Reductions

How do we show that a problem is easy?

Reduce each instance to one (or more) instances of a known easy problem.

\[ I_1 \in P_1 \quad R(I_1) = I_2 \in P_2 \]

To solve \( I_1 \) in \( P_1 \):

1. Apply \( R \) to turn \( I_1 \) into \( I_2 \in P_2 \)
2. Apply algorithm for \( P_2 \) to \( I_2 \).

Cost = cost of applying \( R \) plus cost of applying \( P_2 \) algorithm.
Examples

Reduction to some form of matrix multiplication:
- Transitive closure
- Context-free language recognition

Reduction to linear programming
- Reduction to network flow
- etc.

Cost of reduction?
- Linear time, quadratic time, ...
Positive use of reduction:

? \Rightarrow \text{easy}

Reduce a problem of unknown complexity
to an easy problem

"Negative" use of reduction:

How do we show that a problem of
unknown complexity is hard?

Reduce a hard problem to it

hard \Rightarrow ?

If the questionable problem were easy, so would be
the hard problem XX
Reductions in both directions show computational equivalence (up to the cost of the reduction)

\[ P_1 \iff P_2 \]

both are easy or both are hard

Transitivity of reductions

\[ P_1 \Rightarrow P_2 \Rightarrow P_3 \]

\[ R_1 \quad R_2 \]

Gives a reduction from \( P_1 \) to \( P_3 \)

Cost is cost of \( R_2(R_1(\cdot)) \)

both \( p \)-time, overall \( p \)-time

linear \quad linear
\[ R_1 \quad R_2 \]

\[ n \quad a_n \quad b_n \]

\[ a_n \quad (ab)^n \]

\[ n \quad a_n^2 \quad b_n^2 \]

\[ a_n^2 \quad b(a_n^2)^2 = b^2 a_n^4 \]

\[ a_n^k \quad b_n^k \]

\[ b(b_n^k)^l = b^n \quad n \cdot \frac{k}{l} \]
Satisfiability: Is a Boolean (logical) function true for some choice of variable assignments?

\[(x \lor y) \land (\bar{x} \lor \bar{y})\]  sat: \(x=1, y=0\)

- \(\land\) and
- \(\lor\) or
- \(\neg\) not

\(x\) variable
\(x, \bar{x}\) literal

\((x \lor y \lor z)\) clause: disjunction ("or") of literals
\((x \lor y \lor z) \land (\bar{x} \lor \bar{y} \lor \bar{z})\) conjunctive normal form:
conjunct ("and") of clauses

\(x \lor \bar{x}\) tautology: true for all choices of variables

\(F\) is sat iff \(\overline{F}\) is not a tautology:
can be falsified
Reduction of CNF sat to 3-CNF sat
(at most 3 literals/clause)

\[(xv y v z v w v u v v) \Rightarrow \]
\[(xv y v a ) \land (\bar{a} v z v b ) \land (\bar{b} v w v c ) \]
\[\land (\bar{c} v u v v)\]

Needs \(k - 3\) extra vars per clause of length \(k\).
Graph coloring reducible to SAT, and vice-versa (p-time reductions)

Must phrase graph coloring as a yes-no question: can graph G be colored with k colors?

G: n vertices, m edges

F: nk variables $x_{ij}$, one per vertex per color

$x_{ij}$ true iff vertex i colored color j

Clauses:

Each vertex colored:

$\bigvee_{j} (x_{i1} \lor x_{i2} \lor \cdots \lor x_{ik}) \quad i \in V \quad n$

No vertex colored twice:

$\left[ \bigwedge_{j \neq l} (\bar{x}_{ij} \lor \bar{x}_{il}) \quad i \in V, j \neq l \text{ colors} \quad n \binom{k}{2} \right]$

No adjacent vertices the same color:

$\left( \bigvee \bar{x}_{i\ell} \lor \bar{x}_{j\ell} \right) \quad (i,j) \in E, \ell \text{ a color } mk$

$\# \text{ literals} = nk + 2n \binom{k}{2} + 2mk$
Vice-versa: \( (3\text{-sat}) \)

Reduction to 3-coloring

Vertices: \( x, \bar{x}, \text{true}, \text{false}, \text{red}, \) 5 per clause

\( x, \bar{x} \) colored

\( \text{true}, \text{false} \)

Clause \( x \lor \bar{y} \lor \bar{z} \)

"gadget" for each clause

Colorable iff formula satisfiable
Clique $\Rightarrow$ Sat

$x_{ij} \quad i \in V, 1 \leq j \leq k$

$x_{ij} = \text{true} = \text{vertex } i \text{ is } j^{th} \text{ in clique}$

$i, j \in V \text{ not adjacent}$

$(\bar{x}_{ij} \lor \bar{x}_{ij'}) \quad \forall j, j'$

$(\bar{x}_{ij} \lor \bar{x}_{i'j'}) \quad \forall i, i'$

$(x_{ij} \lor x_{i'j'}) \quad \forall j, j'$

$(\bar{x}_{1j} \lor \cdots \lor \bar{x}_{kj}) \quad \forall j$
\text{Sat} \implies \text{Clique}

\text{Given a graph, are there } k \text{ pairwise adjacent vertices?}

One vertex per literal occurrence,
two vertices in different clauses joined by an edge if compatible (not } x, \bar{x})

\[ k = \# \text{ clauses} \]

\[ (x \lor y \lor \bar{z}) \]

\[ \frac{\bar{x}}{\bar{y}} \]

\[ \frac{\bar{z}}{\bar{y}} \]

\[ (\bar{x} \lor y \lor z) \]
Clique $\iff$ Independent set

Are there $k$ pairwise nonadjacent vertices?

Complement graph

Clique $\iff$ Vertex cover

Are there $k$ vertices "covering" all edges

$S$ a vertex cover in $G$ iff

$V - S$ is an independent set in $G$ iff

$V - S$ is a clique in $\overline{G}$
$P =$ problems solvable in $p$-time

$NP =$ yes–no problems s.t. if answer is "yes," can be verified in $p$-time given a $(p \cdot \text{length})$ "proof" (hint).

$p$-time on a Turing machine

or random-access machine