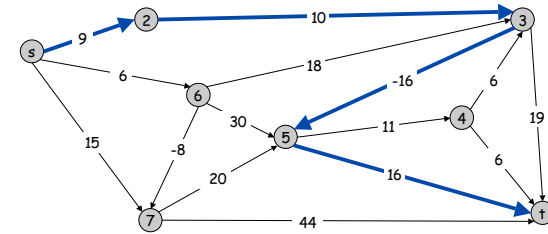


6.8 Shortest Paths

Shortest Paths

Shortest path problem. Given a directed graph $G = (V, E)$, with edge weights c_{vw} , find shortest path from node s to node t .
 allow negative weights

Ex. Nodes represent agents in a financial setting and c_{vw} is cost of transaction in which we buy from agent v and sell immediately to w .

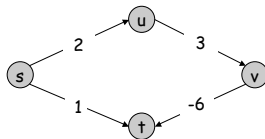


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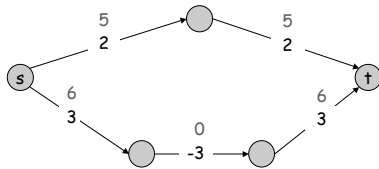
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Shortest Paths: Failed Attempts

Dijkstra. Can fail if negative edge costs.

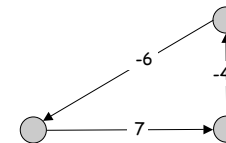


Re-weighting. Adding a constant to every edge weight can fail.

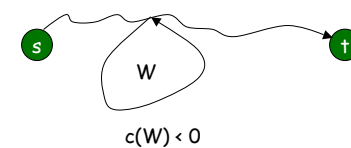


Shortest Paths: Negative Cost Cycles

Negative cost cycle.



Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s - t path; otherwise, there exists one that is simple.



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Shortest Paths: Dynamic Programming

Def. $OPT(i, v)$ = length of shortest v - t path P using at most i edges.

- Case 1: P uses at most $i-1$ edges.
 - $OPT(i, v) = OPT(i-1, v)$
- Case 2: P uses exactly i edges.
 - if (v, w) is first edge, then OPT uses (v, w) , and then selects best w - t path using at most $i-1$ edges

$$OPT(i, v) = \begin{cases} 0 & \text{if } i=0 \\ \min \left\{ OPT(i-1, v), \min_{(v,w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Remark. By previous observation, if no negative cycles, then $OPT(n-1, v)$ = length of shortest v - t path.

Shortest Paths: Implementation

```
Shortest-Path( $G, t$ ) {  
  foreach node  $v \in V$   
     $M[0, v] \leftarrow \infty$   
   $M[0, t] \leftarrow 0$   
  
  for  $i = 1$  to  $n-1$   
    foreach node  $v \in V$   
       $M[i, v] \leftarrow M[i-1, v]$   
      foreach edge  $(v, w) \in E$   
         $M[i, v] \leftarrow \min \{ M[i, v], M[i-1, w] + c_{vw} \}$   
}
```

Analysis. $\Theta(mn)$ time, $\Theta(n^2)$ space.

Finding the shortest paths. Maintain a "successor" for each table entry.

5

6

Shortest Paths: Practical Improvements

Practical improvements.

- Maintain only one array $M[v]$ = shortest v - t path that we have found so far.
- No need to check edges of the form (v, w) unless $M[w]$ changed in previous iteration.

Theorem. Throughout the algorithm, $M[v]$ is length of some v - t path, and after i rounds of updates, the value $M[v]$ is no larger than the length of shortest v - t path using $\leq i$ edges.

Overall impact.

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but substantially faster in practice.

Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path( $G, s, t$ ) {  
  foreach node  $v \in V$  {  
     $M[v] \leftarrow \infty$   
    successor[ $v$ ]  $\leftarrow \phi$   
  }  
  
   $M[t] = 0$   
  for  $i = 1$  to  $n-1$  {  
    foreach node  $w \in V$  {  
      if ( $M[w]$  has been updated in previous iteration) {  
        foreach node  $v$  such that  $(v, w) \in E$  {  
          if ( $M[v] > M[w] + c_{vw}$ ) {  
             $M[v] \leftarrow M[w] + c_{vw}$   
            successor[ $v$ ]  $\leftarrow w$   
          }  
        }  
      }  
    }  
    If no  $M[w]$  value changed in iteration  $i$ , stop.  
  }  
}
```

7

8

6.9 Distance Vector Protocol

Distance Vector Protocol

Communication network.

- Nodes \approx routers.
- Edges \approx direct communication link.
- Cost of edge \approx delay on link. \leftarrow naturally nonnegative, but Bellman-Ford used anyway!

Dijkstra's algorithm. Requires global information of network.

Bellman-Ford. Uses only local knowledge of neighboring nodes.

Synchronization. We don't expect routers to run in lockstep. The order in which each `foreach` loop executes is not important. Moreover, algorithm still converges even if updates are asynchronous.

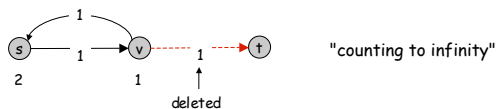
Distance Vector Protocol

Distance vector protocol.

- Each router maintains a vector of shortest path lengths to every other node (distances) and the first hop on each path (directions).
- Algorithm: each router performs n separate computations, one for each potential destination node.
- "Routing by rumor."

Ex. RIP, Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

Caveat. Edge costs may **change** during algorithm (or fail completely).



Path Vector Protocols

Link state routing.

- Each router also stores the entire path. \leftarrow not just the distance and first hop
- Based on Dijkstra's algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).

6.10 Negative Cycles in a Graph

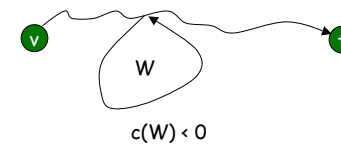
Detecting Negative Cycles

Lemma. If $OPT(n,v) = OPT(n-1,v)$ for all v , then no negative cycles.
Pf. Bellman-Ford algorithm.

Lemma. If $OPT(n,v) < OPT(n-1,v)$ for some node v , then (any) shortest path from v to t contains a cycle W . Moreover W has negative cost.

Pf. (by contradiction)

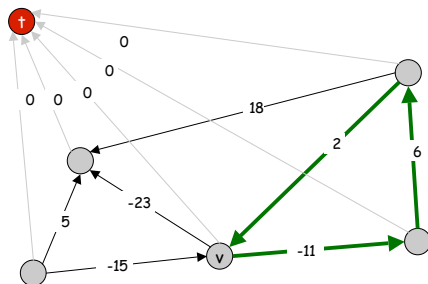
- Since $OPT(n,v) < OPT(n-1,v)$, we know P has exactly n edges.
- By pigeonhole principle, P must contain a directed cycle W .
- Deleting W yields a v - t path with $< n$ edges $\Rightarrow W$ has negative cost.



Detecting Negative Cycles

Theorem. Can detect negative cost cycle in $O(mn)$ time.

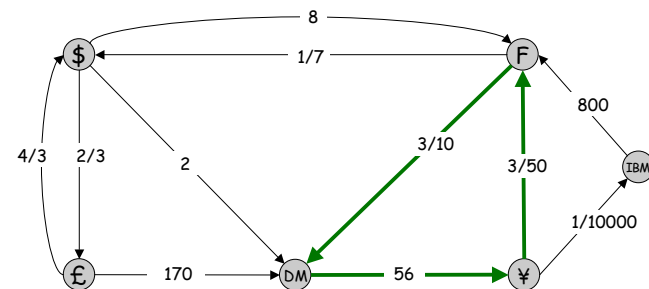
- Add new node t and connect all nodes to t with 0-cost edge.
- Check if $OPT(n, v) = OPT(n-1, v)$ for all nodes v .
 - if yes, then no negative cycles
 - if no, then extract cycle from shortest path from v to t



Detecting Negative Cycles: Application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



Detecting Negative Cycles: Summary

Bellman-Ford. $O(mn)$ time, $O(m + n)$ space.

- Run Bellman-Ford for n iterations (instead of $n-1$).
- Upon termination, Bellman-Ford successor variables trace a negative cycle if one exists.
- See p. 288 for improved version and early termination rule.