COS 521: Advanced Algorithm Design
Homework 1
Due: Tue, Mar 9

Collaboration Policy: You may collaborate with other students on problems 3 and 4. You must do problems 1 and 2 on your own. Collaboration is limited to discussion of ideas only, and you should write up the solutions entirely *on your own* and *list your collaborators* as well as *cite any references* you may have used.

1. **Definition 0.1** *A family of hash functions $H = \{h : M \to N\}$ is said to be a* perfect hash family *if for each set $S \subseteq M$ of size $s \leq n$, there exists a hash function $h \in H$ that is perfect for S.*

   Assuming that $n = s$, show that any perfect hash family must have size $2^{\Omega(s)}$.

2. Consider the following generalization of perfect hash functions.

   Definition 0.2 *Let $b_i(h, S) = |\{x \in S | h(x) = i\}$. A hash function h is b-*perfect *for S if $b_i(h, S) \leq b$ for each i. A family of hash functions $H = \{h : M \to N\}$ is said to be a b-*perfect hash family *if for each $S \subseteq M$ of size s there exists a hash function $h \in H$ that is b-perfect for S.*

   Show that there exists a b-perfect hash family H such that $b = O(\log n)$ and $|H| \leq m$, for any $m \geq n$.

3. Give an algorithm to find an intial basic feasible solution for the Simplex algorithm. (*Hint:* Use the Simplex algorithm itself).

4. Consider the following optimization problem with *robust conditions*:

   $$\min\{c^T x : x \in \Re^n; Ax \geq b \text{ for any } A \in F\},$$

   where $b \in \Re^m$ and F is a set of $m \times n$ matrices:

   $$F = \{A; \forall i, j; a_{ij}^{min} \leq a_{ij} \leq a_{ij}^{max}\}.$$

   (a) Considering F as a polytope in $\Re^{m \times n}$, what are the vertices of F ?

   (b) Show that instead of conditions for all $A \in F$, it is enough to consider the vertices of F. Write the resulting linear program. What is its size ? Is this poynomial in the size of the inpu, namely m, n and the sizes of b, c, a_{ij}^{min} and a_{ij}^{max} ?

   (c) Derive a more efficient description of the linear program: Write the conditions on x given by one row of A, for all choices of A. Formulate the condition as a linear program. Use duality and formulate the original problem as a linear program. What is the size of this one ? Is this polynomial in the size of the input ?