10. Extending the Limits of Tractability

Coping With NP-Completeness

Q. Suppose I need to solve an NP-complete problem. What should I do?
A. Theory says you're unlikely to find poly-time algorithm.

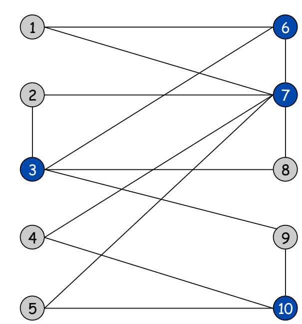
Must sacrifice one of three desired features.
- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

This lecture. Solve some special cases of NP-complete problems that arise in practice.

10.1 Finding Small Vertex Covers

Vertex Cover

VERTEX COVER: Given a graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge (u, v) either $u \in S$, or $v \in S$, or both.



k = 4
S = { 3, 6, 7, 10 }

Finding Small Vertex Covers

Q. What if k is small?

Brute force. $O(k\,n^{k+1})$.

- Try all $C(n, k) = O(n^k)$ subsets of size k.
- Takes $O(k\,n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on k, e.g., to $O(2^k\,k\,n)$.

Ex. n = 1,000, k = 10.
Brute. $k\,n^{k+1} = 10^{34} \Rightarrow$ infeasible.
Better. $2^k\,k\,n = 10^7 \Rightarrow$ feasible.

Remark. If k is a constant, algorithm is poly-time; if k is a small constant, then it's also practical.

Finding Small Vertex Covers

Claim. Let u-v be an edge of G. G has a vertex cover of size $\le$ k iff at least one of $G - \{u\}$ and $G - \{v\}$ has a vertex cover of size $\le$ k-1.

(delete v and all incident edges)

Pf. $\Rightarrow$

- Suppose G has a vertex cover S of size $\le$ k.
- S contains either u or v (or both). Assume it contains u.
- $S - \{u\}$ is a vertex cover of $G - \{u\}$.

Pf. $\Leftarrow$

- Suppose S is a vertex cover of $G - \{u\}$ of size k-1.
- Then $S \cup \{u\}$ is a vertex cover of G. ▪

Claim. If G has a vertex cover of size k, it has $\le k(n-1)$ edges.
Pf. Each vertex covers at most n-1 edges. ▪

Finding Small Vertex Covers: Algorithm

Claim. The following algorithm determines if G has a vertex cover of size $\le$ k in $O(2^k\,kn)$ time.

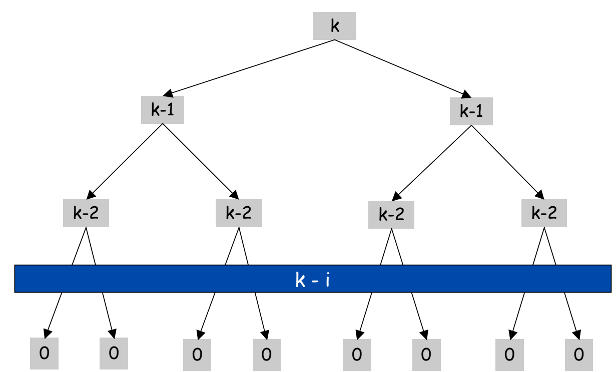
```
boolean Vertex-Cover(G, k) {
   if (G contains no edges)   return true
   if (G contains ≥ kn edges) return false

   let (u, v) be any edge of G
   a = Vertex-Cover(G - {u}, k-1)
   b = Vertex-Cover(G - {v}, k-1)
   return a or b
}
```

Pf.

- Correctness follows previous two claims.
- There are $\le 2^{k+1}$ nodes in the recursion tree; each invocation take $O(kn)$ time. ▪

Finding Small Vertex Covers: Recursion Tree

$$T(n, k) \le \begin{cases} cn & \text{if } k = 1 \\ 2T(n, k-1) + ckn & \text{if } k > 1 \end{cases} \Rightarrow T(n, k) \le 2^k\, c\, k\, n$$



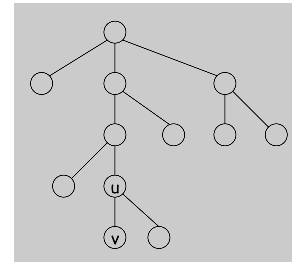
10.2 Solving NP-Hard Problems on Trees

Independent Set on Trees

Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two share an edge.

Fact. A tree has at least two leaf nodes.

↑ degree = 1



Key observation. If v is a leaf, there exists a maximum size independent set containing v.

Pf. Consider a max cardinality independent set S.

- If $v \in S$, we're done.
- If $u \notin S$ and $v \notin S$, then $S \cup \{ v \}$ is independent $\Rightarrow$ S not maximum.
- IF $u \in S$ and $v \notin S$, then $S \cup \{ v \} - \{ u \}$ is independent. ▪

Independent Set on Trees: Greedy Algorithm

Theorem. The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

```
Independent-Set-In-A-Forest(F) {
   S ← φ
   while (F has at least one edge) {
      Let e = (u, v) be an edge such that v is a leaf
      Add v to S
      Delete from F nodes u and v, and all edges
         incident to them.
   }
   return S
}
```

Pf. Correctness follows from the previous key observation. ▪

Remark. Can implement in O(n) time by considering nodes in postorder.

Weighted Independent Set on Trees

Weighted independent set on trees. Given a tree and node weights $w_v > 0$, find an independent set S that maximizes $\Sigma_{v \in S} w_v$.

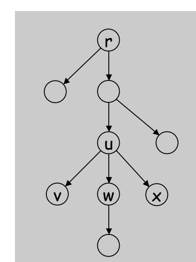
Observation. If (u, v) is an edge such that v is a leaf node, then either OPT includes u, or it includes all leaf nodes incident to u.

Dynamic programming solution. Root tree at some node, say r.

- OPT_{in} (u) = max weight independent set rooted at u, containing u.
- OPT_{out}(u) = max weight independent set rooted at u, not containing u.

$$OPT_{in}(u) = w_u + \sum_{v \in \text{children}(u)} OPT_{out}(v)$$

$$OPT_{out}(u) = \sum_{v \in \text{children}(u)} \max \{OPT_{in}(v),\ OPT_{out}(v)\}$$



children(u) = { v, w, x }

Independent Set on Trees: Greedy Algorithm

Theorem. The dynamic programming algorithm find a maximum weighted independent set in trees in O(n) time.

```
Weighted-Independent-Set-In-A-Tree(T) {
   Root the tree at a node r
   foreach (node u of T in postorder) {
      if (u is a leaf) {
         M_in [u] = w_u
         M_out[u] = 0
      }
      else {
         M_in [u] = Σ_{v∈children(u)} M_out[v]  +  w_v
         M_out[u] = Σ_{v∈children(u)} max(M_out[v], M_in[v])
      }
   }
   return max(M_in[r], M_out[r])
}
```

↑ ensures a node is visited after all its children

Pf. Takes O(n) time since we visit nodes in postorder and examine each edge exactly once. ▪

10.3 Circular Arc Coloring

Wavelength-Division Multiplexing

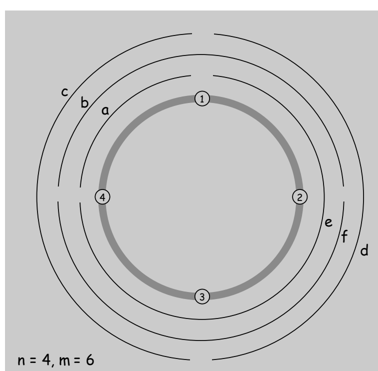
Wavelength-division multiplexing (WDM). Allows m communication streams (arcs) to share a portion of a fiber optic cable, provided they are transmitted using different wavelengths.

Ring topology. Special case is when network is a cycle on n nodes.

Bad news. NP-complete, even on rings.

Brute force. Can determine if k colors suffice in $O(k^m)$ time by trying all k-colorings.

Goal. $O(f(k)) \cdot poly(m, n)$ on rings.



Wavelength-Division Multiplexing

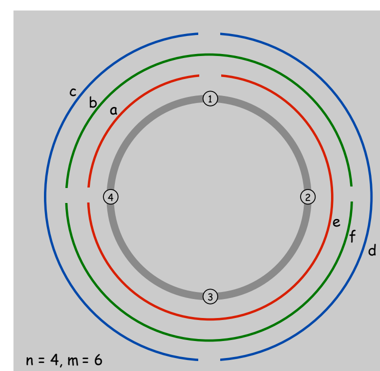
Wavelength-division multiplexing (WDM). Allows m communication streams (arcs) to share a portion of a fiber optic cable, provided they are transmitted using different wavelengths.

Ring topology. Special case is when network is a cycle on n nodes.

Bad news. NP-complete, even on rings.

Brute force. Can determine if k colors suffice in $O(k^m)$ time by trying all k-colorings.

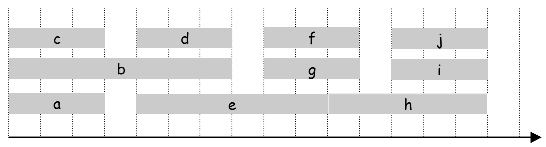
Goal. $O(f(k)) \cdot poly(m, n)$ on rings.



Review: Interval Coloring

Interval coloring. Greedy algorithm finds coloring such that number of colors equals depth of schedule.

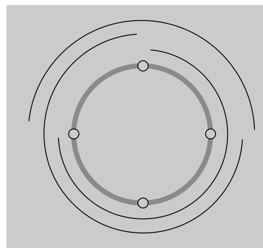
↖ maximum number of streams at one location



Circular arc coloring.

- Weak duality: number of colors ≥ depth.
- Strong duality does not hold.

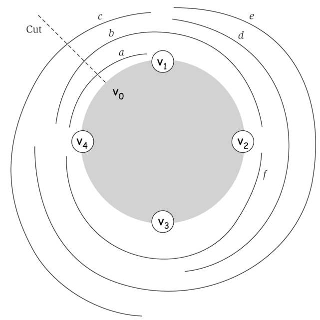
max depth = 2
min colors = 3



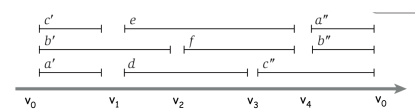
(Almost) Transforming Circular Arc Coloring to Interval Coloring

Circular arc coloring. Given a set of n arcs with depth d ≤ k, can the arcs be colored with k colors?

Equivalent problem. Cut the network between nodes v_1 and v_n. The arcs can be colored with k colors iff the intervals can be colored with k colors in such a way that "sliced" arcs have the same color.



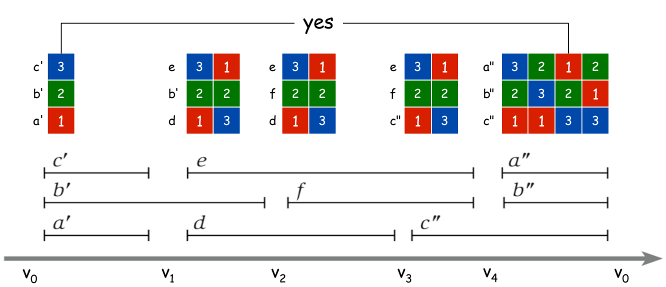
colors of a', b', and c' must correspond to colors of a", b", and c"



Circular Arc Coloring: Dynamic Programming Algorithm

Dynamic programming algorithm.

- Assign distinct color to each interval which begins at cut node v_0.
- At each node v_i, some intervals may finish, and others may begin.
- Enumerate all k-colorings of the intervals through v_i that are consistent with the colorings of the intervals through v_{i-1}.
- The arcs are k-colorable iff some coloring of intervals ending at cut node v_0 is consistent with original coloring of the same intervals.



Circular Arc Coloring: Running Time

Running time. O(k! · n).

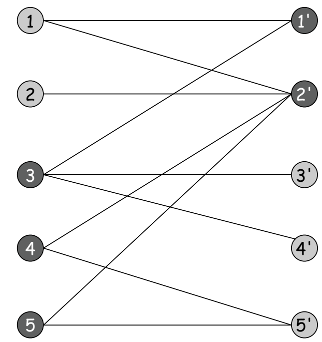
- n phases of the algorithm.
- Bottleneck in each phase is enumerating all consistent colorings.
- There are at most k intervals through v_i, so there are at most k! colorings to consider.

Remark. This algorithm is practical for small values of k (say k = 10) even if the number of nodes n (or paths) is large.

Vertex Cover in Bipartite Graphs

Vertex Cover

Vertex cover. Given an undirected graph G = (V, E), a vertex cover is a subset of vertices $S \subseteq V$ such that for each edge $(u, v) \in E$, either $u \in S$ or $v \in S$ or both.

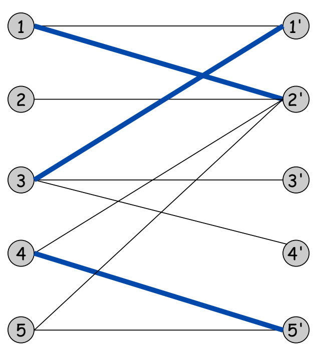


S = { 3, 4, 5, 1', 2' }
|S| = 5

Vertex Cover

Weak duality. Let M be a matching, and let S be a vertex cover. Then, $|M| \leq |S|$.

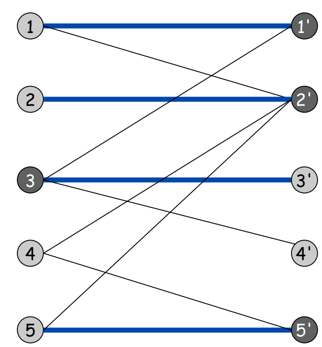
Pf. Each vertex can cover at most one edge in any matching.



M = 1-2', 3-1', 4-5'
|M| = 3

Vertex Cover: König-Egerváry Theorem

König-Egerváry Theorem. In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.



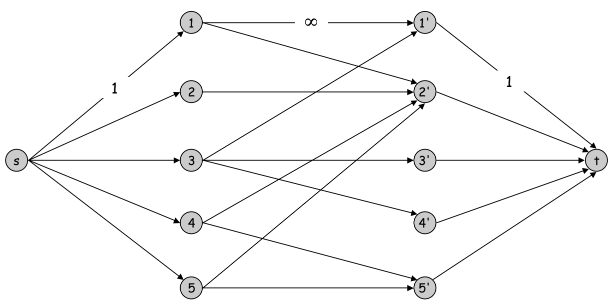
S* = { 3, 1', 2', 5'}
|S*| = 4

M* = 1-1', 2-2', 3-3', 5-5'
|M*| = 4

Vertex Cover: Proof of König-Egerváry Theorem

König-Egerváry Theorem. In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching M and cover S such that |M| = |S|.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.



Vertex Cover: Proof of König-Egerváry Theorem

König-Egerváry Theorem. In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching M and cover S such that |M| = |S|.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$, $R_B = R \cap B$.

- Claim 1. $S = L_B \cup R_A$ is a vertex cover.
  - consider $(u, v) \in E$
  - $u \in L_A$, $v \in R_B$ impossible since infinite capacity
  - thus, either $u \in L_B$ or $v \in R_A$ or both

- Claim 2. |S| = |M|.
  - max-flow min-cut theorem $\Rightarrow$ |M| = cap(A, B)
  - only edges of form (s, u) or (v, t) contribute to cap(A, B)
  - $|M| = \text{cap}(A, B) = |L_B| + |R_A| = |S|$. ▪

Register Allocation

Register Allocation

Register. One of k of high-speed memory locations in computer's CPU.
(k: say 32)

Register allocator. Part of an optimizing compiler that controls which variables are saved in the registers as compiled program executes.

Interference graph. Nodes are "live ranges" (variables or temporaries). There is an edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin, 1982] Can solve register allocation problem iff interference graph is k-colorable.

Spilling. If graph is not k-colorable (or we can't find a k-coloring), we "spill" certain variables to main memory and swap back as needed.
(typically infrequently used variables that are not in inner loops)

A Useful Property

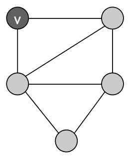
Remark. Register allocation problem is NP-hard.

Useful fact. If a node v in graph G has fewer than k neighbors, G is k-colorable iff G – { v } is k-colorable.

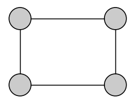
(delete v and all incident edges)

Pf. Delete node v from G and color G – { v }.

- If G – { v } is not k-colorable, then neither is G.
- If G – { v } is k-colorable, then there is at least one remaining color left for v. ▪



k = 3



k = 2

G is 2-colorable even though all nodes have degree 2

Chaitin's Algorithm

Vertex coloring heuristic. [Kempe 1879, Chaitin 1982]

- Repeat until G is empty.
  - pick a node v with fewer than k neighbors (say, node with fewest neighbors)
  - push v on stack
  - delete v and all its incident edges
- Repeat until the stack is empty.
  - pop next node v from the stack
  - assign v a color different from its neighboring nodes which have already been colored

Theorem. If algorithm never encounters a graph where all nodes have degree ≥ k, then it produces a k-coloring.

Practice. This algorithm (and variants) are extremely effective and widely used in real compilers for register allocation.