COS 521: Advanced Algorithm Design
Homework 4
Due: Mon, May 10

Collaboration Policy: You can collaborate in groups *of at most two students*. If you do refer to any sources, indicate this on your homework.

1. Given a graph $G(V, E)$, define the sparsest cut to be

$$\alpha(G) = \min_{S \subset V} \frac{|E(S, \bar{S})|}{|S||\bar{S}|}$$

Consider the following LP

$$\min \sum_{(i,j)\in E} x_{ij}$$

$$
\begin{align}
x_{ij} &= x_{ji} \tag{1}\\
x_{ij} + x_{jk} &\le x_{ik} \tag{2}\\
\sum_{i<j} x_{ij} &= 1 \tag{3}\\
x_{ij} &\ge 0
\end{align}
$$

(i) Show that this is a relaxation for sparsest cut.

(ii) If the distance function x_{ij} embeds isometrically into ℓ_1, show that the optimum of the LP is equal to the value of the sparsest cut. (Note: if the distance function x_{ij} can be mapped to ℓ_1, this implies that the vertices can be mapped to R^d for some d such that for all i, j, the ℓ_1 distance between the images of i and j is exactly $x_i j$.)

2. Recall von Neumann's minimax theorem:

$$\max_p \min_q p^T M q = \min_q \max_p p^T M q$$

where p and q are vectors with non-negative entries that sum up to 1. Prove the minimax theorem via LP duality.

3. Consider a uniform rooted tree of height h. All nodes have three children. Every leaf has a Boolean value associated with it. The value of any internal node is the majority of the value of its three children. We wish to determine the value of the root by evaluating as few leaves as possible.

(i) Show that for any deterministic algorithm, there is an instance that forces it to read the values of all $n = 3^h$ leaves.

(ii) Consider the following randomized algorithm to evaluate the tree: The algorithm evaluates the tree in a depth first fashion picking the order of the children of every node at random. If the value of a node is already determined, any unevaluated subtrees of that node are skipped. Determine the expected number of leaves read by this algorithm (worst case over all instances).

(iii) (Optional) Prove a lower bound on the expected number of queries made by a randomized algorithm for this problem.

4. Let $G(V, W)$ be a weighted graph on n vertices where c_{ij} denotes the weight on edge $\{i, j\}$. The isoperimetric constant $\gamma(G)$ of the graph is defined to be

$$\min \sum_{(i,j) \in E} c_{ij}(x_i - x_j)^2,$$

over all $x = (x_1, \ldots, x_n)$ satisfying $\sum x_i^2 = 1$ and $\sum x_i = 0$.

Show that $\gamma(G)/n$ is the optimum of the following SDP:

$$\min_{(i,j) \in E} c_{ij}(v_i - v_j)^2$$
$$\sum_{i,j}(v_i - v_j)^2 \ = \ 1$$