expected equilibrium payoffs, is satisfied. *Q.E.D.*

Proof of Proposition 2. Consider an auction in which the object is awarded to the highest bidder. The surplus generated by the auction is $S = E(v(Y_1, Y_2))$, which is divided between the seller and the bidders: $S = \pi_0 + \pi_{bidders}$. The *ex ante* expected payoffs for the bidders are $\sum_{i=1}^n \int_0^1 \pi_i^*(s)\,\mathrm{d}F(s)$, where $\pi_i^*(s)$ is the expected equilibrium payoff of bidder i when her signal is s. A simple envelope theorem argument shows that $\pi_i^*(s)$ satisfies

$$\frac{\partial \pi_i^*(s)}{\partial s} \;=\; \int_0^s \frac{\partial v(s,z)}{\partial s}\,\mathrm{d}F(z)^{n-1} \;=\; \frac{\partial}{\partial s}\int_0^s (v(s,z) - v(z,z))\,\mathrm{d}F(z)^{n-1}.$$

Hence,

$$\pi_i^*(s) \;=\; \pi_i^*(0) + \int_0^s (v(s,z) - v(z,z))\,\mathrm{d}F(z)^{n-1}, \tag{A.2}$$

which is the standard incentive compatability constraint (Myerson, 1981). The sum of bidders' *ex ante* expected equilibrium payoffs can now be written as

$$\begin{aligned}\pi_{bidders} &= \sum_{i=1}^{n}\pi_i^*(0) + n\int_0^1\int_0^s (v(s,z) - v(z,z))\,\mathrm{d}F(z)^{n-1}\mathrm{d}F(s)\\ &= \sum_{i=1}^{n}\pi_i^*(0) + n\int_0^1\int_z^1 (v(s,z) - v(z,z))\,\mathrm{d}F(s)\,\mathrm{d}F(z)^{n-1},\end{aligned}$$

where we changed the order of integration. Expanding the second term on the right side yields two terms, one of which is

$$n\,(n-1)\int_0^1\int_z^1 v(s,z)\,f(s)\,f(z)\,F(z)^{n-2}\,\mathrm{d}s\,\mathrm{d}z \;=\; E(v(Y_1,Y_2)),$$

where we used $f_{Y_1,Y_2}(s,z) = n(n-1)F(z)^{n-2}f(z)f(s)$, see Mood, Graybill, and Boes (1963). The other term is

$$n\,(n-1)\int_0^1 v(z,z)\,f(z)\,F(z)^{n-2}\,(1-F(z))\,\mathrm{d}z \;=\; E(v(Y_2,Y_2)).$$

To summarize, in any auction in which the object is awarded to the highest bidder, bidders' *ex ante* expected payoffs are

$$\pi_{bidders} \;=\; \sum_{i=1}^{n}\pi_i^*(0) + E(v(Y_1,Y_2)) - E(v(Y_2,Y_2)), \tag{A.3}$$

and the seller's payoff is therefore

$$\pi_0 \;=\; E(v(Y_2,Y_2)) - \sum_{i=1}^{n}\pi_i^*(0). \tag{A.4}$$

Hence, the all-pay-all auction, in which $\pi_i^*(0) = 0$ for all i, gives the seller the highest possible payoff (ignoring reserve prices and/or entry fees). *Q.E.D.*

Proof of Proposition 3. We first derive the equilibrium bidding function, $B_1(\cdot)$. Consider a bidder with signal x who bids as if she has signal y and who faces opponents that bid according to $B_1(\cdot)$. Her expected payoff is

$$\pi(B_1(y)|x) \;=\; \int_0^y (v(x,z) - (1-\alpha)B_1(y))\,\mathrm{d}F(z)^{n-1} + \alpha \int_y^1 B_1(z)\,\mathrm{d}F(z)^{n-1}.$$

Evaluating the derivative of the right side with respect to y at $y = x$ yields the first-order condition

$$(v(x,x) - B_1(x))\,(F(x)^{n-1})' - (1-\alpha)\,{B_1}'(x)\,F(x)^{n-1} \;=\; 0.$$

It is straightforward to integrate this equation to obtain

$$B_1(s) \;=\; \int_0^s v(z,z)\,\mathrm{d}F_\alpha(z|s), \tag{A.5}$$

where $F_\alpha(z|s) \equiv (F(z)/F(s))^{\frac{n-1}{1-\alpha}}$ and we assumed that $\alpha \leq 1$. This result is a simple extension of Engelbrecht-Wiggans' (1994) result for two-bidders. Note that F_α first-order stochastically dominates $F_{\alpha'}$ for all $\alpha \geq \alpha'$. Since $v(\cdot,\cdot)$ is increasing in its first argument and non-decreasing in its second argument, an increase in α thus raises $\int_0^s v(z,z)dF_\alpha(z|s)$. Hence, an increase in α results in higher bids and higher revenues. For $\alpha = 0$, the optimal bids in (A.5) reduce to the standard optimal bids for the first price auction, yielding revenue $R_1 = E(v(Y_2, Y_2))$. When $\alpha = 1$, the distribution $F_\alpha(z|s)$ is degenerate and puts all probability mass at $z = s$. In this case $B_1(s) = v(s,s)$ and $R_1 = E(v(Y_1, Y_1))$. *Q.E.D.*

Proof of Proposition 4. Again we follow Engelbrecht-Wiggans (1994) in our derivation of the bidding function, $B_2(\cdot)$. Consider a bidder with signal x who bids as if she has signal y and who faces opponents that bid according to $B_2(\cdot)$. Her expected payoff is

$$\begin{aligned}
\pi(B_2(y)|x) \;&=\; \int_0^y (v(x,z) - (1-\alpha)B_2(z))\,\mathrm{d}F(z)^{n-1} \\
&+\; \alpha\,(n-1)\,B_2(y)\,(1-F(y))\,F(y)^{n-2} \\
&+\; \alpha\,(n-1)\,(n-2)\int_y^1 B_2(z)\,f(z)\,(1-F(z))\,F(z)^{n-3}\mathrm{d}z.
\end{aligned} \tag{A.6}$$

Evaluating the derivative of the right side with respect to y at $y = x$ yields the first-order condition

$$(v(x,x) - B_2(x))\,f(x) + \alpha\,{B_2}'(x)\,(1-F(x)) \;=\; 0,$$

independent of the number of bidders. It is straightforward to integrate this equation to obtain

$$B_2(s) \;=\; \int_s^1 v(z,z)\,\mathrm{d}G_\alpha(z|s), \tag{A.7}$$

where $G_\alpha(z|s) \equiv 1 - (\frac{1-F(z)}{1-F(s)})^{\frac{1}{\alpha}}$. Note that G_α first-order stochastically dominates $G_{\alpha'}$ for all $\alpha \geq \alpha'$. Since $v(\cdot,\cdot)$ is increasing in its first argument and non-decreasing in its second argument, an increase in α thus raises $\int_s^1 v(z,z)dG_\alpha(z|s)$. Hence, an increase in α results in higher bids and higher revenues. For $\alpha = 0$, the distribution $G_\alpha(z|s)$ is degenerate and puts all probability mass at $z = s$. In this case $B_2(s) = v(s,s)$ and $R_2 = E(v(Y_2,Y_2))$. When $\alpha = 1$, the optimal bids are given by $B_2(s) = \int_s^1 v(z,z)\,f(z)\,\mathrm{d}z/(1-F(s))$. Hence, expected revenue is

$$\begin{aligned}
R_2 &= n(n-1)\int_0^1 B_2(s)\,f(s)\,F(s)^{n-2}(1-F(s))\,\mathrm{d}s \\
&= n(n-1)\int_0^1\int_s^1 v(z,z)\,f(z)\,f(s)\,F(s)^{n-2}\,\mathrm{d}z\,\mathrm{d}s \\
&= n(n-1)\int_0^1\int_0^z v(z,z)\,f(z)\,f(s)\,F(s)^{n-2}\,\mathrm{d}s\,\mathrm{d}z \\
&= n\int_0^1 v(z,z)\,f(z)\,F(z)^{n-1}\,\mathrm{d}z \\
&= E(v(Y_1,Y_1)).
\end{aligned}$$

Finally, for $\alpha = \infty$, the distribution $G_\alpha(z|s)$ is degenerate and puts all probability mass at $z = 1$. In this case $B_2(s) = v(1,1)$ and $R_2 = v(1,1)$. *Q.E.D.*

Proof of Proposition 5. Our proof follows that of Bulow, Huang, and Klemperer (1999) who show that a second price auction results in a higher revenue when $n = 2$ and $\alpha < \frac{1}{2}$. The revenue of any auction in which the object is awarded to the highest bidder is given by $R = (E(v(Y_2,Y_2)) - n\pi^*(0))/(1-n\alpha)$, see (A.4). Hence, the difference between the revenue of a first and second price auction is

$$\Delta \;\equiv\; R_2 - R_1 \;=\; \frac{n(\pi_1^*(0) - \pi_2^*(0))}{1 - n\,\alpha},$$

where $\pi_1^*(0)$ $(\pi_2^*(0))$ is the expected equilibrium payoff of a bidder with signal $s = 0$ in a first (second) price auction.

First, assume that $\alpha < 1$ so that the bidding function (A.5) of the first price auction is well defined. The expected equilibrium payoff of a bidder with signal 0 in a first price auction is the expected value of the highest of others' bids: $\pi_1^*(0) = \alpha \int_0^1 B_1(s)\,\mathrm{d}F(s)^{n-1}$, where $B_1(s)$ is given by (A.5). Since $F_\alpha(z|s) \equiv (F(z)/F(s))^{\frac{n-1}{1-\alpha}}$ we have

$$\begin{aligned}
\pi_1^*(0) &= (\alpha-1)\int_0^1\int_0^s v(z,z)\,\mathrm{d}F(z)^{\frac{n-1}{1-\alpha}}\,\mathrm{d}F(s)^{-\frac{\alpha(n-1)}{1-\alpha}} \\
&= (\alpha-1)\int_0^1\int_z^1 v(z,z)\,\mathrm{d}F(s)^{-\frac{\alpha(n-1)}{1-\alpha}}\,\mathrm{d}F(z)^{\frac{n-1}{1-\alpha}} \\
&= (n-1)\int_0^1 v(z,z)\,F(z)^{n-2}\,(1-F(z)^{\frac{\alpha(n-1)}{1-\alpha}})\,\mathrm{d}F(z).
\end{aligned}$$

Similarly, in a second price auction, $\pi_2^*(0)$ is the expected value of the second-highest of others' bids: $\pi_2^*(0) = \alpha(n-1)(n-2)\int_0^1 B_2(s)f(s)F(s)^{n-3}(1-F(s))\mathrm{d}s$, where $B_2(s)$ is given by (A.7). Since $G_\alpha(z|s) \equiv 1 - (\frac{1-F(z)}{1-F(s)})^{\frac{1}{\alpha}}$ we have

$$\begin{aligned}
\pi_2^*(0) &= (n-1)\int_0^1\int_s^1 v(z,z)\,(1-F(z))^{\frac{1-\alpha}{\alpha}}\,(1-F(s))^{\frac{\alpha-1}{\alpha}}\,\mathrm{d}F(z)\,\mathrm{d}F(s)^{n-2} \\
&= (n-1)\int_0^1 v(z,z)\,(1-F(z))^{\frac{1-\alpha}{\alpha}}\,\Big(\int_0^z (1-F(s))^{\frac{\alpha-1}{\alpha}}\,\mathrm{d}F(s)^{n-2}\Big)\,\mathrm{d}F(z).
\end{aligned}$$

Let us define the new integration variable, $t = F(z)$, and $\phi(t) = v(F^{-1}(t), F^{-1}(t))$, which is strictly increasing in t. Then $\Delta = n(n-1)\int_0^1 \phi(t)\delta(t)\,\mathrm{d}t$, where

$$\delta(t) \equiv \Big(t^{n-2}\,(1-t^{\frac{\alpha(n-1)}{1-\alpha}}) - (1-t)^{\frac{1-\alpha}{\alpha}}\int_0^t (1-x)^{\frac{\alpha-1}{\alpha}}\,\mathrm{d}x^{n-2}\Big)/(1-n\alpha).$$

Note that $\delta(0) = \delta(1) = 0$ and that $\delta(t)$ has expected value zero, i.e. $\int_0^1 \delta(t)\mathrm{d}t = 0$. The proof that $\Delta > 0$ follows by showing that $\delta(t)$ is negative for $t < t^*$ and positive for $t > t^*$ for some $0 < t^* < 1$. To establish this property we determine the sign of $\delta'(t)$ whenever $\delta(t) = 0$:

$$\mathrm{sign}\Big\{\delta'(t)\Big|_{\delta(t)=0}\Big\} = \mathrm{sign}\Big\{\frac{\alpha\,(n-2+\alpha)\,(1-t) + (1-\alpha)^2\,(t - t^{\frac{1-n\alpha}{1-\alpha}})}{n\alpha - 1}\Big\}. \qquad \text{(A.8)}$$

Let $h(t)$ be the argument of the sign function on the right side of (A.8). It is easy to verify that $h(t)$ is concave in t, with $h(1) = 0$ and $h(0) < 0$ for $\alpha < 1$, so that $\delta(t)$ is negative for small t. Hence, when the graph of $\delta(t)$ crosses zero the first time, say at t^*, its slope must be positive, i.e. $h(t^*) > 0$. Now suppose $\delta(t)$ is zero at multiple points $0 < t_1 < \cdots < t_n < 1$, which can occur only if the sign of $h(t)$ alternates. However, concavity of $h(t)$ together with $h(1) = 0$ implies that $h(t) > 0$ for all $t^* < t < 1$. Hence, t^* is the unique point at which $\delta(t) = 0$ is zero and at that point the slope of $\delta(t)$ is positive. This implies that $\delta(t)$ is negative for $t < t^*$ and positive for $t > t^*$.

Finally, when $\alpha > 1$, bids in a first price auction diverge to infinity while they are finite in a second price auction. Hence, the first price auction yields more revenue. *Q.E.D.*

Proof of Proposition 6. Consider a standard auction format in which only the winner pays. When $\alpha = 1$, the winning bidder's *net* payment is zero. For a bidder with signal $s = 1$, who wins for sure, the expected payoff is therefore $E(v(1, y_1))$ where y_1 is the highest signal of the others. From (A.2) we also have

$$\pi^*(1) \;=\; \pi^*(0) + E(v(1, y_1)) - E(v(y_1, y_1)).$$

Hence $\pi^*(0) = E(v(y_1, y_1))$, or

$$\begin{aligned}
\pi^*(0) \;&=\; (n-1) \int_0^1 v(z, z)\, f(z)\, F(z)^{n-2}\, \mathrm{d}z\, \mathrm{d}z \\
&=\; (n-1) \int_0^1 v(z, z)\, f(z) \Big(F(z)^{n-1} + F(z)^{n-2}(1 - F(x))\Big)\, \mathrm{d}z \\
&=\; \frac{1}{n}\Big((n-1)\, E(v(Y_1, Y_1)) + E(v(Y_2, Y_2))\Big).
\end{aligned}$$

Finally, revenue is given by $R = (E(v(Y_2, Y_2)) - n\pi^*(0))/(1 - n\alpha)$, see the proof of Proposition 2, so $R = E(v(Y_1, Y_1))$ when $\alpha = 1$. *Q.E.D.*

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