COS 226 Lecture 22: Mincost Flow

MAXFLOW: assign flows to edges that
- equalize inflow and outflow at every vertex
- maximize total flow through the network

MINCOST MAXFLOW: find the BEST maxflow

Mincost maxflow is important for two primary reasons

it is a GENERAL PROBLEM-SOLVING MODEL
- solves (through reduction) numerous practical problems

it is TRACTABLE and PRACTICAL
- we know fast algorithms that solve mincost flow problems
- basic data structures play a critical role

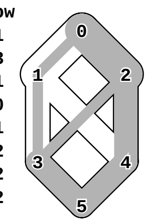
One step closer to a single ADT for combinatorial problems

Mincost flow

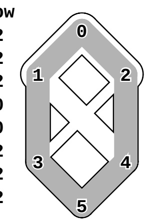
Add COST to each edge in a flow network
FLOW COST: sum of cost*flow over all edges

Maxflows have different costs

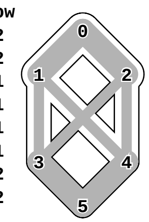
	cap	cost	flow	flow	flow
0-1	3	3	1	2	2
0-2	3	1	3	2	2
1-3	2	1	1	2	1
1-4	2	1	0	0	1
2-3	1	4	1	0	1
2-4	2	2	2	2	1
3-5	2	2	2	2	2
4-5	2	1	2	2	2



cost: 21



cost: 20



cost: 22

MINCOST FLOW: find a minimal-cost maxflow

Distribution problem

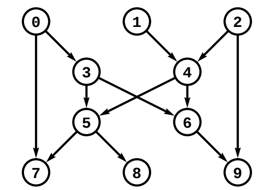
SUPPLY vertices (produce goods)
DEMAND vertices (consume goods)
DISTRIBUTION points (transfer goods)

Feasible flow problem
- Can we make supply to meet demand?

Distribution problem
- Add costs, find the lowest-cost way

Ex: Walmart
Ex: McDonald's

```
supply          channels
   0:  3           0-3:  2
   1:  4           0-7:  1
   2:  6           1-4:  5
distribution       2-4:  3
   3               2-9:  1
   4               3-5:  3
   5               3-6:  4
   6               4-5:  2
demand             4-6:  1
   7:  7           5-7:  6
   8:  3           5-8:  3
   9:  4           6-9:  4
```

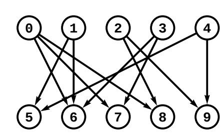


THM: Feasible flow reduces to maxflow
THM: Distribution reduces to mincost maxflow
Proof: Add source to provide supply, sink to take demand

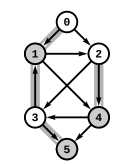
Transportation problem

No distribution points
- feasibility: is there a way?
- transportation: find best way

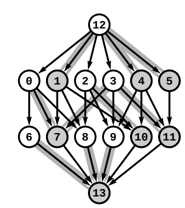
```
supply          channels
   0:  3           0-6:  2
   1:  4           0-7:  1
   2:  6           0-8:  5
   3:  3           1-6:  3
   4:  2           1-5:  1
demand             2-8:  3
   5:  6           2-9:  4
   6:  6           3-6:  2
   7:  7           3-7:  1
   8:  3           4-9:  6
   9:  4           4-5:  3
```




```
0-1  2
0-2  3
1-2  3
1-4  2
2-3  2
2-4  1
3-1  3
3-5  2
4-3  3
4-5  3
```




```
0-6  25   12-0  5    6-13  5
1-7  25   12-1  5    7-13  5
2-8  25   12-2  3    8-13  3
3-9  25   12-3  5    9-13  5
4-10 25   12-4  6   10-13  6
5-11 25   12-5  0   11-13  0
0-7  2
0-8  3
1-8  3
1-10 2
2-9  2
2-10 1
3-7  3
3-11 2
4-9  3
4-11 3
```



Seems easier, but that is not the case (!)
THM: Maxflow reduces to maxflow for acyclic networks
THM: Transportation reduces to mincost maxflow

Mincost flow reductions

SHORTEST PATHS
MAXFLOW
DISTRIBUTION and TRANSPORTATION

ASSIGNMENT
Minimal weight matching in weighted bipartite graph

MAIL CARRIER
Find a cyclic path that includes each edge AT LEAST once

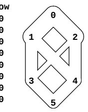
SCHEDULING (example)
Given a sport's league schedule, which teams are eliminated?

POINT MATCHING
Given two sets of N points, find minimal-distance pairing

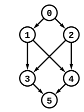
ALL of these problems reduce to mincost flow

Cycle canceling example

	cap	cost	flow
0-1	3	3	0
0-2	3	1	0
1-3	2	1	0
1-4	2	1	0
2-3	1	4	0
2-4	2	2	0
3-5	2	2	0
4-5	2	1	0



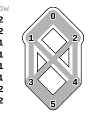
0-1	3
0-2	3
1-3	2
1-4	2
2-3	1
2-4	2
3-5	2
4-5	2



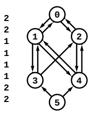
initial maxflow

	cap	cost	flow
0-1	3	3	2
0-2	3	1	2
1-3	2	1	1
1-4	2	1	1
2-3	1	4	1
2-4	2	2	1
3-5	2	2	2
4-5	2	1	2

total cost: 22



0-1	1	1-0	2
0-2	1	2-0	2
1-3	1	3-1	1
1-4	1	4-1	1
		3-2	1
2-4	1	4-2	1
		5-3	2
		5-4	2

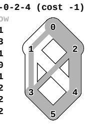


negative cycles: 4-1-0-2-4
3-2-0-1-3
3-2-4-1-3

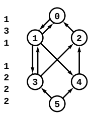
augment +1 on 4-1-0-2-4 (cost -1)

	cap	cost	flow
0-1	3	3	1
0-2	3	1	3
1-3	2	1	1
1-4	2	1	0
2-3	1	4	1
2-4	2	2	2
3-5	2	2	2
4-5	2	1	2

total cost: 21



0-1	2	1-0	1
		2-0	3
1-3	1	3-1	1
1-4	2		
		3-2	1
		4-2	2
		5-3	2
		5-4	2

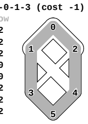


negative cycle: 3-2-0-1-3

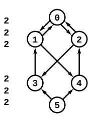
augment +1 on 3-2-0-1-3 (cost -1)

	cap	cost	flow
0-1	3	3	2
0-2	3	1	2
1-3	2	1	2
1-4	2	1	0
2-3	1	4	0
2-4	2	2	2
3-5	2	2	2
4-5	2	1	2

total cost: 20



0-1	1	1-0	2
0-2	1	2-0	2
		3-1	2
1-4	2		
2-3	1		
2-4	1	4-2	2
		5-3	2
		5-4	2



Cycle canceling

RESIDUAL NETWORK
for each edge in original network
- flow f in edge u-v with capacity c and cost x

define TWO edges in residual network
- FORWARD edge: capacity c-f and cost x in edge u-v
- BACKWARD edge: capacity f and cost -x in edge v-u

THM: A maxflow is mincost iff
there are NO negative-cost cycles in its residual network

GENERIC method for solving mincost flow problems:

start with ANY maxflow
REPEAT until no negative cycles are left
- increase the flow along ANY negative cycle

Implementation: use Bellman-Ford to find negative cycles

Cycle canceling implementation

```
void addflow(link u, int d)
  { u->flow += d; u->dup->flow -=d; }
int GRAPHmincost(Graph G, int s, int t)
  { int d, x, w; link u, st[maxV];
    GRAPHmaxflow(G, s, t);
    while ((x = GRAPHnegcycle(G, st)) != -1)
      {
        u = st[x]; d = Q;
        for (w=u->dup->v; w != x; w=u->dup->v)
        { u = st[w]; d = ( Q > d ? d : Q ); }
        u = st[x]; addflow(u, d);
        for (w=u->dup->v; w != x; w=u->dup->v)
        { u = st[w]; addflow(u, d); }
      }
    return GRAPHcost(G);
  }
```

Cycle canceling analysis

No need to compute initial maxflow

- use dummy edge from sink to source that carries maxflow

THM: Generic cycle canceling alg takes $O(VE^2CM)$ time

Proof:

- each edge has at most capacity C and cost M
- total cost could be ECM
- each augment reduces cost by at least 1
- Bellman-Ford takes O(VE) time

There exist $O(VE^2\log^2 V)$ cycle-canceling implementations

- mincost maxflow is therefore TRACTABLE

EXTREMELY pessimistic UPPER bounds

- not useful for predicting performance in practice
- algs that achieve such bounds would be useless
- algs are typically fast on practical problems

Network simplex algorithm

An implementation of the cycle-canceling algorithm

Identify negative cycles quickly by

- maintaining a tree data structure
- reweighting costs at vertices

Edge classification

- EMPTY
- FULL
- PARTIAL

FEASIBLE SPANNING TREE

- Any spanning tree that contains all the partial edges

VERTEX POTENTIALS

- a set of vertex weights (vertex-indexed array phi)

Network simplex concepts (continued)

REDUCED COST (reweighted edge cost)

- c*(u, v) = c(u, v) - (phi(u) - phi(v))

VALID vertex potentials for a spanning tree

- all tree edges have reduced cost 0

ELIGIBLE EDGE

- nontree edge that creates negative cycle with tree edges

THM: A nontree edge is eligible iff it is either

- a full edge with positive reduced cost, or
- an empty edge with negative reduced cost

Proof:

- cycle cost equals cycle reduced cost
- edge cost is negative of cycle reduced cost
  (since reduced costs of tree edges are all zero)

THEREFORE, it is easy to identify eligible edges

Network simplex algorithm

still a generic algorithm for the mincost flow problem

start with ANY feasible spanning tree
REPEAT until no eligible edges are left

- ensure that vertex potentials are valid
- add to the tree an eligible edge
- increase the flow along the negative cycle formed
- remove from the tree an edge that is filled or emptied

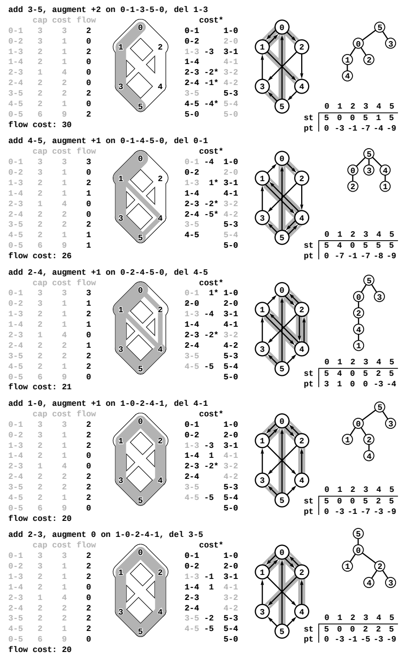
Problem: could have zero flow on cycle

THM: IF the algorithm terminates, it computes a maxflow

Implementation challenges

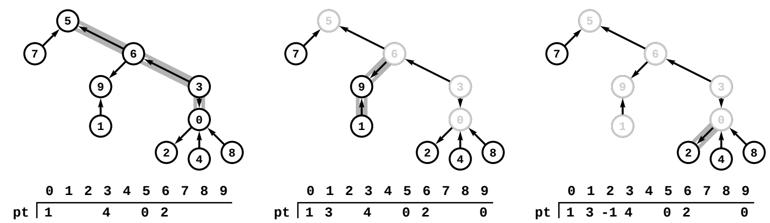
- cope with zero-flow cycles
- strategy to choose eligible edges
- data structure to represent tree

Network simplex example



Computing vertex potentials (example)

	cost
3-0	3
1-9	3
0-2	2
3-6	2
4-0	3
7-5	2
8-0	2
6-9	2
6-5	2



Feasible spanning tree data structure

Operations to support

- compute valid vertex potentials
- find cycle created by nontree edge
- replace tree edge by nontree edge

use PARENT-LINK representation!

to compute vertex potentials

- start with root at potential 0
- for each vertex

  follow parent links to vertex with known potential

  (recursively) set each vertex potential on path

  to make reduced edge costs 0

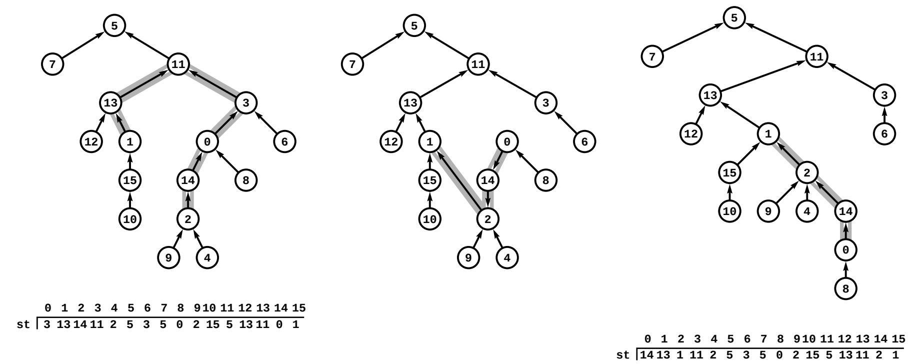
to follow cycle created by nontree edge u-v

- follow parent links from each to their LCA

to delete nontree edge that fills or empties

- REVERSE the parent links from u or v

Spanning tree update example



Network simplex basic implementation

```
#define R(u)  u->cost - phi[u->v]+phi[u->dup->v]
int GRAPHmincost(Graph G, int s, int t)
{ int v; link u, x, st[maxV];
   GRAPHinsertE(G, EDGE(t, s, M, 0, C));
   initialize(G, s, t, st);
   for (valid = 1; valid++; )
   {
      for (v = 0; v < G->V; v++)
         phi[v] = phiR(st, v);
      for (v = 0, x = G->adj[v]; v < G->V; v++)
         for (u = G->adj[v]; u!=NULL; u = u->next)
              if (R(u) < R(x)) x = u;
      if (R(x) == 0) break;
      update(st, augment(st, x), x);
   }
   return GRAPHcost(G);
}
```


Network simplex variations

OBJECTIVES

- guarantee terminimation
- reduce number of iterations
- reduce cost per iteration

Eligible edge selection strategies

- random
- find next
- queue of eligible edges

Lazy vertex potential calculation

Tree representations

- triply-linked, threaded

Guided by practical performance, not worst-case bounds

- DATA STRUCTURES are the key to good performance

Different implementations for different reductions??

BOTTOM LINE

- accessible code for powerful problem-solving model