COS 451 – Assignment (due April 10, 2003)

1 Decomposing Polygons

Let P be a simple polygon with n vertices, $v_1, \ldots, v_n$ in clockwise order. An optimal convex decomposition of P (abbreviated OCD) is any partition of P into a minimum number of convex polygons (called *parts*).

1. Prove that in some cases it might be necessary to introduce new vertices in the interior of P in order to obtain an OCD.
2. Prove that there always exists an OCD of P with no interior part (a part is called interior it contains no point on the boundary of P).

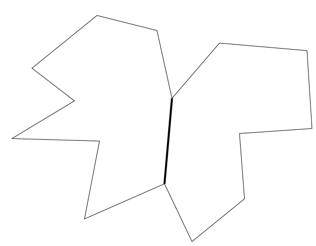

Figure 1: *A link*

Let a *link* be a segment connecting two reflex vertices v_i, v_j such that $v_i v_j$ lies completely inside P and resolves both reflex angles (Fig.1). A convex decomposition of P is called *regular* if it can be obtained by drawing segments in this manner: first draw a collection of non-intersecting links, then iterate on the following process. As long as there is still some unresolved reflex vertex, pick one of them and draw a segment from it along a *good* direction until you hit a point previously drawn or a point on the boundary of the polygon (Fig.2). This point is *not* allowed to be a reflex vertex. A direction is *good* if it resolves the reflex angle of the vertex in question.

3. Prove that if it is possible to draw k non-intersecting links, then a convex decomposition can be obtained with at most $c - k + 1$ parts (c is the number of reflex angles in P).
4. Describe an $O(n^3)$ algorithm for computing a regular convex decomposition with a minimum number of parts (hint: think of dynamic programming).

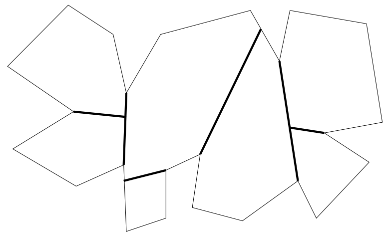

Figure 2: *A regular decomposition with two links*

2 Furthest Neighbors

Given n points in the plane sorted by x-coordinates, find an $O(n)$ time algorithm for computing the two points whose interdistance is maximum. Establish the correctness of your algorithm.

3 Triangulation

A polygon is called hollow if it has polygonal holes (Fig.3). Prove that to compute a triangulation of a hollow n-gon requires $\Omega(n \log n)$ time in the worst case; hint: assume that the number h of holes is $\Omega(n)$. For the general case where h is arbitrary (it could be zero), can you establish a worst-case lower bound on the time for triangulation as a function of both n and h.

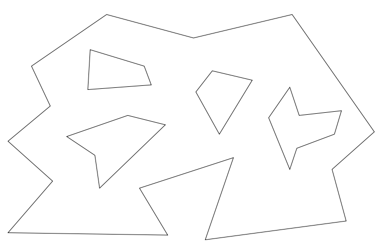

Figure 3: *A hollow polygon with* $n = 35$ *and* $h = 3$

4 Intersecting Convex Polygons

Design and analyze a linear time algorithm for computing the intersection of two convex n-gons. Extra credit: Implement it and test it thoroughly.

5 A Containment Problem

Let P and Q be two convex polygons with respectively p and q vertices. Find an $O(p + q)$ time algorithm for deciding whether P can be made to fit inside Q, allowing only translations.

6 Point-Set Triangulation

Here is a problem that has eluded satisfactory answers for a long time. Now is your chance to have a crack at it: Given n points in the plane (assumed to be in general position) a triangulation of the points is a triangulation of their convex hull such that a point is a vertex of the triangulation if and only if it is one of the n points. We'd like to find the triangulation that minimizes the added length of the edges. It is tempting to conjecture that the Delaunay triangulation is just what we want. Disprove this conjecture.

7 On a Problem of Area vs. Perimeter

Give a simple, yet rigorous proof that if a convex polygon P lies entirely inside another convex polygon Q, the perimeter of P cannot exceed that of Q.

8 Equidecomposibility

Prove that a disk (i.e. a circle and its interior) cannot be equidecomposed with a square of equal area, using a finite number of pieces. Assume that all cuts are piecewise smooth. You are allowed to make reasonable assumptions to make this problem tractable.

9 Ant Trails

Write a program that starts with a large polygon (preferably not simple) and connects the midpoints of the edges in sequence, thus producing a new polygon of the same size. Erase the old polygon and display. Iterate at infinitum. (Keep rescaling because it shrinks fast.) What's happening? Can you suggest any mathematical explanation?