COS 226 Lecture 21: Network Flow

Classical problem-solving model (1940s)

OPERATIONS RESEARCH

Modern implementations benefit from
- Graph algorithm technology
- PQ and data structure design

Researchers still seek efficient algorithms
- many variations
- many practical applications

Optimal solutions still not known

Network flow

NETWORK: weighted digraph

Abstraction for material FLOWING through the edges
- interpret edge weights as CAPACITIES

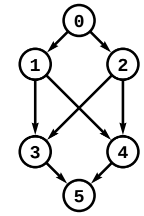
Ex: oil flowing in pipes
Ex: commodities flowing on roads and rails
Ex: bits flowing in Internet

SOURCE: node where all material originates
SINK: node where all material goes

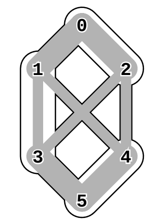
MAXFLOW PROBLEM: assign flows to edges that
- equalize inflow and outflow at every vertex
- maximize total flow through the network

Flow network example

	cap
0-1	2
0-2	3
1-3	3
1-4	1
2-3	1
2-4	1
3-5	2
4-5	3



	cap	flow
0-1	2	2
0-2	3	2
1-3	3	1
1-4	1	1
2-3	1	1
2-4	1	1
3-5	2	2
4-5	3	2



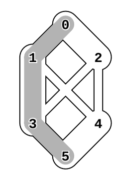
Increasing flow in a network

AUGMENTING PATH: source-sink path for increasing flow

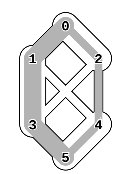
Easy case:
- ADD flow to each edge on the path

Ex: 0-1-3-5, then 0-2-4-5

	cap	flow
0-1	2	2*
0-2	3	0
1-3	3	2
1-4	1	0
2-3	1	0
2-4	1	0
3-5	2	2
4-5	3	0

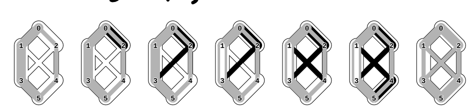


	cap	flow
0-1	2	2*
0-2	3	1
1-3	3	2
1-4	1	0
2-3	1	0
2-4	1	1*
3-5	2	2
4-5	3	1

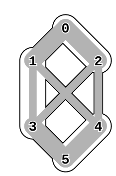


More complicated case:
- REMOVE flow from one or more edges

Ex: 0-2-3-1-4-5



	cap	flow
0-1	2	2
0-2	3	2
1-3	3	2
1-4	1	1*
2-3	1	1*
2-4	1	1*
3-5	2	2
4-5	3	2*



Ford-Fulkerson algorithm

GENERIC method for solving maxflow problems

start with 0 flow everywhere
REPEAT until no augmenting paths are left
- increase the flow along ANY augmenting path

Problem 0:
- Does this process lead to the maximum flow?

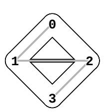
Problem 1: fill in unspecified details
- How do we find an augmenting path?

Problem 2:
- Cost can be proportional to max capacity

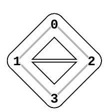
Bad case for generic FF

BAD NEWS
- number of augmenting paths could be huge
- proportional to max edge capacity!

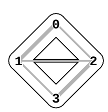
	cap	flow
0-1	X	1
0-2	X	0
1-2	1	1*
1-3	X	0
2-3	X	1



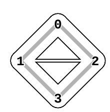
	flow
0-1	1
0-2	1
1-2	0
1-3	1
2-3	1



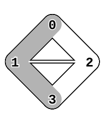
	flow
0-1	2
0-2	1
1-2	1*
1-3	1
2-3	2



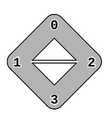
	flow
0-1	2
0-2	2
1-2	0
1-3	2
2-3	2



	cap	flow
0-1	X	X*
0-2	X	0
1-2	1	0
1-3	X	X*
2-3	X	0



	flow
0-1	X*
0-2	X*
1-2	0
1-3	X*
2-3	X*



GOOD NEWS
- always possible to avoid this case

Maxflow-mincut theorem

CUT: set of edges separating source from sink

THM: maxflow is equivalent to mincut
Proof: [see text]

THM: Ford-Fulkerson method gives maximum flow
Proof sketch:
- if there is no augmenting path,
  identify the first full forward
  or empty backward edge on every path
- that set of edges defines a min cut

AUGMENTING-PATH ALG: specific method for finding a path

Design goals:
- find paths quickly
- use as few iterations as possible

Edmonds-Karp algorithms

Idea 1: use BFS to find augmenting path
Idea 2: find path that increases the flow

BOTH easy to implement with standard PFS (!)

RESIDUAL NETWORK
for each edge in original network
- flow x in edge u-v with capacity c

define TWO edges in residual network
- FORWARD edge: flow c-x in edge u-v
- BACKWARD edge: flow -x in edge v-u

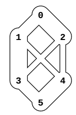
easy implicit implementation:

```
#define Q (u->cap < 0 ? -u->flow : u->cap - u->flow)
```

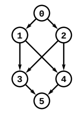
Graph search in residual network finds augmenting path

Residual networks

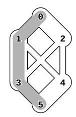
	cap	flow
0-1	2	0
0-2	3	0
1-3	3	0
1-4	1	0
2-3	1	0
2-4	1	0
3-5	2	0
4-5	3	0



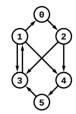
0-1	2		
0-2	3		
1-3	3		
1-4	1		
2-3	1		
2-4	1		
3-5	2		
4-5	3		



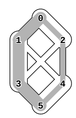
	cap	flow
0-1	2	2
0-2	3	0
1-3	3	2
1-4	1	0
2-3	1	0
2-4	1	0
3-5	2	2
4-5	3	0



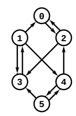
		1-0	2
0-2	3		
1-3	1	3-1	2
1-4	1		
2-3	1		
2-4	1		
		5-3	2
4-5	3		



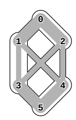
	cap	flow
0-1	2	2
0-2	3	1
1-3	3	2
1-4	1	0
2-3	1	0
2-4	1	1
3-5	2	2
4-5	3	1



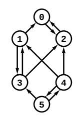
		1-0	2
0-2	2	2-0	1
1-3	1	3-1	2
1-4	1		
2-3	1		
		4-2	1
		5-3	2
4-5	2	5-4	1



	cap	flow
0-1	2	2
0-2	3	2
1-3	3	1
1-4	1	1
2-3	1	1
2-4	1	1
3-5	2	2
4-5	3	2



		1-0	2
0-2	2	2-0	1
1-3	2	3-1	1
		4-1	1
		3-2	1
		4-2	1
		5-3	2
4-5	1	5-4	2



Network flow implementation

Tricky code for sparse graphs

- TWO edge representations with links to each other
- st array has links to edge representations

```
void GRAPHmaxflow(Graph G, int s, int t)
{ int x, d;
   link st[maxV];
   while ((d = GRAPHpfs(G, s, t, st)) != 0)
     for (x = t; x != s; x = st[x]->dup->v)
       { st[x]->flow += d; st[x]->dup->flow -= d; }
}
```

To make GRAPHsearch find shortest aug path

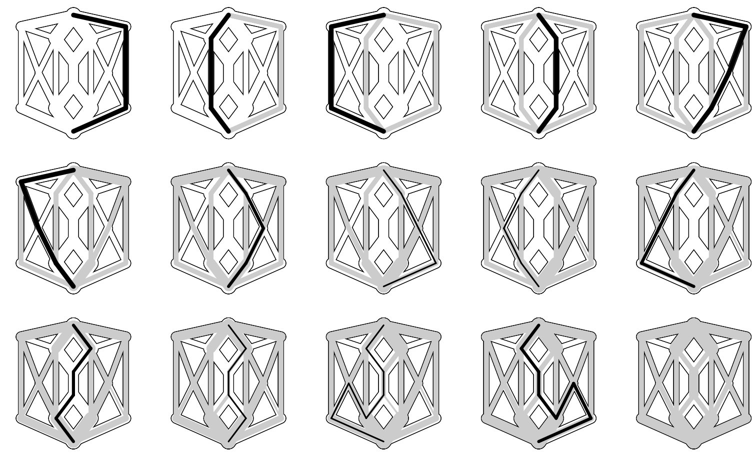
```
#define P G->V - cnt
```

To make GRAPHsearch find max capacity aug path

```
#define P ( Q > wt[v] ? wt[v] : Q )
```

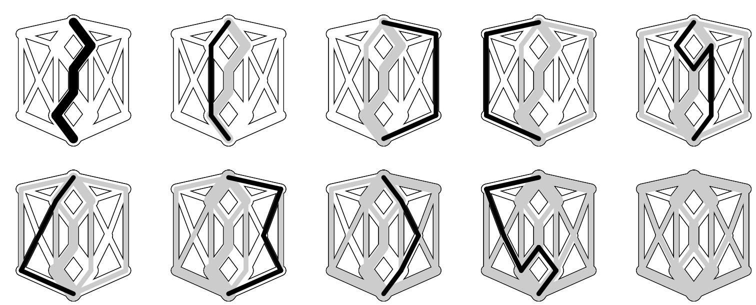

Shortest augmenting paths example

Path lengths increase



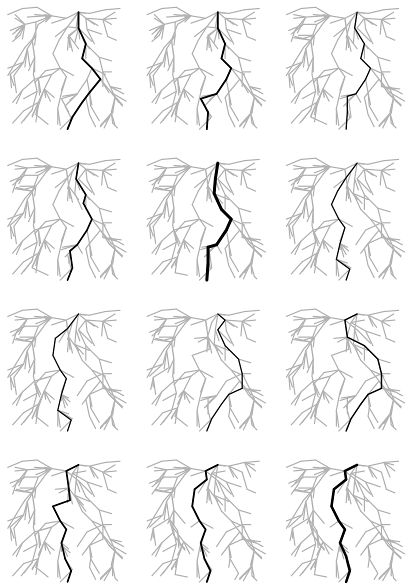
Max capacity augmenting paths example

Path capacities decrease

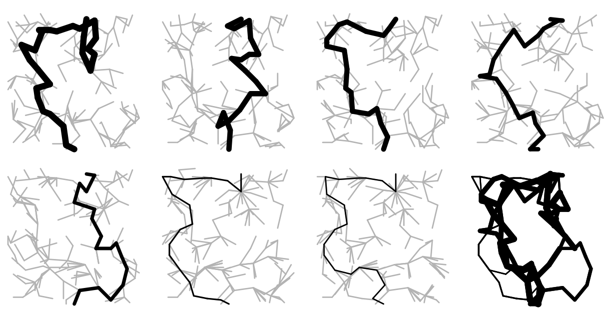


Fewer iterations, lower cost per iteration

Shortest augmenting paths (larger example)



Max capacity augmenting paths (larger example)



Analysis of network flow algorithms

THM: ANY FF alg takes O(VEM) time

Proof:

- mincut capacity less than VM
- aug path increases flow through cut by at least 1
- graph search takes O(E) time

THM: Shortest aug-path alg takes O(VE^2) time

Proof:

- aug paths increase in length
- at most E paths for each of V lengths
- total of at most VE aug paths
- graph search takes O(E) time

THM: Max-capacity aug-path alg takes
O(E^2 lg V lg M) time

Proof: [see text]

Network-flow algorithms

best known worst-case running times

- 1970 V^2 E
- 1977 V^2 E^(1/2)
- 1978 V^3
- 1978 V^(5/3) E^(2/3)
- 1980 V E log V
- 1986 V E log(V^2/E)

generally NOT relevant in practice

- most improvements are for dense graphs (rare in practice)
- worst-case bounds are overly pessimistic
- simple (but not dumb) algorithms may be preferred in practic

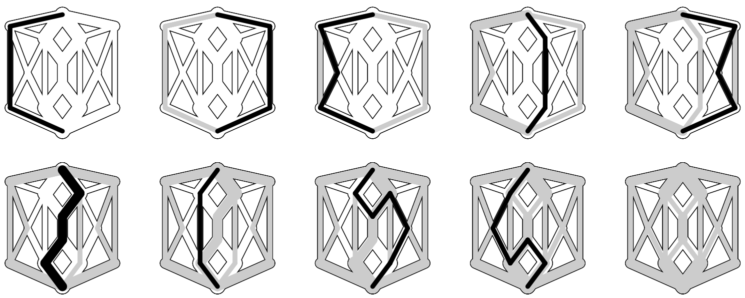
SPARSE GRAPHS

- shortest: O(V^3)
- max capacity: O(V^2 lg V lg M)

BUT research is justified:

- simple O(E) algorithm could still exist!

Random augmenting paths example



Matching

MATCHING: set of edges with no vertex included twice

MAXIMUM MATCHING: no matching contains more edges

BIPARTITE GRAPH

- two sets of vertices
- all edges connect vertex in one set to vertex in the other

BIPARTITE MATCHING: maximum matching in bipartite graph

What does matching have to do with maxflow??

- bipartite matching REDUCES to maxflow
- we can use maxflow to solve it!

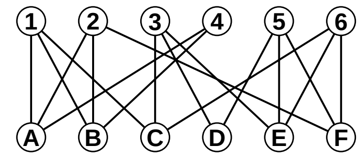
Bipartite matching example

Job Placement

- companies make job offers
- students have job choices

BIPARTITE MATCHING

- can we fill every job?
- can we employ every student?



- **Alice**
  - **Adobe**
  - **Apple**
  - **HP**
- **Bob**
  - **Adobe**
  - **Apple**
  - **Yahoo**
- **Carol**
  - **HP**
  - **IBM**
  - **Sun**
- **Dave**
  - **Adobe**
  - **Apple**
- **Eliza**
  - **IBM**
  - **Sun**
  - **Yahoo**
- **Frank**
  - **HP**
  - **Sun**
  - **Yahoo**

- Adobe
  - Alice
  - Bob
  - Dave
- Apple
  - Alice
  - Bob
  - Dave
- HP
  - Alice
  - Carol
  - Frank
- IBM
  - Carol
  - Eliza
- Sun
  - Carol
  - Eliza
  - Frank
- Yahoo
  - Bob
  - Eliza
  - Frank

Equivalent: Find maximal subset with no dups in

- 1A 1B 1C 2A 2B 2E 3C 3D 3E 4A 4B 5D 5E 5F 6C 6E 6F

Bipartite matching reduction to maxflow

Standard reduction (see lecture 20)

- given an instance of bipartite matching
- transform it to a maxflow problem
- solve the maxflow problem
- transform maxflow solution to bipartite matching solution

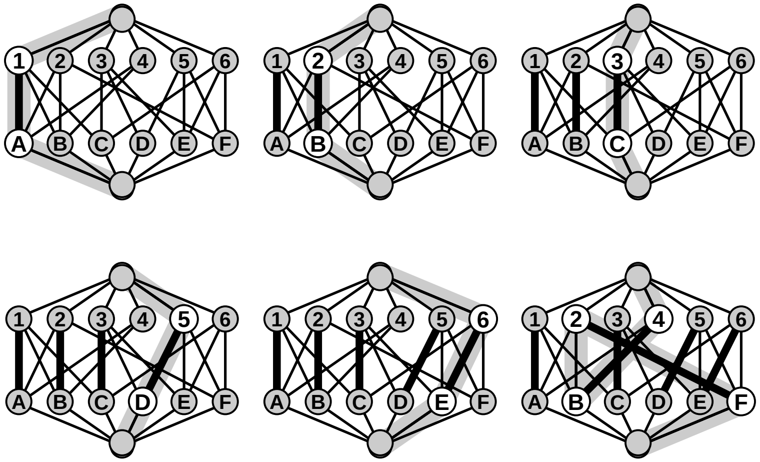
Transformation:

- keep all edges and vertices
- add SOURCE connected to all vertices in one set
- add SINK connected to all nodes of second type
- set all capacities to 1

full edges in maxflow solution give matching solution

NOTE: maxflow easier in unit-capacity networks

Bipartite matching reduction example



SOLUTION: 1-A 2-F 3-C 4-B 5-D 6-E

Alice-Adobe Bob-Yahoo Carol-HP Dave-Apple Eliza-IBM Frank-Sun

Maxflow problem-solving model

Many practical problems reduce to maxflow problems

- merchandise distribution
- matching
- scheduling
- communications networks

Maxflow algorithms provide effective solutions

NEXT STEP: add OPTIMIZATION

- multiple maxflows, in general
- which one is best??

MINCOST FLOW

- generalizes maxflow and shortest paths
- large number of practical applications
- challenge to develop efficient alg/implementation

[stay tuned]