4.1 - 4.2 Analysis of Algorithms

Overview

Analysis of algorithms. Framework for comparing algorithms and predicting performance.

Scientific method.
- Observe some feature of the universe.
- Hypothesize a model that is consistent with observation.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate the theory by repeating the previous steps until the hypothesis agrees with the observations.

Universe = computer itself.

Algorithmic Successes

N-body Simulation.
- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: N log N steps, enables new research.


Andrew Appel
PU '81

Discrete Fourier transform.
- Break down waveform of N samples into periodic components. Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.
- FFT algorithm: N log N steps, enables new technology.


Freidrich Gauss
1805

Sorting.
- Rearrange N items in ascending order.
- Fundamental information processing abstraction.


Jon von Neumann
IAS 1945

Case Study: Sorting

Sorting problem. Given N items, rearrange them in ascending order.

Applications. Statistics, databases, data compression, computational biology, computer graphics, scientific computing, ...

name
Hauser
Hong
Hsu
Hayes
Haskell
Hanley
Hornet

name
Hanley
Haskell
Hauser
Hayes
Hong
Hornet
Hsu

Insertion Sort

Insertion sort.
- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

```
public static void sort(double[] a) {
   for (int i = 0; i < a.length; i++) {
      for (int j = i; j > 0; j--) {
         if (less(a[j], a[j-1]))
            exch(a, j, j-1);
         else break;
      }
   }
}
```


Helper Functions

Sorting helper functions.

- Is real number `x` strictly less than `y`?

```
public static boolean less(double x, double y) {
   return (x < y);
}
```

- Swap real numbers stored in `a[i]` and `a[j]`.

```
public static void exch(double[] a, int i, int j) {
   double swap = a[i];
   a[i] = a[j];
   a[j] = swap;
}
```

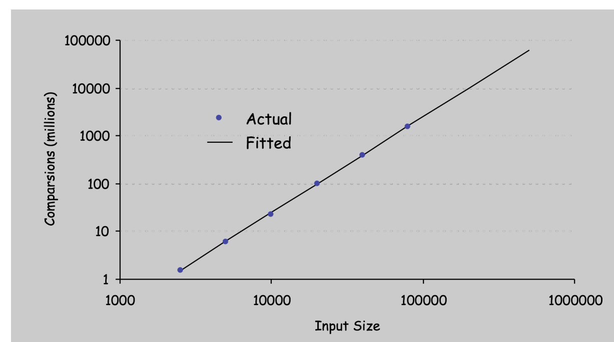

Insertion Sort: Observation

Observe and tabulate running time for various values of N.
- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.
- Timing: Skagen wristwatch.

N	Comparisons	Time
5,000	6.2 million	0.13 seconds
10,000	25 million	0.43 seconds
20,000	99 million	1.5 seconds
40,000	400 million	5.6 seconds
80,000	1600 million	23 seconds

Insertion Sort: Experimental Hypothesis

Data analysis. Plot # comparisons vs. input size on log-log scale.



slope

Regression. Fit line through data points $\approx$ a N^b.
Hypothesis. # comparisons grows quadratically with input size $\approx N^2/4$.

Insertion Sort: Prediction and Verification

Experimental hypothesis. # comparisons ≈ $N^2/4$.

Prediction. 400 million comparisons for N = 40,000.

Observations.

N	Comparisons	Time
40,000	401.3 million	5.595 sec
40,000	399.7 million	5.573 sec
40,000	401.6 million	5.648 sec
40,000	400.0 million	5.632 sec

Agrees.

Prediction. 10 billion comparisons for N = 200,000.

Observation.

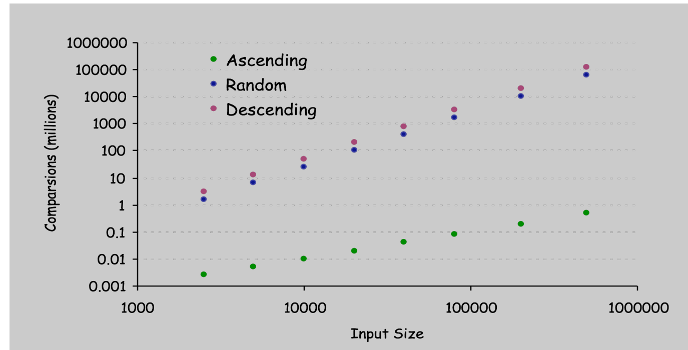
N	Comparisons	Time
200,000	9.997 billion	145 seconds

Agrees.

Insertion Sort: Validation

Number of comparisons depends on input family.

- Ascending: N.
- Random: $N^2/4$.
- Descending: $N^2/2$.



Insertion Sort: Theoretical Hypothesis

Experimental hypothesis.

- Measure running times, plot, and fit curve.
- Model useful for predicting, but not for explaining.

Theoretical hypothesis.

- Analyze algorithm to estimate # comparisons as a function of:
  - number of elements N to sort
  - average or worst case input
- Model useful for predicting and explaining.
- Model is independent of a particular machine or compiler.

Difference. Theoretical model can apply to machines not yet built.

Insertion Sort: Theoretical Hypothesis

Worst case. (descending)

- Iteration i requires i comparisons.
- Total = 0 + 1 + 2 + . . . + N-2 + N-1 = N (N-1) / 2.

E	F	G	H	I	J	D	C	B	A
						i			

Average case. (random)

- Iteration i requires i/2 comparisons on average.
- Total = 0 + 1/2 + 2/2 + . . . + (N-1)/2 = N (N-1) / 4.

A	C	D	F	H	J	E	B	I	G
						i			

Insertion Sort: Theoretical Hypothesis

Theoretical hypothesis.

Analysis	Comparisons	Stddev
Worst	$N^2 / 2$	NA
Average	$N^2 / 4$	$1/6\ N^{3/2}$
Best	N	NA

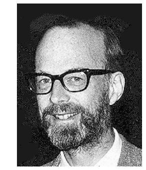
Validation. Theory agrees with observations.

Remark. Supercomputer can't rescue a bad algorithm.

Computer	Comparisons Per Second	Thousand	Million	Billion
laptop	10^7	instant	1 day	3 centuries
super	10^{12}	instant	1 second	2 weeks

Quicksort

Quicksort.
- Partition array so that:
  - some partitioning element `a[i]` is in its final position
  - no larger element to the left of `i`
  - no smaller element to the right of `i`



C. A. R. Hoare

Q	U	I	C	K	S	O	R	T	I	S	C	O	O	L

Quicksort

Quicksort.
- Partition array so that:
  - some partitioning element `a[i]` is in its final position
  - no larger element to the left of `i`
  - no smaller element to the right of `i`

partitioning element

Q	U	I	C	K	S	O	R	T	I	S	C	O	O	L
I	C	K	I	C	L	Q	U	S	O	R	T	S	O	O

≤ L

≥ L

partitioned array

Quicksort

Quicksort.
- Partition array so that:
  - some partitioning element `a[i]` is in its final position
  - no larger element to the left of `i`
  - no smaller element to the right of `i`
- Sort each "half" recursively.

partitioning element

Q	U	I	C	K	S	O	R	T	I	S	C	O	O	L
C	C	I	I	K	L	O	O	O	Q	R	S	S	T	U

sort each piece

Quicksort: Java Implementation

Quicksort.

- Partition array so that:
  - some partitioning element a[m] is in its final position
  - no larger element to the left of m
  - no smaller element to the right of m
- Sort each "half" recursively.

```
public static void quicksort(double[] a, int left, int right) {
   if (right <= left) return;
   int i = partition(a, left, right);
   quicksort(a, left, i-1);
   quicksort(a, i+1, right);
}
```


Quicksort : Implementing Partition

Q. How to partition in-place efficiently?


```
public static int partition(double[] a, int left, int right) {
   int i = left - 1;
   int j = right;

   while(true) {
      while (less(a[++i], a[right]))     find item on left to swap
         if (i == right) break;
      while (less(a[right], a[--j]))     find item on right to swap
         if (j == left) break;
      if (i >= j) break;                 check if pointers cross
      exch(a, i, j);                     swap
   }
   exch(a, i, right);                    swap with partitioning element
   return i;                             return index where crossing occurs
}
```


Quicksort: Observation

Observe and tabulate running time for various values of N.

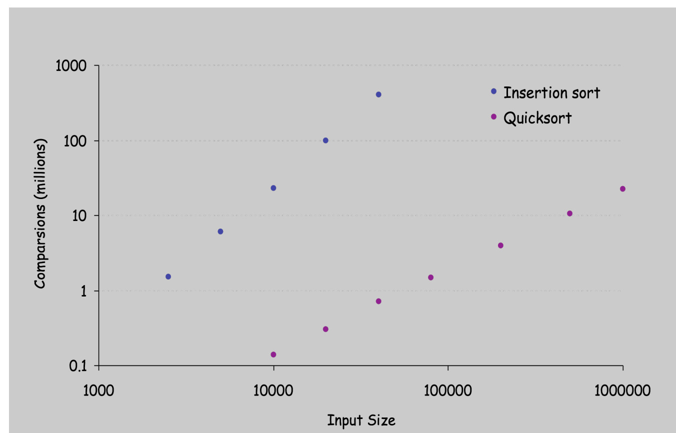
- Data source: first N words of Charles Dicken's life work.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.

N	Comparisons	Time
200,000	4.5 million	0.10 sec
400,000	9.5 million	0.23 sec
1 million	26 million	0.47 sec
2 million	55 million	0.96 sec
4 million	120 million	2.0 sec
8 million	240 million	4.2 sec

Remark. Takes 1.8 seconds to generate input of size 8 million!

Quicksort: Preliminary Hypothesis

Experimental hypothesis. Number of comparisons ≈ 30N.



Quicksort: Prediction and Verification

Experimental hypothesis. Number of comparisons ≈ 30N.

Prediction. 120 million comparisons for N = 4 million.

Observations.

N	Comparisons	Time
4 million	112.9 million	2.04 sec
4 million	116.7 million	2.07 sec
4 million	116.8 million	2.02 sec

Agrees.

Prediction. 600 million comparisons for N = 20 million.

Observations.

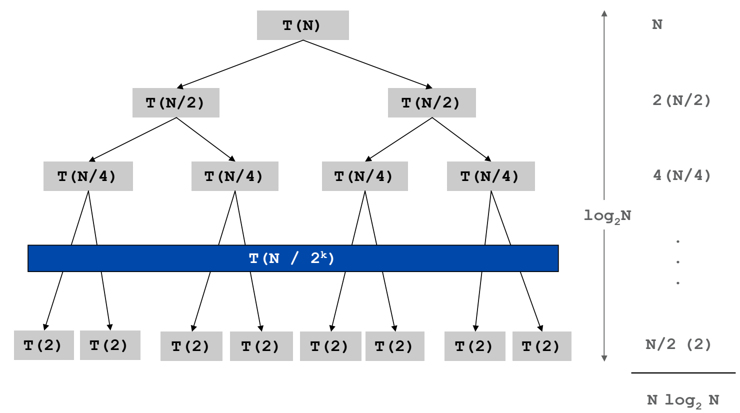
N	Comparisons	Time
20 million	638 million	11.1 sec
100 million	3.6 billion	60.6 sec

Not quite.

Quicksort: Hypothesis

Quicksort. To sort N elements, need to do N comparisons, and then quicksort two files of (roughly) size N/2.

for partitioning



Quicksort: Theoretical Hypothesis

Average case. [random]

a more careful analysis

- Number of comparisons ≈ 2 N ln N (stddev ≈ 0.65N).
- Number of exchanges ≈ 1/3 N ln N.

Worst case. Number of comparisons ≈ $1/2\ N^2$.

Validation.

- Partition on random element to to eliminate worst case.
- Theory now agrees with observations.

Lesson

Lesson. Great algorithms can be more powerful than supercomputers.

Computer	Comparisons Per Second	Insertion	Quicksort
laptop	10^7	3 centuries	3 hours
super	10^{12}	2 weeks	instant

N = 1 billion

Scientific Method

Scientific method applies to estimate running time.

- Experimental analysis: not difficult to perform experiments.
- Theoretical analysis: may require advanced mathematics.
- Small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

N

```
for (int i = 0; i < N; i++)
   ...
```

N^2

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

$\log_2 N$

```
while (N > 1) {
   N = N / 2;
   ...
}
```

2^N

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

$N \log_2 N$

```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```


Order of Growth

Asymptotic running time.

- Estimate time as a function of input size N.
- Ignore lower order terms and leading coefficients.
  - when N is large, terms are negligible
  - when N is small, we don't care
- Ex: $6N^3 + 17N^2 + 56$ is asymptotically proportional to N^3.

Donald Knuth
Turing award '74

Complexity	Description	When N doubles, running time
1	Constant algorithm is independent of input size.	does not change
log N	Logarithmic algorithm gets slightly slower as N grows.	increases by a constant
N	Linear algorithm is optimal for processing N inputs.	doubles
N log N	Linearithmic algorithm scales to huge N.	slightly more than doubles
N^2	Quadratic algorithm is impractical for large N.	quadruples
2^N	Exponential algorithm is not usually practical.	squares!

Summary

How can I evaluate the performance of my algorithm?

- Computational experiments.
- Theoretical analysis.

What if it's not fast enough?

- Understand why.
- Buy a faster computer.
- Find a better algorithm in a textbook.
- Discover a new algorithm.

Attribute	Better Machine	Better Algorithm
Cost	$$$ or more.	$ or less.
Applicability	Makes "everything" run faster.	Does not apply to some problems.
Improvement	Quantitative improvements.	Dramatic qualitative improvements possible.