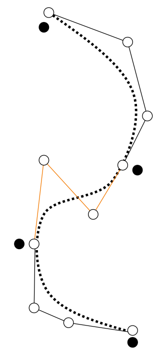
Parametric Curves & Surfaces

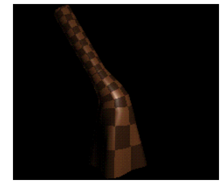
Adam Finkelstein
Princeton University
COS 426, Spring 2002

Overview



Part 1: Curves

Part 2: Surfaces



Przemyslaw Prusinkiewicz

Curves

- Splines: mathematical way to express curves
- Motivated by “loftsman’s spline”
  - Long, narrow strip of wood/plastic
  - Used to fit curves through specified data points
  - Shaped by lead weights called “ducks”
  - Gives curves that are “smooth” or “fair”
- Have been used to design:
  - Automobiles
  - Ship hulls
  - Aircraft fuselage/wing

Many applications in graphics

- Fonts ABC
- Animation paths
- Shape modeling
- etc…

Animation
(Angel, Plate 1)

Shell
(Douglas Turnbull, CS 426, Fall99)

Goals

- Some attributes we might like to have:
  - Predictable control
  - Multiple values
  - Local control
  - Versatility
  - Continuity
- We’ll satisfy these goals using:
  - Piecewise
  - Parametric
  - Polynomials

Parametric curves

A parametric curve in the plane is expressed as:

$$x = x(u)$$
$$y = y(u)$$

Example: a circle with radius r centered at origin:

$$x = r \cos u$$
$$y = r \sin u$$

In contrast, an implicit representation is:

$$x^2 + y^2 = r^2$$

Parametric polynomial curves

- A parametric polynomial curve is described:

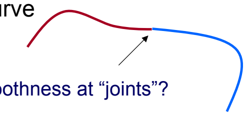
$$x(u) = \sum_{i=0}^{n} a_i u^i$$

$$y(u) = \sum_{i=0}^{n} b_i u^i$$

- Advantages of polynomial curves
  - Easy to compute
  - Infinitely differentiable

Piecewise parametric polynomials

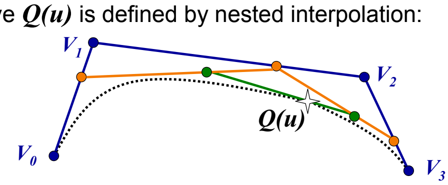
- Use different polynomial functions on different parts of the curve
  - Provides flexibility
  - How do you guarantee smoothness at "joints"? (*continuity*)



- In the rest of this lecture, we'll look at:
  - Bézier curves: general class of polynomial curves
  - Splines: ways of putting these curves together

Bézier curves

- Developed independently in 1960s by
  - Bézier (at Renault)
  - deCasteljau (at Citroen)
- Curve $Q(u)$ is defined by nested interpolation:



V_i's are control points
$\{V_0, V_1, \ldots, V_n\}$ is control polygon

Basic properties of Bézier curves

- Endpoint interpolation:

$$Q(0) = V_0$$
$$Q(1) = V_n$$

- Convex hull:
  - Curve is contained within convex hull of control polygon
- Symmetry

$$Q(u) \text{ defined by } \{V_0, \ldots, V_n\} \equiv Q(1-u) \text{ defined by } \{V_n, \ldots, V_0\}$$

Explicit formulation

- Let's indicate level of nesting with superscript j:
- An explicit formulation of $Q(u)$ is given by:

$$V_i^j = (1-u)V_i^{j-1} + uV_{i+1}^{j-1}$$

- Case n=2 (quadratic):

$$\begin{aligned} Q(u) &= V_0^2 \\ &= (1-u)V_0^1 + uV_1^1 \\ &= (1-u)[(1-u)V_0^0 + uV_1^0] + u[(1-u)V_1^0 + uV_2^0] \\ &= (1-u)^2 V_0^0 + 2u(1-u)V_1^0 + u^2 V_2^0 \end{aligned}$$

More properties

- General case: Bernstein polynomials

$$Q(u) = \sum_{i=0}^{n} V_i \binom{n}{i} u^i (1-u)^{n-i}$$

- Degree: polynomial of degree n
- Tangents: $Q'(0) = n(V_1 - V_0)$
  $Q'(1) = n(V_n - V_{n-1})$

Cubic curves

- From now on, let's talk about cubic curves (n=3)
- In CAGD, higher-order curves are often used
- In graphics, piecewise cubic curves will do
  - Specified by points and tangents
  - Allows specification of a curve in space
- All these ideas generalize to higher-order curves

Matrix form

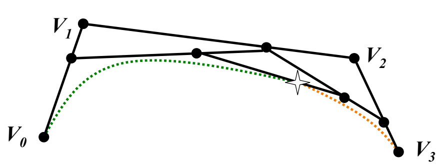
Bézier curves may be described in matrix form:

$$Q(u) = \sum_{i=0}^{n} V_i \binom{n}{i} u^i (1-u)^{n-i}$$
$$= (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u)V_2 + u^3 V_3$$
$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{M}_{\mathbf{Bezier}}} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Display

Q: How would you draw it using line segments?

A: Recursive subdivision!



Display

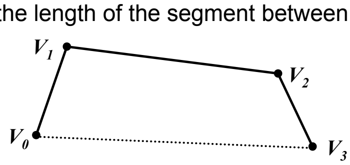
Pseudocode for displaying Bézier curves:

procedure Display($\{V_i\}$):
  if $\{V_i\}$ flat within ε
  then
    output line segment V_0V_n
  else
    subdivide to produce $\{L_i\}$ and $\{R_i\}$
    Display($\{L_i\}$)
    Display($\{R_i\}$)
  end if
end procedure

Flatness

Q: How do you test for flatness?

A: Compare the length of the control polygon to the length of the segment between endpoints

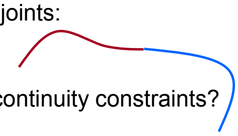


$$\frac{|V_1 - V_0| + |V_2 - V_1| + |V_3 - V_2|}{|V_3 - V_0|} < 1 + \varepsilon$$

(…or, compare dot products…)

Splines

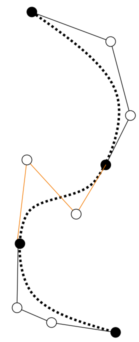
- For more complex curves, piece together Béziers
- We want continuity across joints:
  - Positional (C^0) continuity
  - Derivative (C^1) continuity



- Q: How would you satisfy continuity constraints?
- Q: Why not just use higher-order Bézier curves?
- A: Splines have several of advantages:
  - Numerically more stable
  - Easier to compute
  - Fewer bumps and wiggles

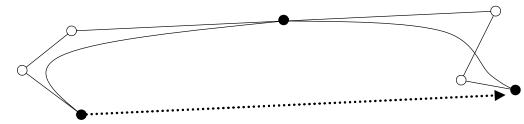
Catmull-Rom splines

- Properties
  - Interpolate control points
  - Have C^0 and C^1 continuity
- Derivation
  - Start with joints to interpolate
  - Build cubic Bézier between each joint
  - Endpoints of Bézier curves are obvious
- What should we do for the other Bézier control points?



Catmull-Rom Splines

- Catmull & Rom use:
  - half the magnitude of the vector between adjacent CP's



- Many other formulations work, for example:
  - Use an arbitrary constant τ times this vector
  - Default is 1/2
  - Gives a "tension" control
  - Could be adjusted for each joint

Matrix formulation

Convert from Catmull-Rom CP's to Bezier CP's:

$$\begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 0 & 6 & 0 & 0 \\ -1 & 6 & 1 & 0 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Exercise: Derive this matrix.
(Hint: in this case, τ is not 1/2.)

Properties

- Catmull-Rom splines have these attributes:
  - C1 continuity
  - Interpolation
  - Locality of control
  - No convex hull property

  (Proof left as an exercise.)

B-splines

- We still want local control
- Now we want C^2 continuity
- Give up interpolation
- It turns out we get convex hull property
- Constraints:
  - Three continuity conditions at each joint j
    - » Position of two curves same
    - » Derivative of two curves same
    - » Second derivatives same
  - Local control
    - » Each joint affected by 4 CPs

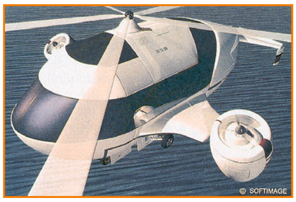
Matrix formulation for B-splines

- Grind through some messy math to get:

$$Q(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Curved Surfaces

- Motivation
  - Exact boundary representation for some objects
  - More concise representation than polygonal mesh



H&B Figure 10.46

Curved Surfaces

- What makes a good surface representation?
  - Accurate
  - Concise
  - Intuitive specification
  - Local support
  - Affine invariant
  - Arbitrary topology
  - Guaranteed continuity
  - Natural parameterization
  - Efficient display
  - Efficient intersections

Curved Surface Representations

- Polygonal meshes
- Subdivision surfaces
- Parametric surfaces
- Implicit surfaces

Curved Surface Representations

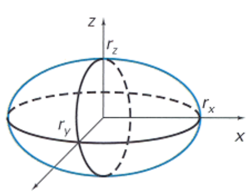
- Polygonal meshes
- Subdivision surfaces
- Parametric surfaces
- Implicit surfaces

Parametric Surfaces

- Boundary defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

- Example: ellipsoid

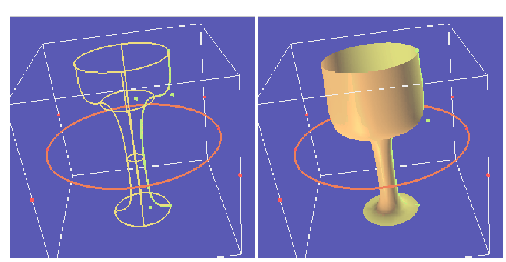
$$x = r_x \cos\phi \cos\theta$$
$$y = r_y \cos\phi \sin\theta$$
$$z = r_z \sin\phi$$



H&B Figure 10.10

Surface of revolution

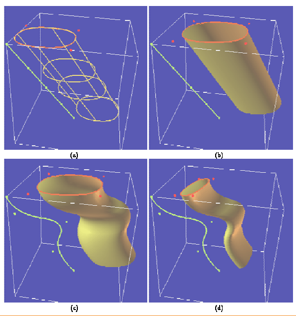
- Idea: take a curve and rotate it about an axis



Demetri Terzopoulos

Swept surface

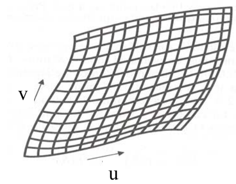
Idea: sweep one curve along path of another curve



Demetri Terzopoulos

Parametric Surfaces

Advantage: easy to enumerate points on surface.

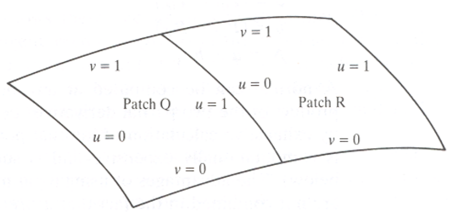


Disadvantage: need piecewise-parametric surface to describe complex shape.

FvDFH Figure 11.42

Piecewise Parametric Surfaces

Surface is partitioned into parametric patches:

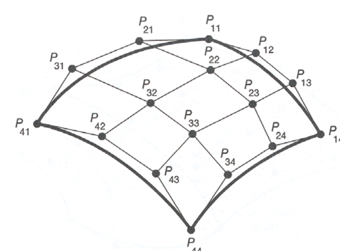


Same ideas as parametric splines!

Watt Figure 6.25

Parametric Patches

- Each patch is defined by blending control points

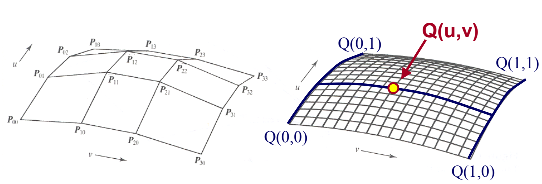


Same ideas as parametric curves!

FvDFH Figure 11.44

Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Bicubic Patches

Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:

$$Q(u,v) = \mathbf{UM}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}$$

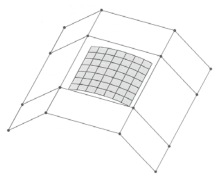
$$\mathbf{U} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bezier, B-spline, etc.)

B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\text{B-Spline}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}^{\mathbf{T}}_{\text{B-Spline}}\mathbf{V}$$

$$\mathbf{M}_{\text{B-Spline}} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \end{bmatrix}$$

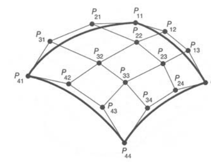


Watt Figure 6.28

Bezier Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\text{Bezier}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}^{\mathbf{T}}_{\text{Bezier}}\mathbf{V}$$

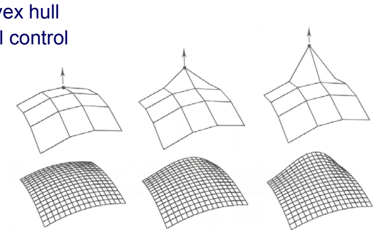
$$\mathbf{M}_{\text{Bezier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



FvDFH Figure 11.42

Bezier Patches

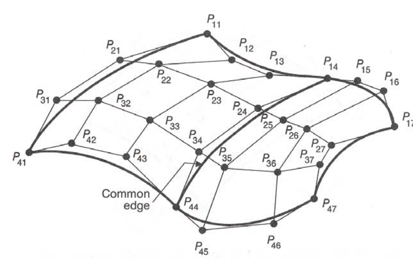
- Properties:
  - Interpolates four corner points
  - Convex hull
  - Local control



Watt Figure 6.22

Bezier Surfaces

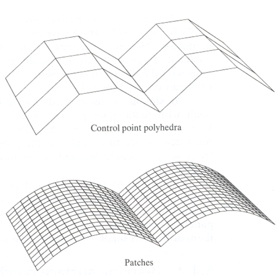
- Continuity constraints are similar to the contraints Bezier splines



FvDFH Figure 11.43

Bezier Surfaces

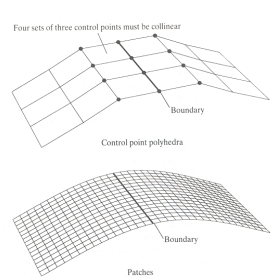
- C^0 continuity requires aligning boundary curves



Watt Figure 6.26a

Bezier Surfaces

- C^1 continuity requires aligning boundary curves <u>and</u> derivatives (a reason to prefer subdiv. surf.)

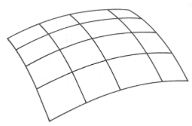


Watt Figure 6.26b

Drawing Bezier Surfaces

- Simple approach is to loop through uniformly spaced increments of u and v

```
DrawSurface(void)
{
    for (int i = 0; i < imax; i++) {
        float u = umin + i * ustep;
        for (int j = 0; j < jmax; j++) {
            float v = vmin + j * vstep;
            DrawQuadrilateral(...);
        }
    }
}
```

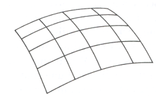


Watt Figure 6.32

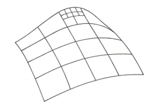
Drawing Bezier Surfaces

- Better approach is to use adaptive subdivision:

```
DrawSurface(surface)
{
    if Flat (surface, epsilon) {
        DrawQuadrilateral(surface);
    }
    else {
        SubdivideSurface(surface, ...);
        DrawSurface(surfaceLL);
        DrawSurface(surfaceLR);
        DrawSurface(surfaceRL);
        DrawSurface(surfaceRR);
    }
}
```



Uniform subdivision

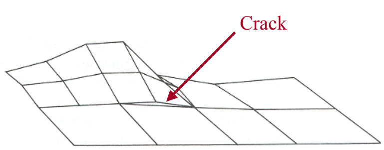


Adaptive subdivision

Watt Figure 6.32

Drawing Bezier Surfaces

- One problem with adaptive subdivision is avoiding cracks at boundaries between patches at different subdivision levels



Avoid these cracks by adding extra vertices and triangulating quadrilaterals whose neighbors are subdivided to a finer level.

Watt Figure 6.33

Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface
  - Possible to describe complex shapes
- Disadvantages:
  - Control mesh must be quadrilaterals
  - Continuity constraints difficult to maintain
  - Hard to find intersections

Blender (www.blender.nl)