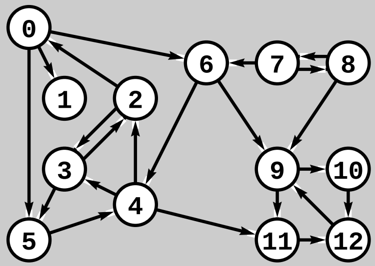
COS 226 Lecture 18: Digraphs and DAGs

DIGRAPH: directed graph

- edge s-t from s to t
- edge t-s from t to s

4-2	11-12
4-11	5-4
2-3	12-9
4-3	0-5
3-2	9-10
3-5	6-4
0-6	9-11
7-8	6-9
0-1	8-9
8-7	7-6
2-0	10-12



Can you get there from here?

Basic definitions

CONNECTIVITY

- path from s to t in undirected graph

REACHABILITY

- directed path from s to t in digraph

STRONG CONNECTIVITY

- directed paths from s to t AND from t to s

Connectivity ADT implementation (last lecture)

- query: O(1)
- preprocessing: O(E)
- space: O(V)

Can we do as well for reachability and strong connectivity?

DFS in a digraph (adjacency lists)

```
void dfsR(Graph G, Edge e, int pre[], int post[])
  { link t; int i, v, w = e.w; Edge x;
    pre[w] = cnt0++;
    for (t = G->adj[w]; t != NULL; t = t->next)
      if (pre[t->v] == -1)
        dfsR(G, EDGE(w, t->v), pre, post);
    post[w] = cnt1++;
  }
void GRAPHsearch(Graph G, int pre[], int post[])
  { int v;
    cnt0 = 0; cnt1 = 0; depth = 0;
    for (v = 0; v < G->V; v++)
      { pre[v] = -1; post[v] = -1; }
    for (v = 0; v < G->V; v++)
      if (pre[v] == -1)
        search(G, EDGE(v, v), pre, post);
  }
```

Need both PREORDER and POSTORDER numbering

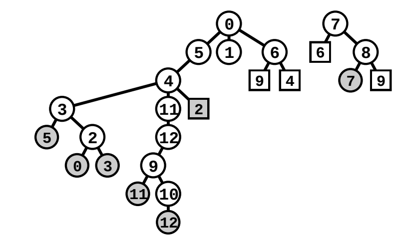
DFS forests

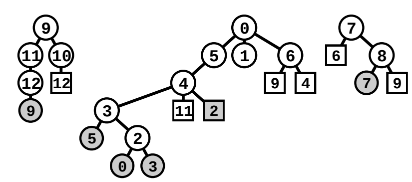
Structure determined by digraph AND search dynamics

- use pre- and post- numbering to distinguish edge types

Edge types

- TREE
- BACK
- DOWN
- CROSS





ONLY the FIRST tree

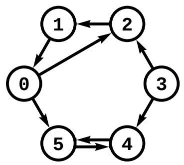
has the set of nodes reachable from its root

Transitive closure

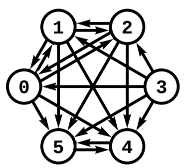
Digraph G

Transitive closure G* has edge from s to t

iff there is a directed path from s to t in G



	0	1	2	3	4	5
0	1	0	1	0	0	1
1	1	1	0	0	0	0
2	0	1	1	0	0	0
3	0	0	1	1	1	0
4	0	0	0	0	1	1
5	0	0	0	0	1	1



	0	1	2	3	4	5
0	1	1	1	0	1	1
1	1	1	1	0	1	1
2	1	1	1	0	1	1
3	1	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1

NOT symmetric

supports O(1) reachability queries with O(V^2) space

Boolean matrices and paths in graphs

Adjacency matrix A

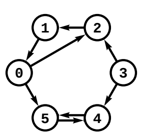
- A[s][t] is 1 iff path from s to t

Square A*A = A^2

- A[s][t] is 1 iff path from s to t of length 2 in A [s-k-t]

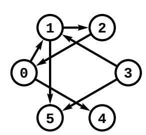
Reflexive square A + A^2

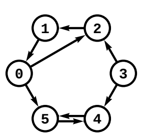
- A[s][t] is 1 iff path from s to t of length < 2 in A



	0	1	2	3	4	5
0	0	0	1	0	0	1
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	1	0
4	0	0	0	0	0	1
5	0	0	0	0	1	0

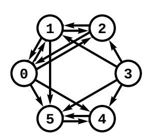
	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	1
2	1	0	0	0	0	0
3	0	1	0	0	0	1
4	0	0	0	0	1	0
5	0	0	0	0	0	1





	0	1	2	3	4	5
0	1	0	1	0	1	1
1	1	1	0	0	0	0
2	0	1	1	0	0	0
3	0	0	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1

	0	1	2	3	4	5
0	1	1	1	0	1	1
1	1	1	1	0	0	1
2	1	1	1	0	0	0
3	0	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1



Transitive closure: A + A^2 + A^3 + A^4 + A^5 + ...

- same as lg V reflexive squares
- [A + (A + A^2)^2 = A + A^2 + A^3 + A^4]

leads to easy V^3 lg V transitive closure algorithm

Warshall's algorithm

Method of choice for transitive closure of a dense graph

- running time proportional to V^3

```
for (k = 0; k < G->V; k++)
  for (s = 0; s < G->V; s++)
    if (G->tc[s][k] == 1)
      for (t = 0; t < G->V; t++)
        if (G->tc[k][t] == 1) G->tc[s][t] = 1;
```

Proof of correctness (induction on k)

- there is a path from s to t (with no nodes > k) if

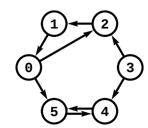
  EITHER

  there is path from s to k (with no nodes > k-1)

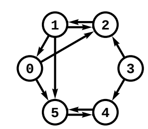
  AND a path from k to t (with no nodes > k-1)

  OR there is a path from s to t (with no nodes > k-1)

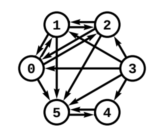
Warshall's algorithm (example)



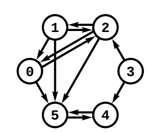
	0	1	2	3	4	5
0	1	0	1	0	0	1
1	1	1	0	0	0	0
2	0	1	1	0	0	0
3	0	0	1	1	1	0
4	0	0	0	0	1	1
5	0	0	0	0	1	1



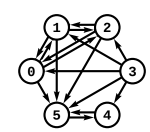
	0	1	2	3	4	5
0	1	0	1	0	0	1
1	1	1	1	0	0	1
2	0	1	1	0	0	0
3	0	0	1	1	1	0
4	0	0	0	0	1	1
5	0	0	0	0	1	1



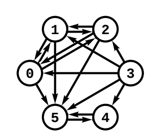
	0	1	2	3	4	5
0	1	1	1	0	0	1
1	1	1	1	0	0	1
2	1	1	1	0	0	1
3	1	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1



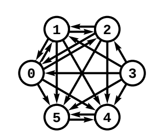
	0	1	2	3	4	5
0	1	0	1	0	0	1
1	1	1	1	0	0	1
2	1	1	1	0	0	1
3	0	0	1	1	1	0
4	0	0	0	0	1	1
5	0	0	0	0	1	1



	0	1	2	3	4	5
0	1	1	1	0	0	1
1	1	1	1	0	0	1
2	1	1	1	0	0	1
3	1	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1



	0	1	2	3	4	5
0	1	1	1	0	0	1
1	1	1	1	0	0	1
2	1	1	1	0	0	1
3	1	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1



	0	1	2	3	4	5
0	1	1	1	0	1	1
1	1	1	1	0	1	1
2	1	1	1	0	1	1
3	1	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1

Transitive closure lower bound

Consider Boolean (0-1) matrices

Premise: Matrix multiplication is not easy
- grade-school algorithm: V^3
- best known: V^c, c>2 [practical?]

THM: Transitive closure is no easier than matrix multiplication

Proof:
- Given a matrix multiplication problem
- can solve it with a TC algorithm

```
I  A  0        I  A  AB
0  I  B   =    0  I  B
0  0  I        0  0  I
```

O(V^2) TC would yield O(V^2) matrix multiply (not likely)

DFS-based transitive closure

Package DFS to implement reachability ADT
- run new DFS for each vertex

```
void TCdfsR(Graph G, int v, int w)
  { link t;
    G->tc[v][w] = 1;
    for (t = G->adj[w]; t != NULL; t = t->next)
      if (G->tc[v][t->v] == 0)
        TCdfsR(G, v, t->v);
  }
void GRAPHtc(Graph G, Edge e)
  { int v, w;
    G->tc = malloc2d(G->V, G->V);
    for (v = 0; v < G->V; v++)
      for (w = 0; w < G->V; w++)
        G->tc[v][w] = 0;
    for (v = 0; v < G->V; v++) TCdfsR(G, v, v);
  }
int GRAPHreach(Graph G, int s, int t)
  { return G->tc[s][t]; }
```

Running time? less than VE (V^2 for sparse graphs)
Violates lower bound? NO (worst case still V^3)

Abstract transitive closure

ADT function for reachability in digraphs

THM: DFS-based transtive closure provides
- VE preprocessing time
- V^2 space
- constant query time

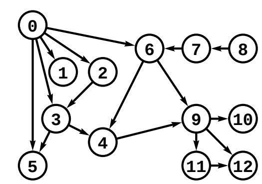
GOAL:
- V^2 (or VE) preprocessing time
- V space
- constant query time

V^2 preprocessing guarantee not likely by TC lower bound

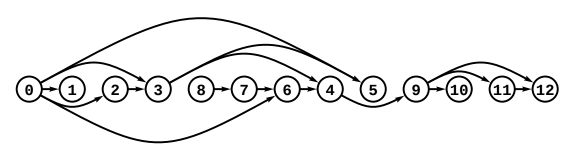
Next attempt:
- is the problem easier if there are no cycles (DAG)??

Topological sort (DAG)

DAG: directed acyclic graph

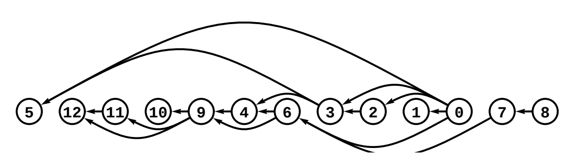


Topological sort: all edges point left to right



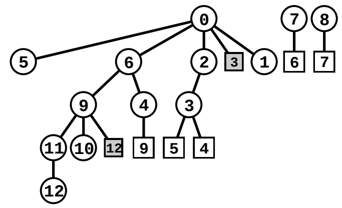
	0	1	2	3	4	5	6	7	8	9	10	11	12
ts	0	1	2	3	8	7	6	4	5	9	10	11	12
tsI	0	1	2	3	7	8	6	5	4	9	10	11	12

Reverse TS: all edges point right to left



DFS topological sort

Easy alg for reverse TS: DFS!
(postorder visit is reverse TS)



	0	1	2	3	4	5	6	7	8	9	10	11	12
post	5	12	11	10	9	4	6	3	2	1	0	7	8

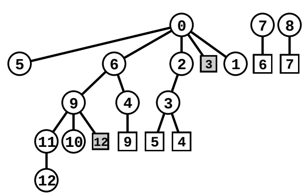
```
void TSdfsR(Graph G, int v, int ts[])
  { int w;
    pre[v] = 0;
    for (w = 0; w < G->V; w++)
      if (G->adj[w][v] != 0)
        if (pre[w] == -1) TSdfsR(G, w, ts);
    ts[cnt0++] = v;
  }
```

Quick hack for arrays:

- switch rows and cols to process reverse

DAG Transitive closure

Compute TC row vectors (in postorder) during reverse TS



5:	0	0	0	0	0	1	0	0	0	0	0	0	0
12:	0	0	0	0	0	0	0	0	0	0	0	0	1
11:	0	0	0	0	0	0	0	0	0	0	0	1	1
10:	0	0	0	0	0	0	0	0	0	0	1	0	0
9:	0	0	0	0	0	0	0	0	0	1	1	1	1
4:	0	0	0	0	1	0	0	0	0	1	1	1	1
6:	0	0	0	0	1	0	1	0	0	1	1	1	1
3:	0	0	0	1	1	1	0	0	0	1	1	1	1
2:	0	0	1	1	1	1	0	0	0	1	1	1	1
1:	0	1	0	0	0	0	0	0	0	0	0	0	0
0:	1	1	1	1	1	1	1	0	0	1	1	1	1
7:	0	0	0	0	1	0	1	1	0	1	1	1	1
8:	0	0	0	0	1	0	1	1	1	1	1	1	1

Good news: can skip down edges
Bad news: there may not be any down edges

DAG transitive closure (code)

```
void TCdfsR(Dag D, int w, int v)
  { int u, i;
    pre[v] = cnt0++;
    for (u = 0; u < D->V; u++)
      if (D->adj[v][u])
        {
          D->tc[v][u] = 1;
          if (pre[u] > pre[v]) continue;
          if (pre[u] == -1) TCdfsR(D, v, u);
          for (i = 0; i < D->V; i++)
            if (D->tc[u][i] == 1)
              D->tc[v][i] = 1;
        }
  }
```

worst-case cost bound: VE (no help!)
actual cost is V(V+ no. of down edges)
V^2 algorithm? lower bound?

Progress report on reachability ADT

Classical TC algs (Warshall) give

- query: O(1)
- preprocessing: O(V^3)
- space: O(V^2)

Reducing preprocessing to O(VE) is easy DFS application

NO PROGRESS on reducing space to O(V)

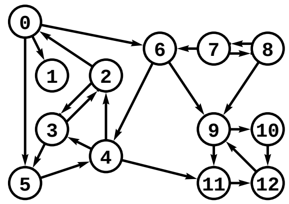
NO PROGRESS on better guarantees EVEN FOR DAGS (!!)

Next attempt:

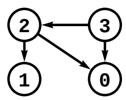
- Is the STRONG reachability problem easier??

Strong components

STRONG COMPONENTS: mutually reachable vertices



	0	1	2	3	4	5	6	7	8	9	10	11	12
sc	2	1	2	2	2	2	2	3	3	0	0	0	0

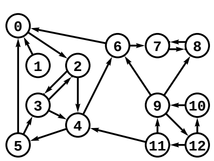


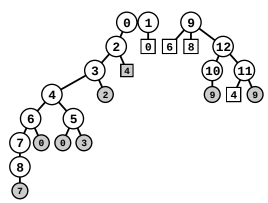
KERNEL DAG

- reachability among strong components
- collapse each strong component to a single vertex

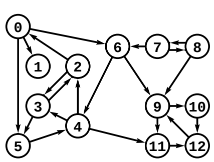
Kosaraju's SC algorithm

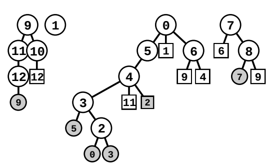
- Run DFS on reverse digraph
- Run DFS on digraph, using reverse postorder from first DFS
  to seek unvisited vertices at top level





	0	1	2	3	4	5	6	7	8	9	10	11	12
post	8	7	6	5	4	3	2	0	1	11	10	12	9





	0	1	2	3	4	5	6	7	8	9	10	11	12
G->sc	2	1	2	2	2	2	2	3	3	0	0	0	0

THM:

- Trees in (second) DFS forest are strong components (!)

Kosaraju's algorithm implementation

Add vertex-indexed array sc to graph representation

Use standard recursive DFS, with postorder numbering

```
void SCdfsR(Graph G, int w)
  { link t;
    G->sc[w] = cnt1;
    for (t = G->adj[w]; t != NULL; t = t->next)
      if (G->sc[t->v] == -1) SCdfsR(G, t->v);
    post[cnt0++] = w;
  }
```

ADT function for constant-time strong reach queries

```
int GRAPHstrongreach(Graph G, int s, int t)
  { return G->sc[s] == G->sc[t]; }
```


Kosaraju's algorithm implementation (continued)

```
int GRAPHsc(Graph G)
  { int i, v; Graph R;
    R = GRAPHreverse(G);
    cnt0 = 0; cnt1 = 0;
    for (v = 0; v < G->V; v++) R->sc[v] = -1;
    for (v = 0; v < G->V; v++)
      if (R->sc[v] == -1) SCdfsR(R, v);
    cnt0 = 0; cnt1 = 0;
    for (v = 0; v < G->V; v++) G->sc[v] = -1;
    for (v = 0; v < G->V; v++) postR[v] = post[v];
    for (i = G->V-1; i >=0; i--)
      if (G->sc[postR[i]] == -1)
        { SCdfsR(G, postR[i]); cnt1++; }
    return cnt1;
  }
```

LINEAR time to find strong components (!!)

Fast abstract transitive closure

1. Find strong components and build kernel DAG
2. Compute TC of kernel DAG
3. Reachability query:

    IF in same strong component, YES

    ELSE check reachability in kernel DAG

Running time depends on graph structure

- density (fast if sparse)
- size of kernel DAG (fast if small)
- cross edges in kernel DAG (fast if few)

Meets performance goals for many graphs

Huge sparse DAG? STILL OPEN

Fast transitive closure implementation

Testimony to benefits of careful ADT design

```
Dag K;
void GRAPHtc(Graph G)
  { int v, w; link t; int *sc = G->sc;
    K = DAGinit(GRAPHsc(G));
    for (v = 0; v < G->V; v++)
      for (t = G->adj[v]; t != NULL; t = t->next)
        DAGinsertE(K, dagEDGE(sc[v], sc[t->v]));
    DAGtc(K);
  }
int GRAPHreach(Graph G, int s, int t)
  { return DAGreach(K, G->sc[s], G->sc[t]); }
```