COS 226 Lecture 15: Geometric algorithms

Important applications involve geometry

- models of physical world
- computer graphics
- mathematical models

Ancient mathematical foundations

Most geometric algorithms less than 25 years old

Knowledge of fundamental algorithms is critical

- use them directly
- use the same design strategies
- know how to compare and evaluate algs

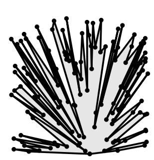
Warning: intuition may mislead

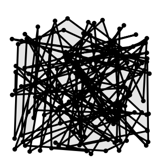
Humans have spatial intuition in 2D and 3D

- computers do not!
- neither have good intuition in high dimensions

Ex: Is a polygon convex?

we think of this   alg sees this   or even this

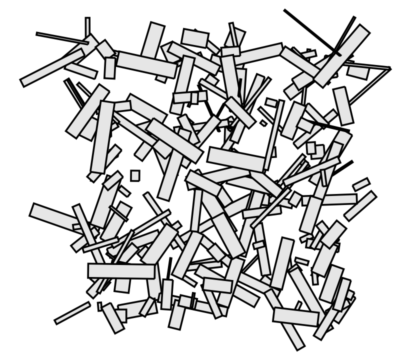




Warning: intuition may mislead (continued)

Ex: Find intersections among set of rectangles

- we think of this   algorithm sees this



Geometric algorithms: overview

New primitives

- points, lines, planes; polygons, circles

Primitive operations

- distance, angles
- "compare" point to line
- do two line segments intersect?

Problems extend to higher dimensions

- (algorithms sometimes do, sometimes don't)

Higher level intrinsic structures arise

Basic problems

- intersection
- proximity
- point location
- range search

Approaches to solving geometric problems

- incremental (brute-force)
- divide-and-conquer
- sweep-line algs
- multidimensional tree structures
- randomized algs
- discretized algorithms
- online and dynamic algs

Algorithm design paradigms

Draw from knowledge about fundamental algs

Move up one level of abstraction

- use fundamental algs and data structures
- know their performance characterisitics

More primitives lead to wider range of problems

Some problems too complex to admit simple algorithms

For many important problems

- classical approaches give good algorithms
- need research to find ``best'' algorithms
- no excuse for using ``dumb'' algorithms

Algorithm design paradigms (continued)

Progression of algorithm design (oversimplified)

all possibilities	double recursion	2^N
brute force	nested for loops	N^2
divide-and-conquer	recursion, trees	N log N
elegant idea	1 "for" loop	N
randomization	random choices	N

Many examples in geometric algorithms

Geometric primitives (2D)

POINT

two numbers (x, y)

LINE

two numbers a and b [ax + by = 1]

LINE SEGMENT

four numbers (x1, y1) (x2, y2)

POLYGON

sequence of points

No shortage of other geometric shapes

TRIANGLE

SQUARE

CIRCLE

- 3D and higher dimensions more complicated

Building algorithms from geometric primitives

First, need good implementations of primitives!

- is polygon simple?
- is point on line?
- is point inside polygon?
- do two line segments intersect?
- do two polygons intersect?

Algorithms search through SETS of primitives

- all points in specified range
- closest pair in set of points
- intersecting pairs in set of line segments
- overlapping areas in set of polygons

Line segment intersection

Do two line segments intersect?

To implement INTERSECT(l_1, l_2)

- use simpler primitive SAME(p_1, p_2, l):
  - Given two points p_1, p_2 and a line l,
  - are p_1 and p_2 on the same side of l?

To implement SAME

- use simpler primitive CCW(p_1, p_2, p_3):
  - Given three points p_1, p_2, p_3,
  - is the route p_1-p_2-p_3 a ccw turn?

two ccw tests to implement SAME

four ccw tests to implement INTERSECT

CCW implementation

compare slopes

- less:
- greater:
- equal: points are collinear

```
#typedef struct point POINT
int ccw(POINT p0, POINT p1, POINT p2)
  {
    int dx1, dx2, dy1, dy2;
    dx1 = p1.x - p0.x; dy1 = p1.y - p0.y;
    dx2 = p2.x - p0.x; dy2 = p2.y - p0.y;
    if (dx1*dy2 > dy1*dx2) return 1;
    if (dx1*dy2 < dy1*dx2) return -1;
    return 0;
  }
```


CCW implementation (continued)

Still not quite right! Bug in degenerate case

- four collinear points
- Does AB intersect CD?
  - on the line in the order ABCD: NO
  - on the line in the order ACDB: YES

Can't just return 0 if dx1*dy2 = dx2*dy1 (see book)

CCW is an important basic primitive

Ex: is point inside convex N-gon? N CCW tests

Lesson:

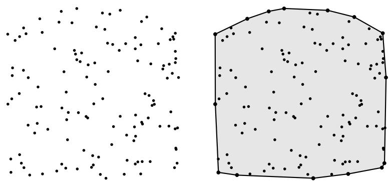
- geometric primitives are tricky to implement
- can't ignore degenerate cases

Convex hull of a point set

Basic property of a set of points

CONVEX HULL:

- smallest convex polygon enclosing the points
- shortest fence surrounding the points
- intersection of halfplanes defined by point pairs



Running time of algorithm can depend on

- N: number of points
- M: number of points on the hull
- point distribution

Package-wrap algorithm

Operates like selection sort

Abstract idea

- sweep line anchored at current point CCW
- first point hit is on hull

Implementation

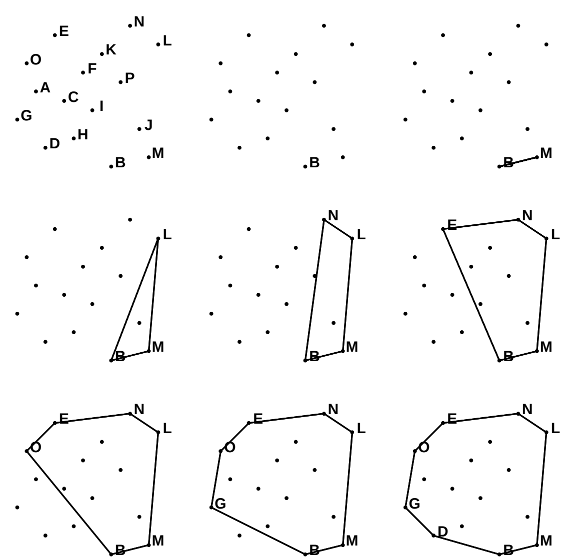
- compute angle to all points
- pick smallest angle larger than current one

Package-wrap implementation

```
int wrap(POINT p[], int N)
  { int i, min, M; float th, v; struct point t;
    for (min = 0, i = 1; i < N; i++)
      if (p[i].y < p[min].y) min = i;
    p[N] = p[min]; th = 0.0;
    for (M = 0; M < N; M++)
    {
      t = p[M]; p[M] = p[min]; p[min] = t;
      min = N; v = th; th = 360.0;
      for (i = M+1; i <= N; i++)
        if (theta(p[M], p[i]) > v)
          if (theta(p[M], p[i]) < th)
            { min = i; th = theta(p[M], p[min]);}
      if (min == N) return M;
    }
  }
```

Use pseudo-angle theta to save time (see text)

Package-wrap example



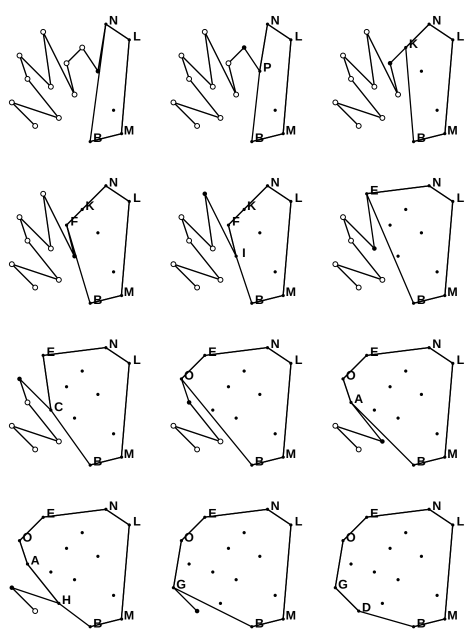
Graham Scan

Sort points on angle with bottom point as origin
- forms simple closed polygon

Proceed through polygon
- discard points that would cause a CW turn

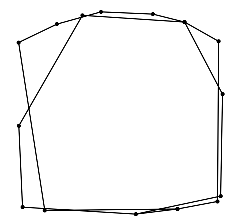
```
int grahamscan(struct point p[], int N)
  { int i, min, M; struct point t;
    for (min = 1, i = 2; i <= N; i++)
      if (p[i].y < p[min].y) min = i;
    for (i = 1; i <= N; i++)
      if (p[i].y == p[min].y)
        if (p[i].x > p[min].x) min = i;
    t = p[1]; p[1] = p[min]; p[min] = t;
    quicksort(p, 1, N);
    p[0] = p[N];
    for (M = 3, i = 4; i <= N; i++)
      {
        while (ccw(p[M],p[M-1],p[i]) >= 0) M--;
        M++; t = p[M]; p[M] = p[i]; p[i] = t;
      }
    return M;
  }
```


Graham scan example

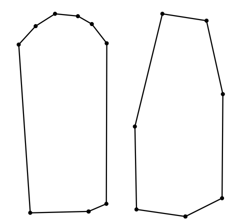


Divide-and-conquer convex hull algorithms

divide points

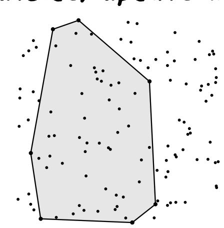


divide space



Incremental convex hull algorithm

Consider next point
- if inside hull of previous points, ignore
- if outside, update hull



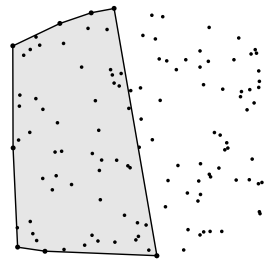
Two subproblems to solve
- test if point inside or outside polygon
- update hull for outside points

Both subproblems
- can be solved by looking at all hull points
- can be improved with binary search

Randomized algorithm
- consider points in random order
- N + M log M

"Sweep line" convex hull algorithm

Sort points on x-coordinate first



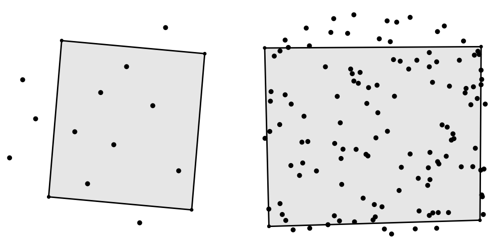
Eliminates "inside" test

Total time proportional to N log N (for sort)

Quick elimination

Improve the performance of any convex hull
- algorithm by quickly eliminating most
- points (known not to be on the hull)

Use points at "corners": max, min x+y, x-y



Check if point inside quadrilateral: four CCW tests
Check if point inside rectangle: four comparisons

Almost all points eliminated if points random
- number of points left proportional to N^(1/2)

LINEAR algorithm

Summary of 2D convex hull algs

Package wrap
- NM

Graham scan
- N log N (sort time)

Divide-and-conquer
- N log N (with work)

Quick elimination
- N (fast average-case)

One-by-one elimination
- N log M

Sweep line
- N log N (sort time)

How many points on the hull?
Worst case: N
Average case: depends on distribution
- uniform in a convex polygon: log N
- uniform in a circle: N^(1/3)

requires understanding of basic properties of DATA

Higher dimensions

Multifaceted (convex) polytope encloses points

NOT a simple object

Ex: N points d dimensions
- d=2: convex hull
- d=3: Euler's formula (v - e + f = 2)
- d>3: exponential number of facets at worst

EXTREME POINTS
- return points on the hull, not necc in order

Package-wrap
Divide-and-conquer
Randomized
Interior elimination

Geometric models of mathematical problems

Impact of geometric algs extends far beyond physical models

Geometric problem

- find point where two lines intersect in 2D
- find point where three planes intersect in 3D

Mathematical equivalent

- solve simultaneous equations
- algorithm: gaussian elimination

Geometric problem

- find convex polytope defined by intersecting half-planes
- find vertex hit by line of given slope moving in from infinity

Mathematical equivalent

- LINEAR PROGRAMMING
- algorithm: SIMPLEX (stay tuned)

Linear programming example

Maximize a+b subject to the constraints

- $b - a < 5$
- $a + 4b < 45$
- $2a + b < 27$
- $3a - 4b < 24$
- $a > 0$
- $b > 0$