Computer Science 345
The Efficient Universe

Homework 1
Due Tuesday, February 21, 2006

No collaboration is permitted for this homework.

It is very important that you provide clear and concise answers. Read your answers to make sure they articulate your understanding. A few sentences or pseudo-code suffice for each of the problems. When you are asked to attempt to do something, please give it your best shot and describe (again clearly and concisely) your thoughts, even if you didn't succeed. No points will be deducted if you don't.

Some of the problems below are designed to illustrate that for many natural problems an exponential time algorithm exists, and represents a "brute-force" solution (in which we basically try out all possibilities), whereas a polynomial time algorithm (if exists) requires an idea which drastically cuts down the search space. Moreover, sometimes problems which have efficient solutions look extremely similar to problems which don't (as far as we know).

A function f on the integers is called a *polynomial* function if there are positive constants A, c such that for every n, $f(n) \leq An^c$. A function g is called *exponential* if for every n $g(n) \leq 2^{f(n)}$ for a polynomial function f.

Problem 1

Assume you have two algorithms A and B for some problem. The running time of A is n^{10}, and the the running time of B is $2^{n/10}$. Which would you rather use? For inputs of size 10?, 100?, 1000? See the pattern[1]? Let t be the time used by any of these algorithms for a certain input length. How will t grow in each, if we doubled that input length?

[1] one feature of technological development is that input size (the amount of data we feed our algorithms) is always growing

Problem 2

Graphs are perhaps the most useful Mathematical model of real-life interactions. It models communication networks, social networks, relations, conflicts and more.

Formally, a graph consists of a finite set V of *vertices* (or nodes), and a set of *edges* E, each of which connecting a pair of vertices.

A graph is k-colorable, if we can assign to every vertex a color from the set $[k] = \{1, 2, \cdots, k\}$, such that the endpoints of every edge are assigned different colors. A graph with a k-coloring is called k colorable, and the smallest k for which this is possible is called the chromatic number of the graph.

A *path* in a graph is a sequence of vertices $v_0, v_1, \cdots, v_t$ such that (v_i, v_{i+1}) is an edge for every i. A cycle is a path as above with $v_0 = v_t$.

A *Hamilton* cycle in a graph is a cycle which traverses every vertex exactly once. An *Euler* cycle is a cycle which traverses every edge exactly once.

The algorithms below should be designed to accept as input a graph, given e.g. by its adjacency matrix.

Definition. The adjacency matrix $A = (a_{ij})$ of a graph G on the vertices $\{1, \ldots, \mathrm{n}\}$ is the $n \times n$ matrix defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are connected by an edge} \\ 0, & \text{otherwise.} \end{cases}$$

- Describe exponential time algorithm to decide if the graph has a k-coloring, and estimate its running time as best you can.
- Attempt to find a polynomial time algorithm for 2-coloring. Attempt to understand what feature of the number 2 makes this work, which is not shared, say by 3 (3-coloring is believed to be much harder).
- Describe exponential time algorithm to test is a graph is Hamiltonian and if it is Eulerian, and analyze their running time as best you can.

- Attempt to find a polynomial time algorithm for to test if a graph is Eulerian.

Problem 3* [This is a bonus (extra credit) problem]

You have a red crayon, and your friend has a blue one. The board consists of 6 dots are drawn on a piece of paper. Red moves first. You two alternate moves - in each move one of you connects a pair of (yet unconnected) dots using your crayon. You lose if you have completed a triangle with your color.
1. Prove that this game can never end with a draw.
2. Prove that either the Red player or the Blue player has a winning strategy (namely a method of winning no matter how the other player behaves).
3. Consider a slightly different game: the player who completes a triangle with his color wins. Prove that the Red player has a winning strategy.
4. Which of the 3 statements above remain true if we have only 5 dots on our board?