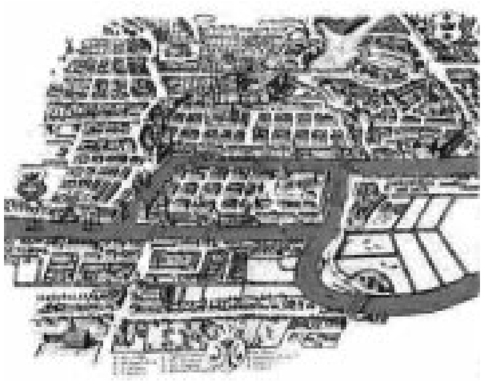
Undirected Graphs



Some of these lecture slides are adapted from material in:
• *Algorithms in C, 3rd Edition, Part 5,* R. Sedgwick.

Princeton University • COS 226 • Algorithms and Data Structures • Spring 2003 • http://www.Princeton.EDU/~cs226

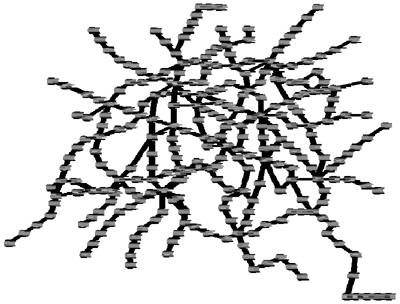
Undirected Graphs

GRAPH. Set of OBJECTS with pairwise CONNECTIONS.

- Interesting and broadly useful abstraction.

Why study graph algorithms?

- Challenging branch of computer science and discrete math.
- Hundreds of graph algorithms known.
- Thousands of practical applications.



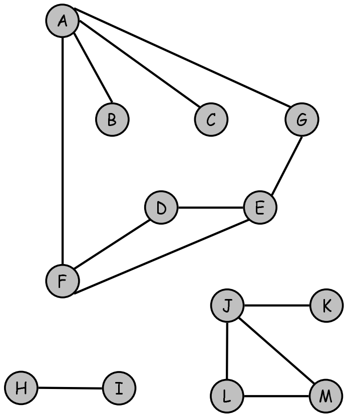
Graphs

Graph	Vertices	Edges
communication	telephones, computers	fiber optic cables
circuits	gates, registers, processors	wires
mechanical	joints	rods, beams, springs
hydraulic	reservoirs, pumping stations	pipelines
financial	stocks, currency	transactions
transportation	street intersections, airports	highways, airway routes
scheduling	tasks	precedence constraints
software systems	functions	function calls
internet	web pages	hyperlinks
games	board positions	legal moves
social relationship	people, actors	friendships, movie casts

Graph Jargon

Terminology.

- Vertex: v.
- Edge: e = v-w.
- Graph: G.
- V vertices, E edges.
- Parallel edge, self loop.
- Directed, undirected.
- Sparse, dense.
- Path, cycle.
- Cyclic path, tour.
- Tree, forest.
- Connected, connected component.



A Few Graph Problems

PATH. Is there a path between s to t?
SHORTEST PATH. What is the shortest path between two vertices?
LONGEST PATH. What is the longest path between two vertices?

CYCLE. Is there a cycle in the graph?
EULER TOUR. Is there a cyclic path that uses each edge exactly once?
HAMILTON TOUR. Is there a cycle that uses each vertex exactly once?

CONNECTIVITY. Is there a way to connect all of the vertices?
MST. What is the best way to connect all of the vertices?
BI-CONNECTIVITY. Is there a vertex whose removal disconnects graph?

PLANARITY. Can you draw the graph in the plane with no crossing edges?
ISOMORPHISM. Do two adjacency matrices represent the same graph?

Graph ADT in C

Typical client program.

- Call `GRAPHinit()` or `GRAPHrand()` to create instance.
- Uses `Graph` handle as argument to ADT functions.
- Calls `Graph` ADT function to do graph processing.

client.c

```
#include <stdio.h>
#include "graph.h"

int main(int argc, char *argv[]) {
   int V = atoi(argv[1]);
   int E = atoi(argv[2]);
   Graph G = GRAPHrand(V, E);
   GRAPHshow(G);
   printf("%d component(s)\n", GRAPHcc(G));
   return 0;
}
```

Graph ADT in C

Standard method to separate clients from implementation.

- Opaque pointer to `Graph` ADT.
- Plus simple `typedef` for `Edge`.

graph.h

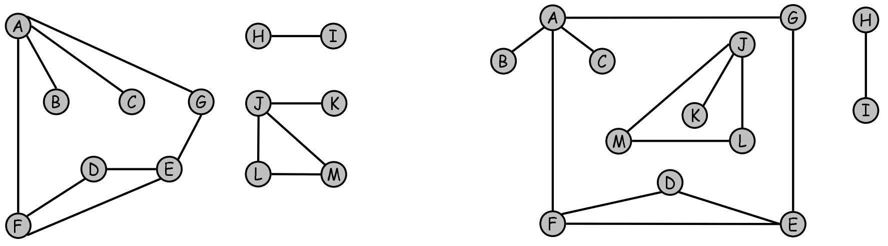
```
typedef struct graph *Graph;
typedef struct { int v, w; } Edge;
Edge EDGEinit(int v, int w);

Graph GRAPHinit(int V);
Graph GRAPHrand(int V, int E);
void  GRAPHdestroy(Graph G);
void  GRAPHshow(Graph G);
void  GRAPHinsertE(Graph G, Edge e);
void  GRAPHremoveE(Graph G, Edge e);
int   GRAPHcc(Graph G);
int   GRAPHisplanar(Graph G);
. . .
```

Graph Representation

Vertex names. (A B C D E F G H I J K L M)

- C program uses integers between `0` and `V-1`.
- Convert via associative indexing symbol table.

Two drawing represent same graph.



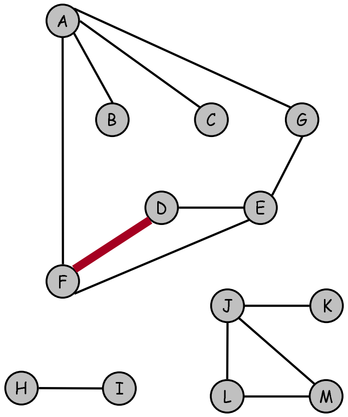
Set of edges representation.

- { A-B, A-G, A-C, L-M, J-M, J-L, J-K, E-D, F-D, H-I, F-E, A-F, G-E }.

Adjacency Matrix Representation

Adjacency matrix representation.

- Two-dimensional V × V array.
- Edge v-w in graph: `adj[v][w] = adj[w][v] = 1`.



		A	B	C	D	E	F	G	H	I	J	K	L	M
0	A	0	1	1	0	0	1	1	0	0	0	0	0	0
1	B	1	0	0	0	0	1	1	0	0	0	0	0	0
2	C	1	0	0	0	0	0	0	0	0	0	0	0	0
3	D	0	0	0	0	1	1	0	0	0	0	0	0	0
4	E	0	0	0	1	0	1	1	0	0	0	0	0	0
5	F	1	1	0	1	1	0	0	0	0	0	0	0	0
6	G	1	1	0	0	1	0	0	0	0	0	0	0	0
7	H	0	0	0	0	0	0	0	0	1	0	0	0	0
8	I	0	0	0	0	0	0	0	1	0	0	0	0	0
9	J	0	0	0	0	0	0	0	0	0	0	1	1	1
10	K	0	0	0	0	0	0	0	0	0	1	0	0	0
11	L	0	0	0	0	0	0	0	0	0	1	0	0	1
12	M	0	0	0	0	0	0	0	0	0	1	0	1	0

Adjacency Matrix

Graph ADT Implementation: Adjacency Matrix

graph.c

```c
#include "graph.h"

struct graph {
   int V;          // # vertices
   int E;          // # edges
   int **adj;      // V × V adjacency matrix
};

Graph GRAPHinit(int V) {
   Graph G = malloc(sizeof *G);
   G->V = V; G->E = 0;
   G->adj = MATRIXinit(V, V, 0);
   return G;
}

void GRAPHinsertE(Graph G, Edge e) {
   int v = e.v, w = e.w;
   if (G->adj[v][w] == 0) G->E++;
   G->adj[v][w] = G->adj[w][v] = 1;
}
```

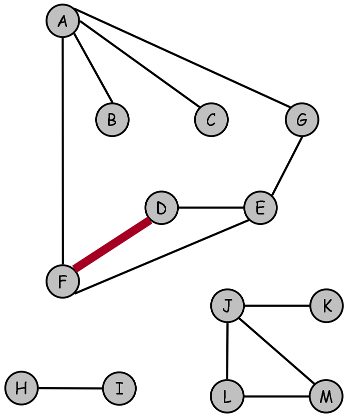
no parallel edges

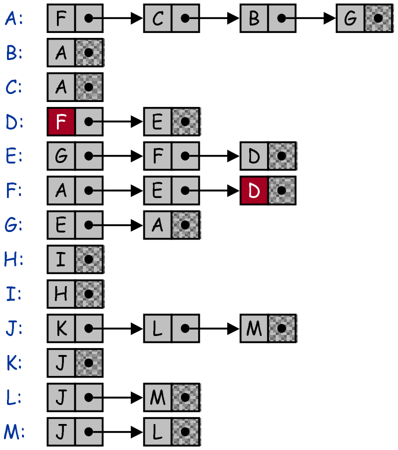
Adjacency List Representation

Vertex indexed array of lists.

- Space proportional to number of edges.
- Two representations of each undirected edge.

Adjacency List





Graph ADT Implementation: Adjacency List

graph.c

```c
#include "graph.h"

typedef struct node *link;
struct node {
   int w;       // current vertex in adjacency list
   link next;   // next node in adjacency list
};

struct graph {
   int V;       // # vertices
   int E;       // # edges
   link *adj;   // array of V adjacency lists
};

link NEWnode(int w, link next) {
   link x = malloc(sizeof *x);
   x->w = w;
   x->next = next;
   return x;
}
```

Adjacency List Graph ADT Implementation

graph.c

```
// initialize a new graph with V vertices
Graph GRAPHinit(int V) {
   int v;
   Graph G = malloc(sizeof *G);
   G->V = V;
   G->E = 0;
   G->adj = malloc(V * sizeof(link));
   for (v = 0; v < V; v++) G->adj[v] = NULL;
   return G;
}

// insert an edge e = v-w into Graph G
void GRAPHinsertE(Graph G, Edge e) {
   int v = e.v, w = e.w;
   G->adj[v] = NEWnode(w, G->adj[v]);
   G->adj[w] = NEWnode(v, G->adj[w]);
   G->E++;
}
```

Graph Representations

Graphs are abstract mathematical objects.

- ADT implementation requires specific representation.
- Efficiency depends on matching algorithms to representations.

Representation	Space	Edge between v and w?	Edge from v to anywhere?	Enumerate all edges
Adjacency matrix	$O(V^2)$	$O(1)$	$O(V)$	$O(V^2)$
Adjacency list	$O(E + V)$	$O(E)$	$O(1)$	$O(E + V)$

Most real-world graphs are sparse ⇒ adjacency list.

Graph Search

Goal. Visit every node and edge in Graph.

A solution. Depth-first search.

- To visit a node v:
  - mark it as visited
  - recursively visit all unmarked nodes w adjacent to v
- To traverse a Graph G:
  - initialize all nodes as unmarked
  - visit each unmarked node

Enables direct solution of simple graph problems.

- Connected components.
- Cycles.

Basis for solving more difficult graph problems.

- Biconnectivity.
- Planarity.

Depth First Search: Connected Components

graph.c (connected components)

```
#define UNMARKED -1

// traverse component of graph
int GRAPHcc(Graph G) {
    int v, id = 0;

    // initialize all nodes as unmarked
    for (v = 0; v < G->V; v++) G->cc[v] = UNMARKED;

    // visit each unmarked node
    for (v = 0; v < G->V; v++)
      if (G->cc[v] == UNMARKED) dfs(G, v, id++);
    return id;
}

// are v and w in the same connected component?
int GRAPHconnect(int v, int w) {
    return G->cc[v] == G->cc[w];
}
```

connected component of v

Depth First Search: Connected Components

Depth First Search: Adjacency Matrix

```
void dfs(Graph G, int v, int id) {
   int w;
   G->cc[v] = id;
   for (w = 0; w < G->V; w++)
      if (G->adj[v][w] == 1 && G->cc[w] == UNMARKED)
         dfs(G, w, id);
}
```

Depth First Search: Adjacency List

```
void dfs(Graph G, int v, int id) {
   link t;
   int w;
   G->cc[v] = id;

   // iterate over all nodes w adjacent to v
   for (t = G->adj[v]; t != NULL; t = t->next) {
      w = t->w;
      if (G->cc[w] == UNMARKED) dfs(G, w, id);
   }
}
```

← idiom

Connected Components

PATHS. Is there a path from s to t?

Method	Preprocess	Query	Space
Union Find	O(E log* V)	O(log* V)	O(V)
DFS	O(E + V)	O(1)	O(V)

UF advantage.

- Dynamic: can intermix query and edge insertion.

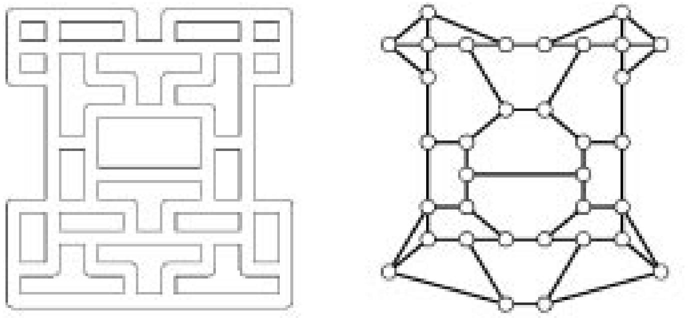
DFS advantage.

- Can get path itself in same running time.
  - maintain parent-link representation of tree
  - change DFS argument to pass EDGE taken to visit vertex
- Extends to more general problems.

Graphs and Mazes

Maze graphs.

- Vertices = intersections
- Edges = hallways.



DFS.

- Mark ENTRY and EXIT halls at each vertex.
- Leave by ENTRY when no unmarked halls.

Breadth First Search

Depth-first search.

- Visit all nodes and edges recursively.
- Put unvisited nodes on a STACK.

Breadth-first search.

- Put unvisited nodes on a QUEUE.

SHORTEST PATH. What is fewest number of edges to get from s to t?

Solution. BFS.

- Initialize `dist[v] = ∞, dist[s] = 0.`
- When considering edge `v-w`:
  - if `dist[w]` is marked, then ignore
  - if `w` not marked, set `dist[w] = dist[v] + 1`

Breadth First Search

Breadth First Search: Adjacency List

```
bfs(Graph G, int s) {
   link t;
   int v, w;
   for (v = 0; v < G->V; v++) G->dist[v] = INFINITY;

   G->dist[s] = 0;
   QUEUEput(s);
   while (!QUEUEempty()) {
      v = QUEUEget();
      for (t = G->adj[v]; t != NULL; t = t->next) {
         w = t->w;
         if (G->dist[w] == INFINITY) {
             G->dist[w] = G->dist[v] + 1;
             QUEUEput(w);
         }
      }
   }
}
```

Related Graph Search Problems

PATHS. Is there a path from s to t?
- Solution: DFS, BFS, any graph search.

SHORTEST PATH. Find shortest path (fewest edges) from s to t.
- Solution: BFS.

CYCLE. Is there a cycle in the graph?
- Solution: DFS. See textbook.

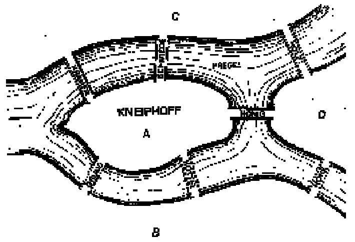
EULER TOUR. Is there a cyclic path that uses each edge exactly once?
- Yes if connected and degrees of all vertices are even.

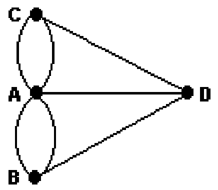
HAMILTON TOUR. Is there a cycle that uses each vertex exactly once?
- Solution: ??? (NP-complete)

Bridges of Königsberg

> "..... in Königsberg in Prussia, there is an island A, called the Kneiphof; the river which surrounds it is divided into two branches ... and these branches are crossed by seven bridges. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he could cross each bridge once and only once....."

- Leonhard Euler, *The Seven Bridges of Königsberg*, 1736.

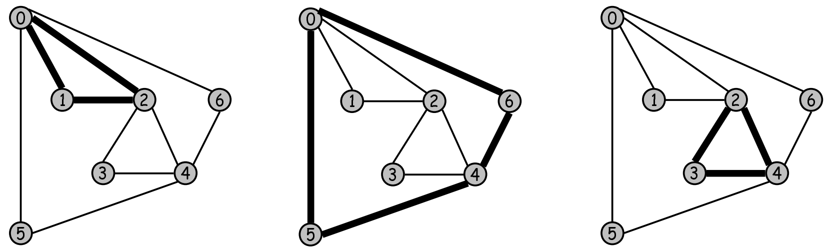




EULER TOUR. Is there a cyclic path that uses each edge exactly once?
- Yes if connected and degrees of all vertices are even.

Euler Tour

How to find an Euler tour (assuming graph is Eulerian).
- Start at some vertex v and repeatedly follow unused edge until you return to v.
  - always possible since all vertices have even degree
- Find additional cyclic paths using remaining edges and splice back into original cyclic path.



Euler Tour

How to find an Euler tour (assuming graph is Eulerian).

- Start at some vertex v and repeatedly follow unused edge until you return to v.
  - always possible since all vertices have even degree
- Find additional cyclic paths using remaining edges and splice back into original cyclic path.

How to efficiently keep track of unused edges?

- Delete edges after you use them.

How to efficiently find and splice additional cyclic paths?

- Push each visited vertex onto a stack.

Euler Tour: Implementation

Euler Tour

```
int euler(Graph G, int v) {
   link t;
   int w;
   for(t = G->adj[v]; t != NULL, v = w) {
      STACKpush(v);
      w = t->w;
      GRAPHremove(G, EDGE(v, w));
   }
   return v;
}
```

delete both copies of edge

```
void GRAPHshowEuler(Graph G, int v) {
   STACKinit(G->E);
   STACKpush(v);
   while ((euler(G, v) == v) && !STACKisempty())
      v = STACKpop();
      printf("%d ", v);
   }
   if (!STACKisempty()) printf("Not Eulerian.\n");
}
```

cyclic path back to initial vertex