8. NP and Computational Intractability

Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns.	Ex.
Greed.	O(n log n) interval scheduling.
Divide-and-conquer.	O(n log n) FFT.
Dynamic programming.	$O(n^2)$ edit distance.
Duality.	$O(n^3)$ bipartite matching.
Reductions.	
Randomization.	

Algorithm design anti-patterns.	
NP-completeness.	$O(n^k)$ algorithm unlikely.
PSPACE-completeness.	$O(n^k)$ certification algorithm unlikely.
Undecidability.	No algorithm possible.

8.1. Polynomial-Time Reductions

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [Cobham 1964, Edmonds 1965, Rabin 1966]
Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Euler cycle	Hamiltonian cycle
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover
Matching	3D-matching
Primality testing	Factoring

Classify Problems

Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.

Provably requires exponential-time.
- Given a Turing machine, does it halt in at most k steps?
- Given a board position in an n-by-n generalization of chess, can black guarantee a win?

Bad news. Huge number of fundamental problems have defied classification for decades.

Worse news. Many were shown to be "computationally equivalent" and intractable for all practical purposes.

Polynomial-Time Reduction

Desiderata'. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

don't confuse with reduces from ↙

Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:
- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

↑ computational model supplemented by special piece of hardware that solves instances of Y in a single step

Notation. $X \le_p Y$.

Remarks.
- We pay for time to write down instances sent to black box ⇒ instances of Y must be of polynomial size.
- Note: Cook reducibility.

↖ in contrast to Karp reductions

Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $X \le_p Y$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.

Establish intractability. If $X \le_p Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.

Establish equivalence. If $X \le_p Y$ and $Y \le_p X$, we use notation $X \equiv_p Y$.

↑ up to cost of reduction

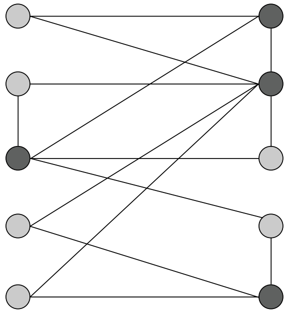
Polynomial-Time Reduction

Basic strategies.
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

Ex. Is there an independent set of size ≥ 6? Yes.
Ex. Is there an independent set of size ≥ 7? No.

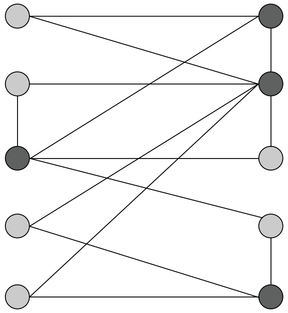


○ independent set

Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

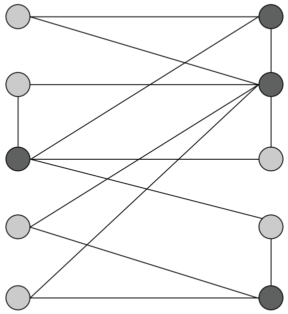
Ex. Is there a vertex cover of size ≤ 4? Yes.
Ex. Is there a vertex cover of size ≤ 3? No.



● vertex cover

Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.
Pf. We show S is an independent set iff $V - S$ is a vertex cover.



○ independent set
● vertex cover

Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.
Pf. We show S is an independent set iff $V - S$ is a vertex cover.

$\Rightarrow$

- Let S be any independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow u \notin S$ or $v \notin S \Rightarrow u \in V - S$ or $v \in V - S$.
- Thus, $V - S$ covers (u, v).

$\Leftarrow$

- Let $V - S$ be any vertex cover.
- Consider two nodes $u \in S$ and $v \in S$.
- Observe that $(u, v) \notin E$ since $V - S$ is a vertex cover.
- Thus, no two nodes in S are joined by an edge $\Rightarrow S$ independent set. ▪

Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of ≤ k of these sets whose union is equal to U?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
S_1 = {3, 7} S_4 = {2, 4}
S_2 = {3, 4, 5, 6} S_5 = {5}
S_3 = {1} S_6 = {1, 2, 6, 7}

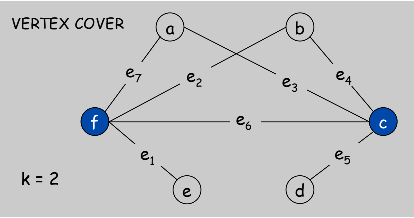
Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\le_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance G = (V, E), k, we construct a set cover instance whose size equals the size of the vertex cover instance.

Construction.

- Create SET-COVER instance:
  - k = k, U = E, $S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size ≤ k iff vertex cover of size ≤ k. ▪



SET COVER

U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
S_a = {3, 7} S_b = {2, 4}
S_c = {3, 4, 5, 6} S_d = {5}
S_e = {1} S_f = {1, 2, 6, 7}

Integer Programming

INTEGER-PROGRAMMING: Given integers a_{ij} and b_i, find integers x_j that satisfy:

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij} x_j & \ge & b_i & 1 \le i \le m \\ x_j & \ge & 0 & 1 \le j \le n \\ x_j & & \text{integral} & 1 \le j \le n \end{array}$$

Claim. VERTEX-COVER $\le_P$ INTEGER-PROGRAMMING.

$$\begin{array}{rcll} \sum_{u \in V} x_u & \le & k & \\ x_u + x_v & \ge & 1 & (u, v) \in E \\ x_u & \ge & 0 & u \in V \\ x_u & & \text{integral} & u \in V \end{array}$$

Polynomial-Time Reduction

Basic strategies.
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.
↑
each corresponding to different variables

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$
Yes: x_1 = true, x_2 = true x_3 = false.

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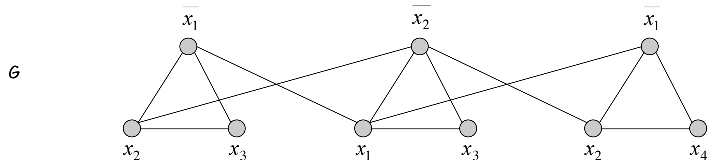
3 Satisfiability Reduces to Independent Set

Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.
Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.
- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.



k = 3

$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

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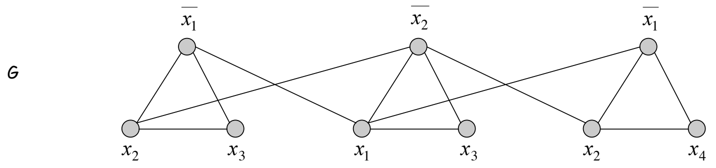
3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable.

Pf. ⇒ Let S be independent set of size k.
- S must contain exactly one vertex in each triangle.
- Set these literals to true. ← and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf ⇐ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪



k = 3

$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\le_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\le_P$ INDEPENDENT-SET.

Transitivity. If $X \le_P Y$ and $Y \le_P Z$, then $X \le_P Z$.
Pf idea. Compose the two algorithms.

Ex: 3-SAT $\le_P$ INDEPENDENT-SET $\le_P$ VERTEX-COVER $\le_P$ SET-COVER.

Self-Reducibility

Decision problem. Does there exist a vertex cover of size $\le k$?
Search problem. Find vertex cover of minimum cardinality.

Self-reducibility. Search problem $\le_P$ decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.

- (Binary) search for cardinality k^* of min vertex cover.
- Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\le k^* - 1$.
  - any vertex in any min vertex cover will have this property
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{v\}$.

↑ delete v and all incident edges