COS 521: Advanced Algorithm Design
Homework 3
Due: Thu, April 15

Collaboration Policy: You can collaborate in groups *of at most two students*. Try to solve the problems without referring to any texts. If you do refer to a text, indicate this on your homework, you might not get full credit.

1. We saw that the bottleneck flow algorithm combined with the augmenting path algorithm gave an $O(\min(mn^{2/3}, m^{5/2}))$ time algorithm for maxflow in a simple graph. Use similar ideas to develop an $O(m\sqrt{n})$ time algorithm for finding a maximum matching in bipartite graphs.

2. A *bridge* in a graph is an edge whose removal disconnects the graph into two components. Prove that a bridgeless cubic graph (all vertices have degree 3) has a perfect matching.

3. In the last homework we defined a cycle cover for a directed graph $G = (V, A)$. It is a collection of cycles such that each vertex is in exactly one cycle. Show that you can test for and find a cycle cover if one exists using an algorithm for matching.

4. Let A be a totally unimodular matrix. Prove that the polyhedron defined by the inequalities $a \le Ax \le b$ and $c \le x \le d$ is integral for all integral vectors a, b, c, d.

5. A $0, 1$ matrix is called a consecutive-ones matrix if in each column all the 1 occur consecutively. Show that a consecutive-one matrix is a network matrix.

6. The vertex cover problem asks for the minimum number of vertices that cover all the edges of a graph. Consider the following LP relaxation for the problem on a graph $G = (V, E)$. There is a variable $x(u)$ for each vertex u. The LP is

$$\min \sum_u x(u) \quad \text{s.t} \quad x(u) + x(v) \ge 1 \quad \forall (u, v) \in E \quad \text{and} \quad x(u) \ge 0 \quad \forall u \in V.$$

   Prove that the vertices of the polyhedron are *half-integral*, that is each coordinate is in $\{0, 1/2, 1\}$. Can you show this for all LPs that have only two variables per inequality?