Longest Increasing Subsequence

Longest Increasing Sequence

Longest increasing sequence. Given a sequence of elements $c_1, c_2, \ldots, c_n$ from a totally ordered set, find the longest increasing subsequence.

Ex: 7 2 8 **1** **3** **4** 10 **6** **9** 5

Maximum Unique Match finder

Application. Part of MUMmer system for aligning entire genomes.

Dynamic programming solution. $O(n^2)$.
LIS is a special case of edit-distance.

- $x = c_1\ c_2 \ldots c_n$
- y = sorted sequence of c_i, removing any duplicates
- mismatch penalty = ∞

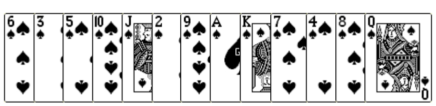
Patience

Patience. Deal cards $c_1, c_2, \ldots, c_n$ into piles according to two rules:

- Can't place a higher valued card onto a lowered valued card.
- Can form a new pile and put a card onto it.

Goal. Form as few piles as possible.

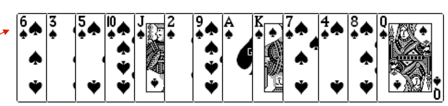
first card to deal

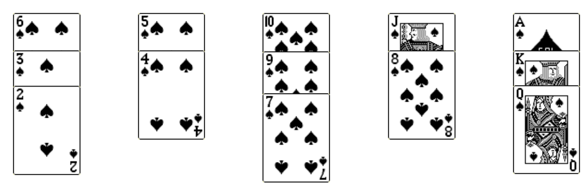


Patience: Greedy Algorithm

Greedy algorithm. Place each card on leftmost pile that fits.

first card to deal

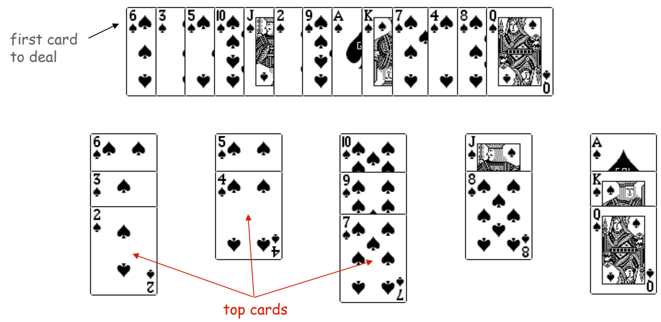




Patience: Greedy Algorithm

Greedy algorithm. Place each card on leftmost pile that fits.

Observation. At any stage during greedy algorithm, top cards of piles increase from left to right.

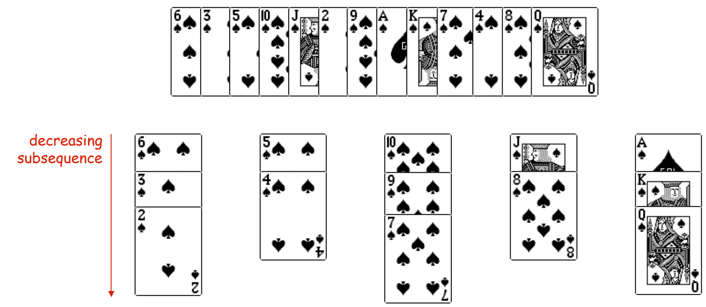


Patience-LIS: Weak Duality

Weak duality. In any legal game of patience, the number of piles ≥ length of any IS.

increasing subsequence

Pf. Cards within a pile form a decreasing subsequence.
An IS can use at most one card from each pile. ▪

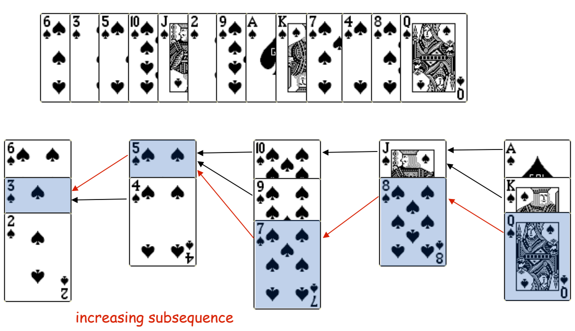


Patience-LIS: Strong Duality

Strong duality. Min number of piles = max length of an IS.
Moreover. Greedy algorithm finds both.

at time of insertion

Pf. Each card maintains a pointer to top card in previous pile. Follow pointers to obtain an IS whose length equals the number of piles.
By weak duality, both are optimal. ▪



Greedy Algorithm: Implementation

Efficient implementation. O(n log n)

- Use n stacks to represent n piles.
- Use binary search to find leftmost legal pile.

```
Create n empty stacks S1, …, Sn

foreach (card c in deck order) {
   Sj ← binary search to find leftmost stack that fits c
   push card c on top of stack Sj
   make card c point to top card of stack Sj-1

}

Form LIS by following back-pointers from top card of
rightmost nonempty stack
```

Patience Sorting

Patience sorting. Deal all cards; repeatedly remove smallest card.

Theorem. Expected number of piles is approximately $2n^{1/2}$ with standard deviation around $n^{1/6}$ if deck is uniformly random.

Remark. An almost-trivial $O(n^{3/2})$ sorting algorithm.

Speculation. [Persi Diaconis] Patience sorting is the fastest way to sort a pile of cards by hand.

Bonus Theorem

Theorem. [Erdös-Szekeres, 1935] A sequence of $n^2 + 1$ distinct real numbers either has an increasing or decreasing subsequence of size $n + 1$.

Pf. (by pigeonhole principle)