COS 521: Advanced Algorithm Design
Homework 1
Due: Thu, April 1

Collaboration Policy: You must do problems 1 to 5 *on your own* without referring to texts/papers. Collaboration on 6 is allowed.

1. In this exercise, the goal is to prove the max flow mincut theorem using LP duality. Write down an LP formulation for the s-t maxflow in a given graph with capacities c_e on edges. Write the dual of this LP and show that it is equal to the mincut between s and t in the graph.

2. Give a bound on the number of cuts of size at most α times the mincut in an undirected graph. (Your score will depend on how tight your bound is). Does your bound work for directed graphs ?

3. Consider the bottleneck flow algorithm to compute a maxflow from s to t in a unit-capacity graph. Show that the algorithm spends $O(mk)$ time to increase the distance from s to t in the residual graph to k. Also show that when the distance between s and t is at least k, the mincut from s to t is $O(\min(n^2/k^2, m/k))$.

4. We discussed the mincost flow algorithm that finds a cycle W such that the cost reduction obtained by augmenting along W is at least $(OPT - C)/(m + n)$ where OPT is the cost of an optimum solution and C is the cost of the current solution. We noticed that finding such a cycle is NP-hard. We sketched an argument to find a set of cycles to obtain the same cost reduction using cycle covers. A cycle cover of a directed graph is a set of cycles such that every vertex is in exactly one of the cycles. A minimum cost cycle cover can be found in polynomial time. Show that you can use this algorithm to find a set of augmentations that reduce the cost of the current solution by at least $(OPT - C)/(m + n)$.

5. Suppose we have a partial order on n elements. We can represent the partial order as a directed acyclic graph. A chain in a partial order is simply a subset of elements that are totally ordered (they induce a path in the graph). An anti-chain is a set of elements that are incomparable (they form an independent set in the graph). Dilworth's theorem for partial orders is the following: in any partial order the minimum number m of disjoint chains which together contain all the elements is equal to the maximum number M of elements contained in an antichain. Prove Dilworth's theorem using the maxflow-micut theorem.

6. **Bonus question:** Some students claimed they could achieve a runtime of $O(m \cdot \text{polylog}(n))$ in the randomized min cut algorithm by a clever implementation of the contraction step. If you come up with such an implementation, you will recieve an automatic A+.