Max Flow, Min Cut

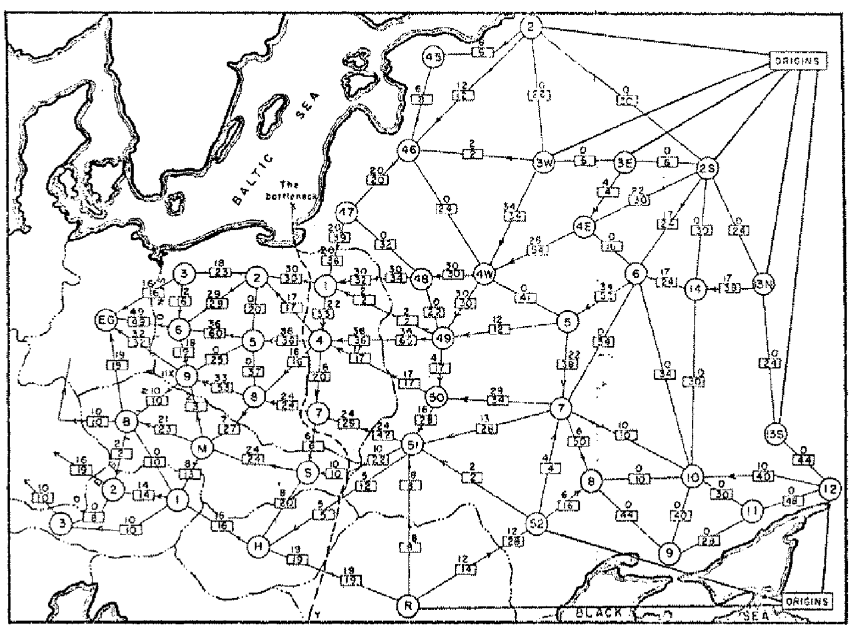
Minimum cut
Maximum flow
Max-flow min-cut theorem
Ford-Fulkerson augmenting path algorithm
Edmonds-Karp heuristics
Bipartite matching

Maximum Flow and Minimum Cut

Max flow and min cut.
- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

Nontrivial applications / reductions.
- Network connectivity.
- Bipartite matching.
- Data mining.
- Open-pit mining.
- Airline scheduling.
- Image processing.
- Project selection.
- Baseball elimination.
- Network reliability.
- Security of statistical data.
- Distributed computing.
- Egalitarian stable matching.
- Distributed computing.
- Many many more . . .

Soviet Rail Network, 1955

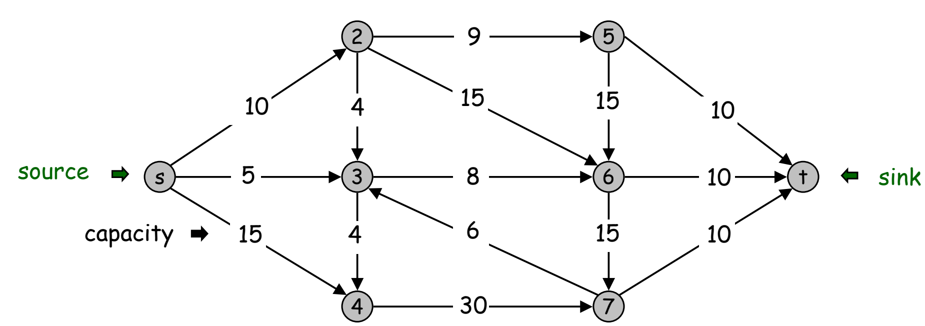


Source: *On the history of the transportation and maximum flow problems.* Alexander Schrijver in Math Programming, 91: 3, 2002.

Minimum Cut Problem

Network: abstraction for material FLOWING through the edges.
- Directed graph.
- Capacities on edges.
- Source node s, sink node t.

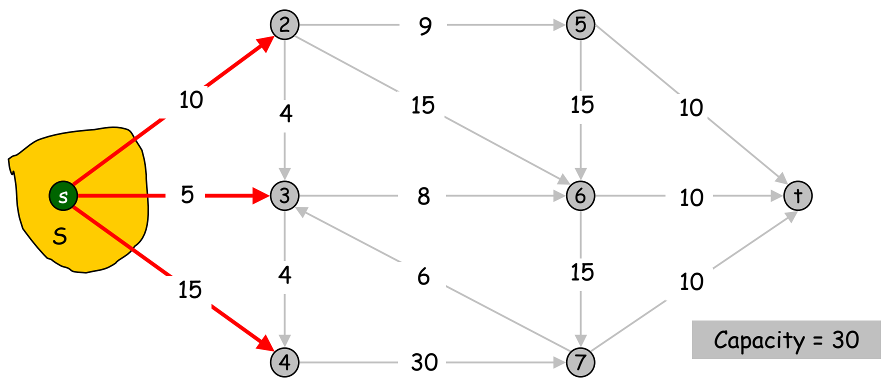
Min cut problem. Delete "best" set of edges to disconnect t from s.



Cuts

A cut is a node partition (S, T) such that s is in S and t is in T.

- capacity(S, T) = sum of weights of edges leaving S.



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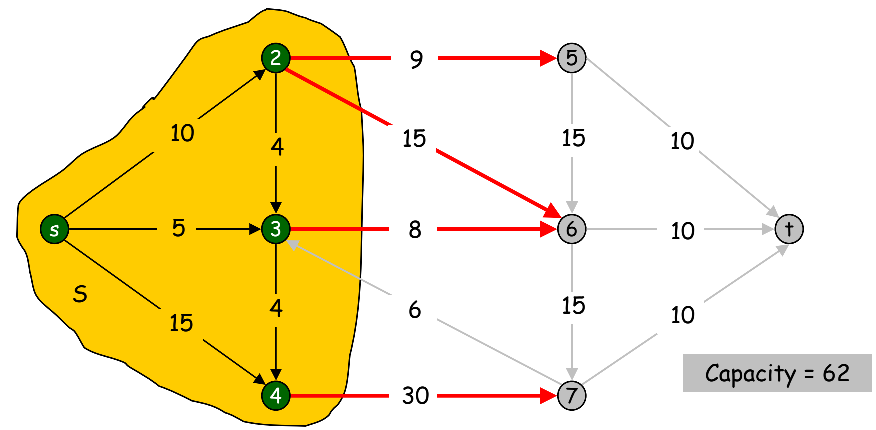

Minimum Cut Problem

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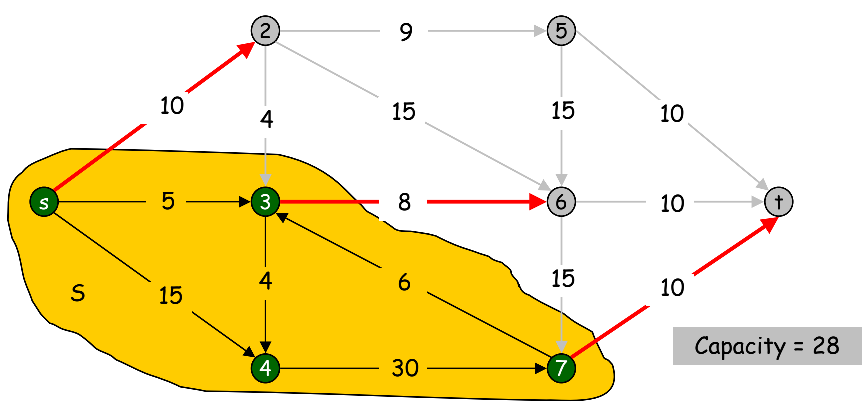
- capacity(S, T) = sum of weights of edges leaving S.

Min cut problem. Find an s-t cut of minimum capacity.



Maximum Flow Problem

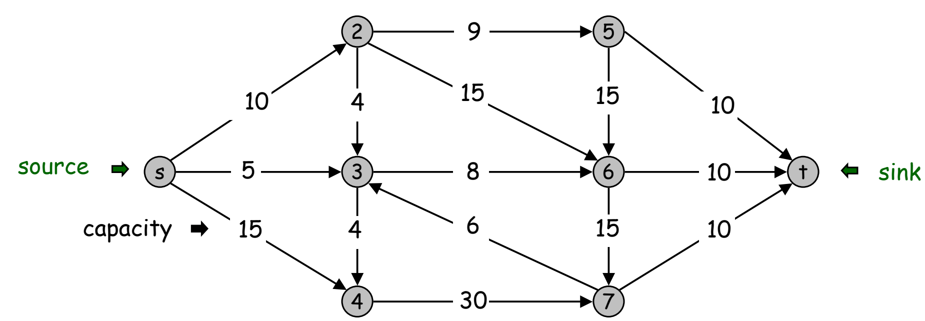
Network: abstraction for material FLOWING through the edges.

- Directed graph.
- Capacities on edges.
- Source node s, sink node t.

same input as min cut problem

Max flow problem. Assign flow to edges so as to:

- Equalize inflow and outflow at every intermediate vertex.
- Maximize flow sent from s to t.

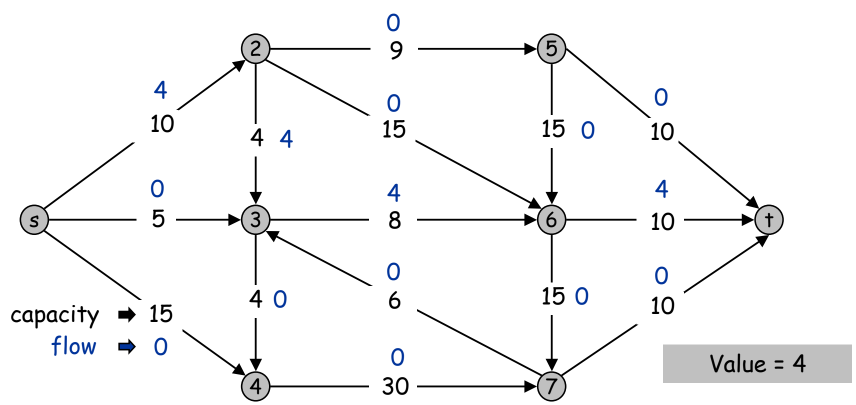


Flows

A flow f is an assignment of weights to edges so that:

- Capacity: $0 \leq f(e) \leq u(e)$.
- Flow conservation: flow leaving v = flow entering v.

except at s or t



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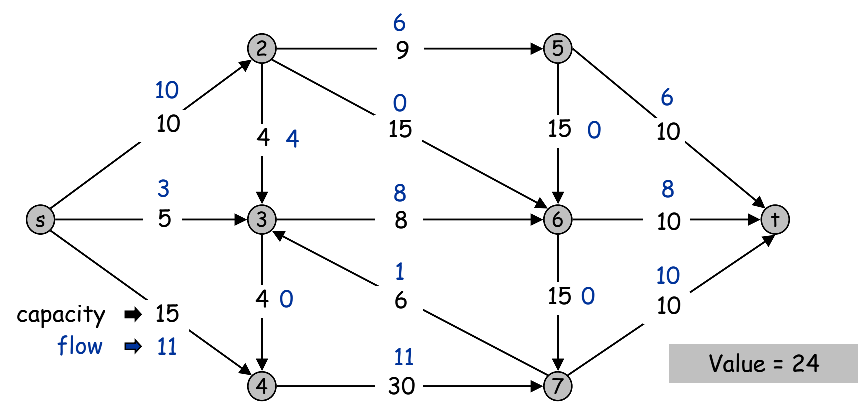
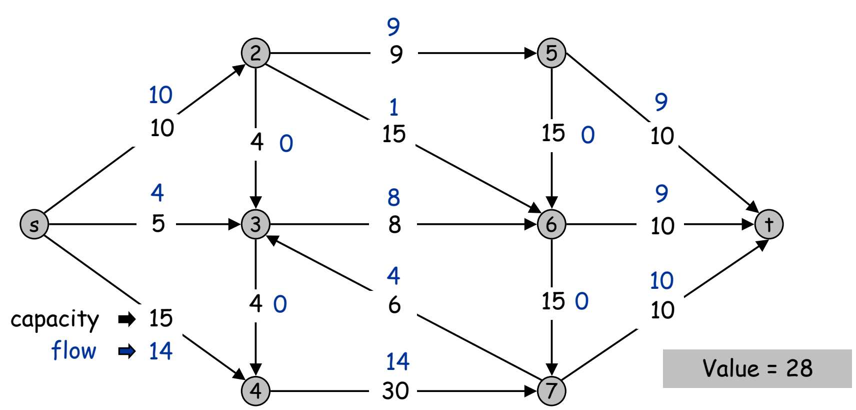


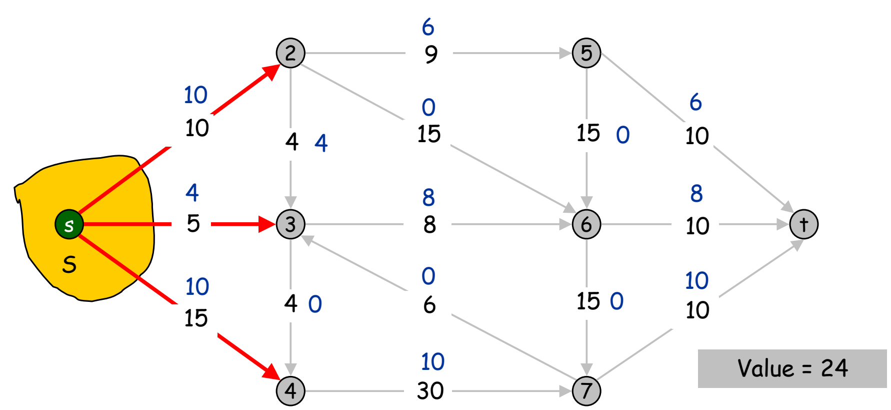
Maximum Flow Problem

Max flow problem: find flow that maximizes net flow into sink.



Flows and Cuts

Observation 1. Let f be a flow, and let (S, T) be any s-t cut. Then, the net flow sent across the cut is equal to the amount reaching t.



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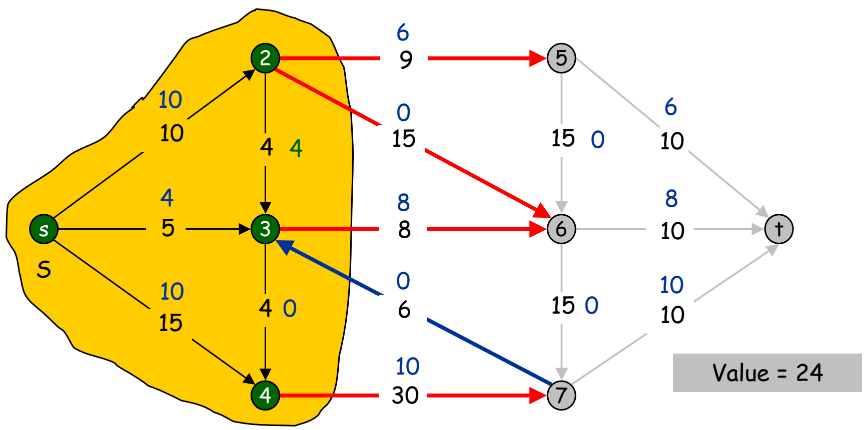

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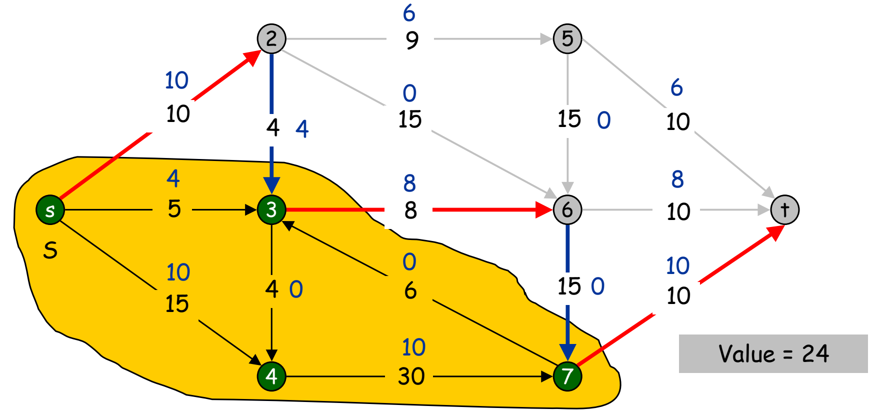
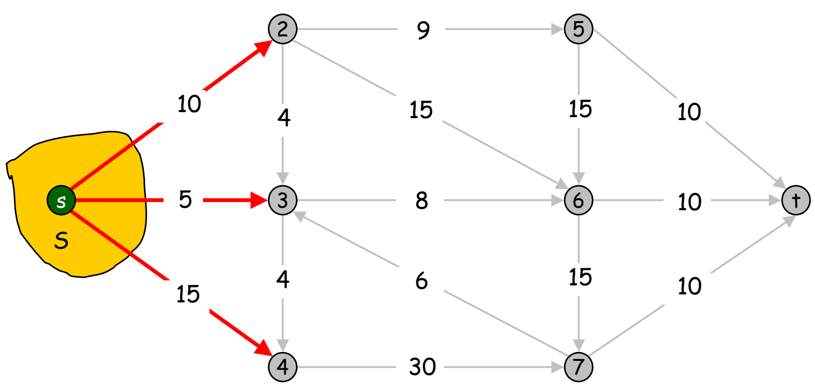


Flows and Cuts

Observation 2. Let f be a flow, and let (S, T) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

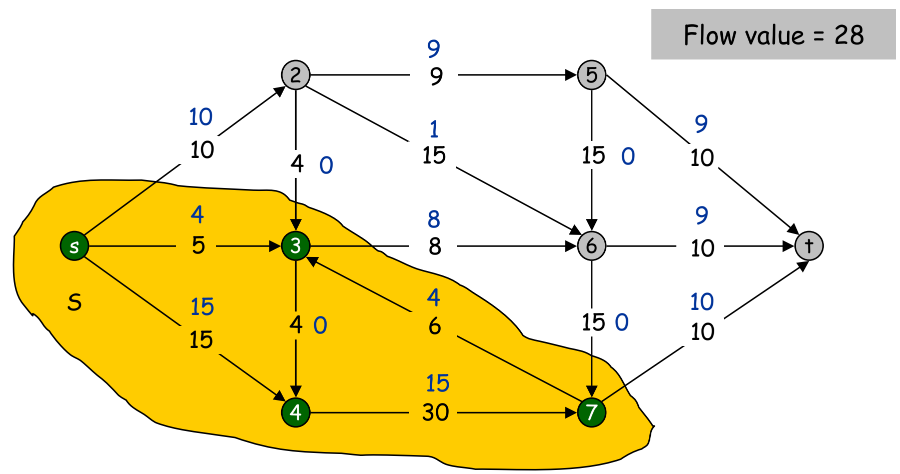
Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30



Max Flow and Min Cut

Observation 3. Let f be a flow, and let (S, T) be an s-t cut whose capacity equals the value of f. Then f is a max flow and (S, T) is a min cut.

Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28

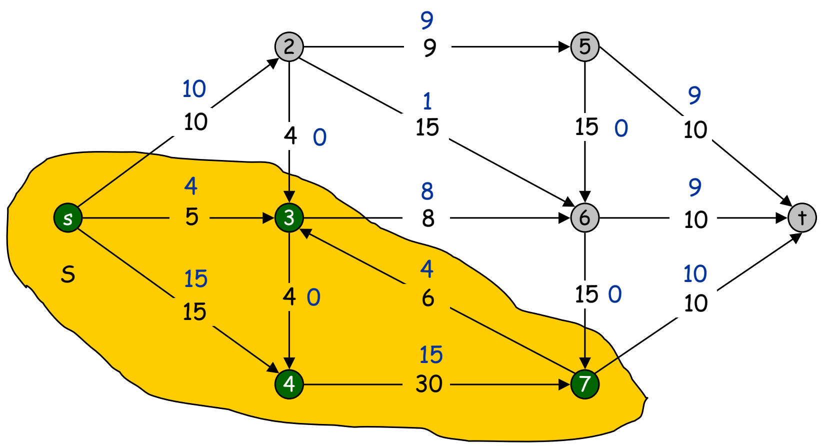


Max-Flow Min-Cut Theorem

Max-flow min-cut theorem. (Ford-Fulkerson, 1956): In any network, the value of max flow equals capacity of min cut.

- Proof IOU: we find flow and cut such that Observation 3 applies.

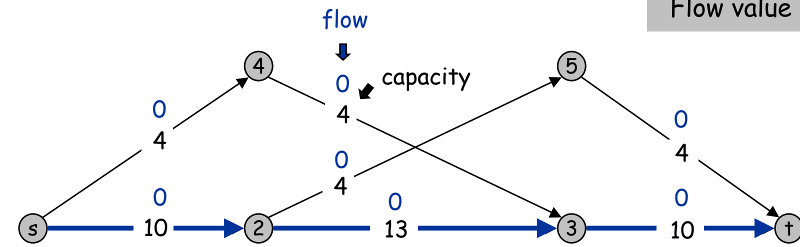
Min cut capacity = 28 ⇔ Max flow value = 28



Towards an Algorithm

Find s-t path where each arc has f(e) < u(e) and "augment" flow along it.

Flow value = 0

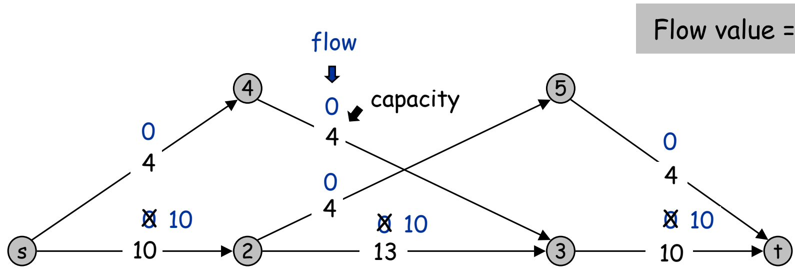


Towards an Algorithm

Find s-t path where each arc has f(e) < u(e) and "augment" flow along it.

- Greedy algorithm: repeat until you get stuck.

Flow value = 10



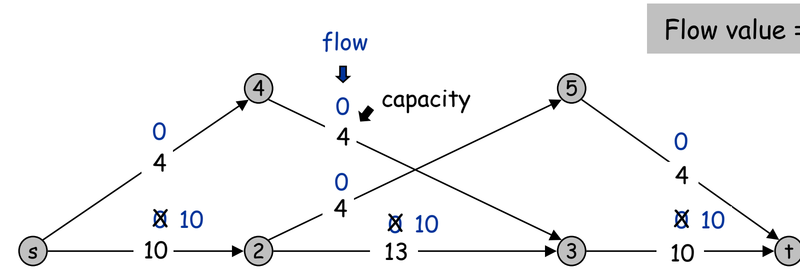
Bottleneck capacity of path = 10

Towards an Algorithm

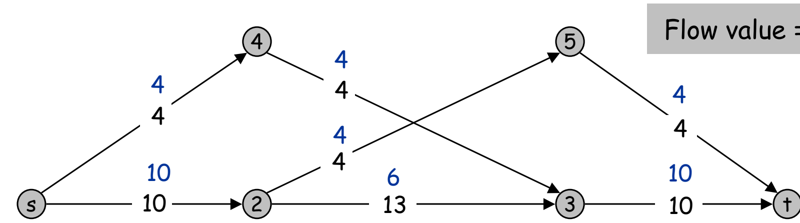
Find s-t path where each arc has f(e) < u(e) and "augment" flow along it.

- Greedy algorithm: repeat until you get stuck.
- Fails: need to be able to "backtrack."

Flow value = 10

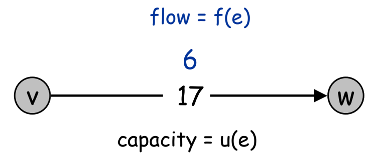


Flow value = 14



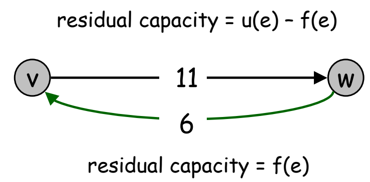
Residual Graph

Original graph.
- Flow f(e).
- Edge e = v-w



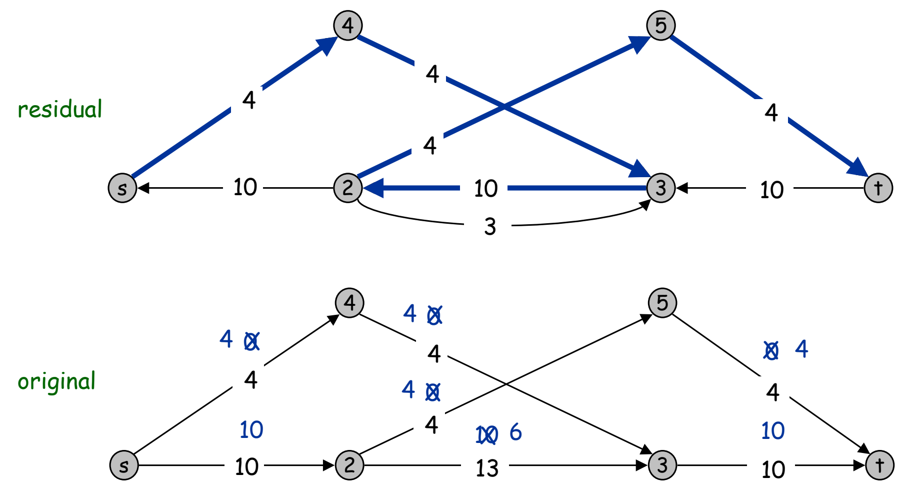
Residual edge.
- Edge e = v-w or w-v.
- "Undo" flow sent.

Residual graph.
- All the edges that have strictly positive residual capacity.



Augmenting Paths

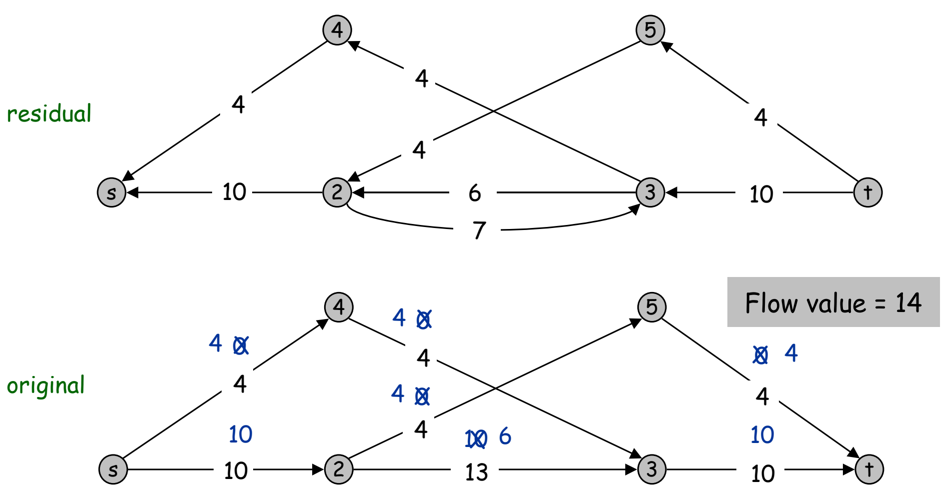
Augmenting path = path in residual graph.
- Increase flow along forward edges.
- Decrease flow along backward edges.



Augmenting Paths

Observation 4. If augmenting path, then not yet a max flow.

Q. If no augmenting path, is it a max flow?



Ford-Fulkerson Augmenting Path Algorithm

Ford-Fulkerson algorithm. Generic method for solving max flow.

```
while (there exists an augmenting path) {
   Find augmenting path P
   Compute bottleneck capacity of P
   Augment flow along P
}
```

Questions.
- Does this lead to a maximum flow? yes
- How do we find an augmenting path? s-t path in residual graph
- How many augmenting paths does it take?
- How much effort do we spending finding a path?

Max-Flow Min-Cut Theorem

Augmenting path theorem. A flow f is a max flow if and only if there are no augmenting paths.

> Max-flow min-cut theorem. The value of the max flow is equal to the capacity of the min cut.

We prove both simultaneously by showing the following are equivalent:

(i) f is a max flow.
(ii) There is no augmenting path relative to f.
(iii) There exists a cut whose capacity equals the value of f.

(i) $\Rightarrow$ (ii) equivalent to not (ii) $\Rightarrow$ not (i), which was Observation 4
(ii) $\Rightarrow$ (iii) next slide
(iii) $\Rightarrow$ (i) this was Observation 3

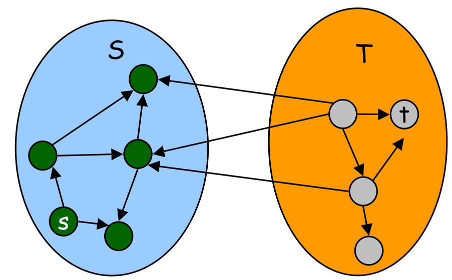
Proof of Max-Flow Min-Cut Theorem

(ii) $\Rightarrow$ (iii). If there is no augmenting path relative to f, then there exists a cut whose capacity equals the value of f.

Proof.

- Let f be a flow with no augmenting paths.
- Let S be set of vertices reachable from s in residual graph.
  - S contains s; since no augmenting paths, S does not contain t
  - all edges e leaving S in original network have f(e) = u(e)
  - all edges e entering S in original network have f(e) = 0

$$\begin{aligned} |f| &= \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ in to } S} f(e) \\ &= \sum_{e \text{ out of } S} u(e) \\ &= \mathit{capacity}(S, T) \end{aligned}$$



residual network

Max Flow Network Implementation

Edge in original graph may correspond to 1 or 2 residual edges.

- May need to traverse edge e = v-w in forward or reverse direction.
- Flow = f(e), capacity = u(e).
- Insert two copies of each edge, one in adjacency list of v and one in w.

```
public class Edge {
   private int v, w;   // from, to
   private int cap;    // capacity from v to w
   private int flow;   // flow from v to w

   public Edge(int v, int w, int cap) { ... }

   public int cap()  { return cap;  }
   public int flow() { return flow; }

   public boolean from(int v) { return this.v == v;            }
   public int other(int v)    { return from(v) ? this.w : this.v;   }
   public int capRto(int v)   { return from(v) ? flow : cap - flow; }
   public void addflowRto(int v, int d) { flow += from(v) ? -d : d; }
}
```


Ford-Fulkerson Algorithm: Implementation

Ford-Fulkerson main loop.

```
// while there exists an augmenting path, use it
while (augpath()) {

   // compute bottleneck capacity
   int bottle = INFINITY;
   for (int v = t; v != s; v = ST(v))
      bottle = Math.min(bottle, pred[v].capRto(v));

   // augment flow
   for (int v = t; v != s; v = ST(v))
      pred[v].addflowRto(v, bottle);

   // keep track of total flow sent from s to t
   value += bottle;
}
```

Ford-Fulkerson Algorithm: Analysis

Assumption: all capacities are integers between 1 and U.

Invariant: every flow value and every residual capacities remain an integer throughout the algorithm.

Theorem: the algorithm terminates in at most $| f^* | \leq V U$ iterations.

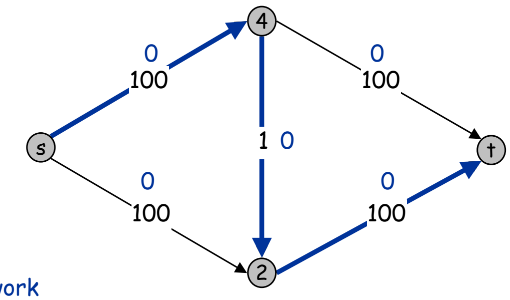
not polynomial in input size!

Corollary: if U = 1, then algorithm runs in $\leq$ V iterations.

Integrality theorem: if all arc capacities are integers, then there exists a max flow f for which every flow value is an integer.

Choosing Good Augmenting Paths

Use care when selecting augmenting paths.



Original Network

Choosing Good Augmenting Paths

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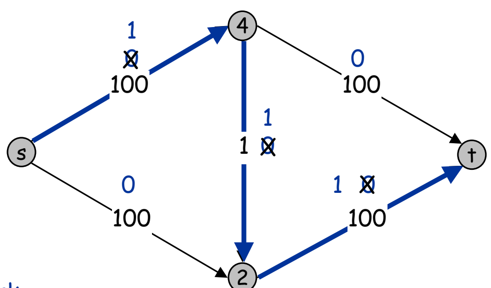

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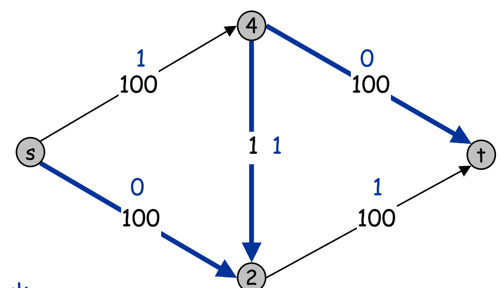

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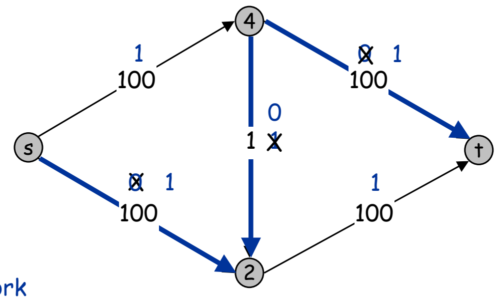

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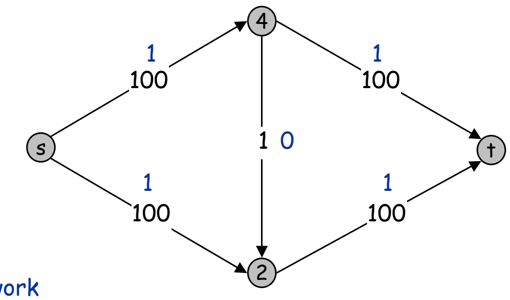


Original Network

200 iterations possible!

Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- Optimal choices for real world problems ???

Design goal is to choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

Choose augmenting path with: Edmonds-Karp (1972)

- Fewest number of arcs. (shortest path)
- Max bottleneck capacity. (fattest path)

Shortest Augmenting Path

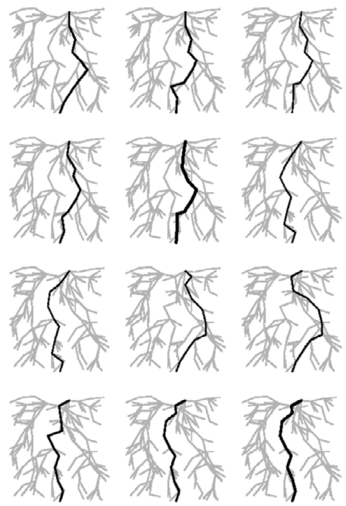
Shortest augmenting path.

- Easy to implement with BFS.
- Finds augmenting path with fewest number of arcs.

```
while (!q.isEmpty()) {
   int v = q.dequeue();
   IntIterator i = G.neighbors(v);
   while(i.hasNext()) {
      Edge e = i.next();
      int w = e.other(v);
      if (e.capRto(w) > 0) {  // is v-w a residual edge?
         if (wt[w] > wt[v] + 1) {
            wt[w] = wt[v] + 1;
            pred[w] = e;      // keep track of shortest path
            q.enqueue(w);
         }
      }
   }
}
return (wt[t] < INFINITY);    // is there an augmenting path?
```

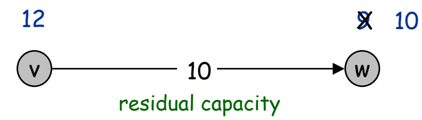
Shortest Augmenting Path Analysis

Length of shortest augmenting path increases monotonically.
- Strictly increases after at most E augmentations.
- At most E V total augmenting paths.
- $O(E^2 V)$ running time.



Fattest Augmenting Path

Fattest augmenting path.
- Finds augmenting path whose bottleneck capacity is maximum.
- Delivers most amount of flow to sink.
- Solve using Dijkstra-style (PFS) algorithm.




```
if (wt[w] < Math.min(wt[v], e.capRto(w)) {
    wt[w] = Math.min(wt[v], e.capRto(w));
    pred[w] = v;
}
```

Finding a fattest path. O(E log V) per augmentation with binary heap.
Fact. O(E log U) augmentations if capacities are between 1 and U.

Choosing an Augmenting Path

Choosing an augmenting path.
- Any path will do $\Rightarrow$ wide latitude in implementing Ford-Fulkerson.
- Generic priority first search.
- Some choices lead to good worst-case performance.
  - shortest augmenting path
  - fattest augmenting path
  - variation on a theme: PFS
- Average case not well understood.

Research challenges.
- Practice: solve max flow problems on real networks in linear time.
- Theory: prove it for worst-case networks.

History of Worst-Case Running Times

Year	Discoverer	Method	Asymptotic Time
1951	Dantzig	Simplex	$E V^2 U$ †
1955	Ford, Fulkerson	Augmenting path	$E V U$ †
1970	Edmonds-Karp	Shortest path	$E^2 V$
1970	Edmonds-Karp	Max capacity	$E \log U (E + V \log V)$ †
1970	Dinitz	Improved shortest path	$E V^2$
1972	Edmonds-Karp, Dinitz	Capacity scaling	$E^2 \log U$ †
1973	Dinitz-Gabow	Improved capacity scaling	$E V \log U$ †
1974	Karzanov	Preflow-push	V^3
1983	Sleator-Tarjan	Dynamic trees	$E V \log V$
1986	Goldberg-Tarjan	FIFO preflow-push	$E V \log (V^2 / E)$
. . .	. . .	. . .	. . .
1997	Goldberg-Rao	Length function	$E^{3/2} \log (V^2 / E) \log U$ † $EV^{2/3} \log (V^2 / E) \log U$ †

† Arc capacities are between 1 and U.

An Application

Jon placement.
- Companies make job offers.
- Students have job choices.

Can we fill every job?

Can we employ every student?

Alice-Adobe
Bob-Yahoo
Carol-HP
Dave-Apple
Eliza-IBM
Frank-Sun

Alice
- **Adobe**
- **Apple**
- **HP**

Bob
- **Adobe**
- **Apple**
- **Yahoo**

Carol
- **HP**
- **IBM**
- **Sun**

Dave
- **Adobe**
- **Apple**

Eliza
- **IBM**
- **Sun**
- **Yahoo**

Frank
- **HP**
- **Sun**
- **Yahoo**

Adobe
- Alice
- Bob
- Dave

Apple
- Alice
- Bob
- Dave

HP
- Alice
- Carol
- Frank

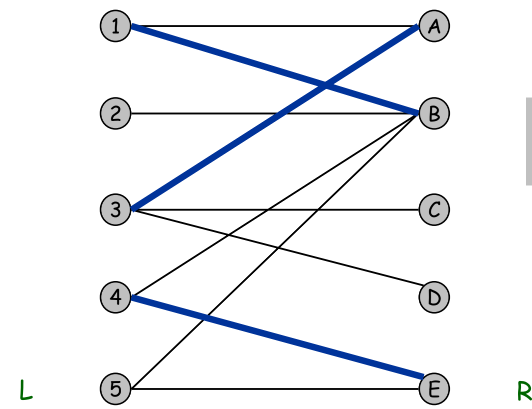
IBM
- Carol
- Eliza

Sun
- Carol
- Eliza
- Frank

Yahoo
- Bob
- Eliza
- Frank

Bipartite Matching

Bipartite matching.
- Input: undirected and bipartite graph G.
- Set of edges M is a matching if each vertex appears at most once.
- Max matching: find a max cardinality matching.



Matching M
1-B, 3-A, 4-E

Bipartite Matching

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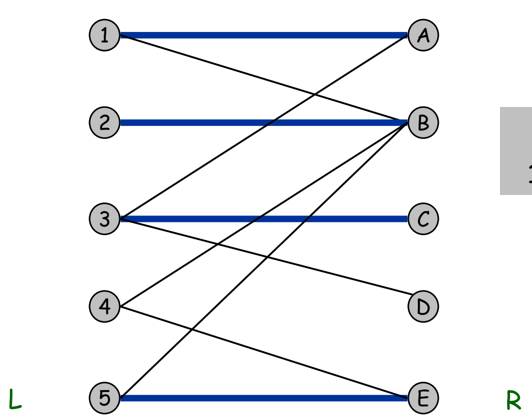
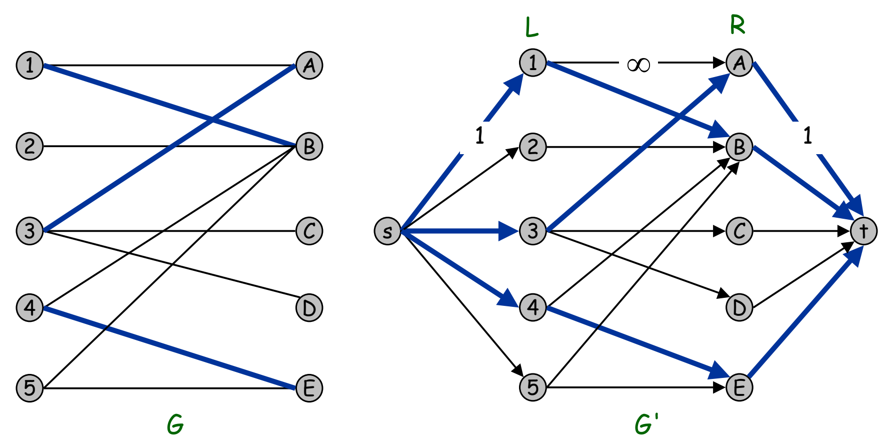


Matching M
1-A, 2-B, 3-C, 4-D

Bipartite Matching

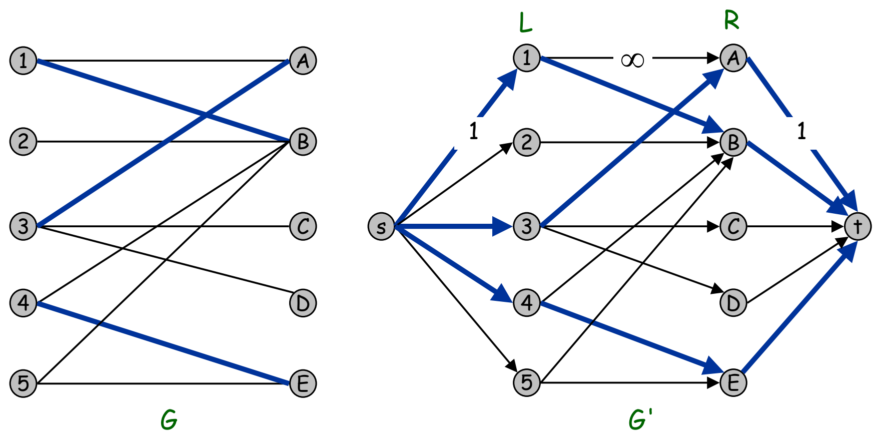
Reduces to max flow.
- Create a directed graph G'.
- Direct all arcs from L to R, and give infinite (or unit) capacity.
- Add source s, and unit capacity arcs from s to each node in L.
- Add sink t, and unit capacity arcs from each node in R to t.



Bipartite Matching: Proof of Correctness

Claim. Matching in G of cardinality k induces flow in G' of value k.

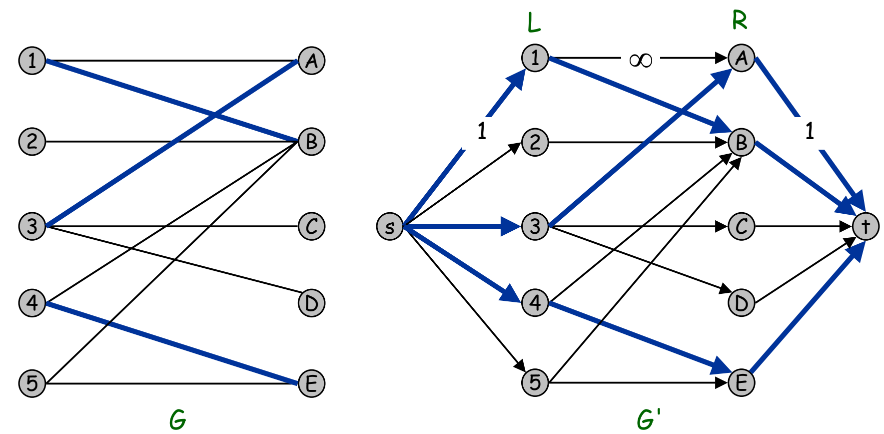
- Given matching M = { 1-B, 3-A, 4-E } of cardinality 3.
- Consider flow f that sends 1 unit along each of 3 paths:
  s-1-B-t s-3-A-t s-4-E-t.
- f is a flow, and has cardinality 3.



Bipartite Matching: Proof of Correctness

Claim. Flow f of value k in G' induces matching of cardinality k in G.

- By integrality theorem, there exists 0/1 valued flow f of value k.
- Consider M = set of edges from L to R with f(e) = 1.
  - each node in L and R incident to at most one edge in M
  - |M| = k



Reduction

Reduction.

- Given an instance of bipartite matching.
- Transform it to a max flow problem.
- Solve max flow problem.
- Transform max flow solution to bipartite matching solution.

Issues.

- How expensive is transformation? O(E + V)
- Is it better to solve problem directly? $O(E\ V^{1/2})$ bipartite matching

Bottom line: max flow is an extremely rich problem-solving model.

- Many important practical problems reduce to max flow.
- We know good algorithms for solving max flow problems.