Subdivision Surfaces

COS598b Geometric Modeling

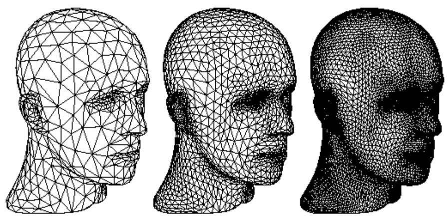
Basic Idea

- Subdivision defines smooth curve or surface as the limit of a sequence of successive refinements.

Examples

- 1D

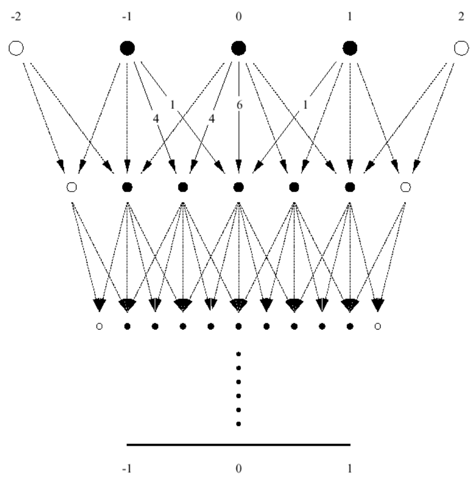
- 2D



Rules

- Efficiency
- Compact Support
- Local Definition
- Affine Invariance
- Simplicity
- Continuity

Cubic Spline Subdivision



Cubic Subdivision Matrix

$$\begin{pmatrix} p_{-2}^{j+1} \\ p_{-1}^{j+1} \\ p_{0}^{j+1} \\ p_{1}^{j+1} \\ p_{2}^{j+1} \end{pmatrix} = 1/8 \begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix} \begin{pmatrix} p_{-2}^{j} \\ p_{-1}^{j} \\ p_{0}^{j} \\ p_{1}^{j} \\ p_{2}^{j} \end{pmatrix}$$

Eigen Analysis

- Cubic Splines
  - Eigen Values

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$$

  - Complete set of eigenvectors

$$(\boldsymbol{x_0}, \boldsymbol{x_0}, \boldsymbol{x_0}, \boldsymbol{x_0}, \boldsymbol{x_0}) = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 0 & \frac{-1}{11} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Eigen Analysis

- **1** is an eigenvector of S with $\lambda_0 = 1$
- Invariance under translation:

$$S(\boldsymbol{p}^j + \boldsymbol{1} * a) = S\boldsymbol{p}^j + S(\boldsymbol{1} * a) = \boldsymbol{p}^{j+1} + S(\boldsymbol{1} * a)$$
$$S(\boldsymbol{1} * a) = \boldsymbol{1} * a$$

Eigen Analysis

$$\boldsymbol{p} = \sum_{i=0}^{n-1} a_i \boldsymbol{x}_i$$

- Applying subdivision matrix S:

$$\mathrm{S}\boldsymbol{p}^0 = \sum_{i=0}^{n-1} a_i \lambda_i \boldsymbol{x}_i$$

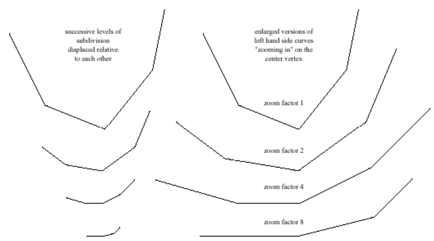
- After j applications:

$$\boldsymbol{p}^j = \mathrm{S}^j \boldsymbol{p}^0 = \sum_{i=0}^{n-1} a_i \lambda_i^j \boldsymbol{x}_i$$

- Greatest eigenvalue can have value 1.

Eigen Analysis

- Repeatedly applying the subdivision matrix to a set of n control points results in the control points converging to a configuration aligned with a tangent vector.



Eigen Analysis

- Show that only one eigenvalue = 1.
- Assume two eigenvalues = 1.
- Limit of the subdivision process is the tangent:

$$\lim_{j\to\infty} S^j p^0 = \lim_{j\to\infty} \sum_{i=0}^{n-1} a_i \lambda_i^j x_i = a_0 x_0 + a_1 x_1$$

- Tangent is not a straight line => only one eigenvalue =1.

Eigen Analysis

Similarly we show that $\lambda_1 > \lambda_i \qquad \forall i > 1$

Make a_0 the origin then we have

$$p^j = \sum_{i=1}^{n-1} a_i \lambda_i^j x_i$$

$$\frac{1}{\lambda_1^j} p^j = a_1 x_1 + \sum_{i=1}^{n-1} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^j x_i$$

If $\lambda_1 = \lambda_2$ then

$$p^j = \sum_{i=1}^{n-1} a_i \lambda_i^j x_i$$

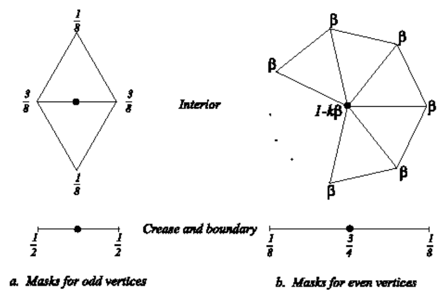
$$\frac{1}{\lambda_1^j} p^j = a_1 x_1 + a_2 x_2 + \sum_{i=2}^{n-1} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^j x_i$$

Summary

- The eigenvectors should form a basis
- **1** is an eigenvector of S with $\lambda_0 = 1$
- The first eigenvalue λ0 = 1
- The second eigenvalue λ1 < 1
- All other eigenvalues should be less than λ1

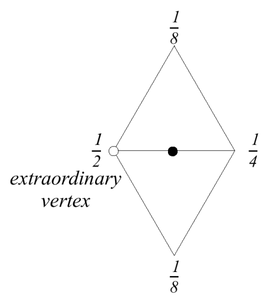
Loop Scheme

- Ordinary vertex subdivision



Loop Scheme

- Extraordinary vertex
  - interior vertex: valence other than 6
  - boundary vertex: valence other than 4



Overview of Subdivision Schemes

- Variational
  - Subdivision rules change based on global energy minimization function
- Stationary

Vertex Insertion		
	Triangular Meshes	Quadrilateral Meshes
Approximating	Loop	Catmull-Clark
Interpolating	Modified Butterfly	Kobbelt

Corner-cutting
Doo-Sabin Midedge

Variational Subdivision

- Multigrid methods:
  - Find $\boldsymbol{p}_i$ such that

$$E_i \boldsymbol{p}_i = \boldsymbol{b}_i$$

$$\boldsymbol{p}_i = S_{i-1} \boldsymbol{p}_{i-1}$$

$$\textit{Set} \quad E_i \boldsymbol{p}_i = U_{i-1} E_{i-1} \boldsymbol{p}_{i-1}$$

$$\textit{where} \quad U_{i-1} = \begin{pmatrix} & 1 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & \\ \cdots\cdot & 0 & 1 & 0 & 0 & \cdots\cdot \\ & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$E_i S_{i-1} = U_{i-1} E_{i-1}$$

Variational Subdivision

- Minimize bending energy functional

$$\mathbf{E}[p_i] = 8^i * \sum_{j=1}^{2^i n-1} \left((p_i)_{j-1} - 2(p_i)_j + (p_i)_{j+1}\right)^2.$$

$$\mathbf{E}[p_i] = {p_i}^T E_i p_i$$

Variational Subdivision

$$8\begin{pmatrix} 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \end{pmatrix} S_{i-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 1 & -4 & 6 \end{pmatrix}$$

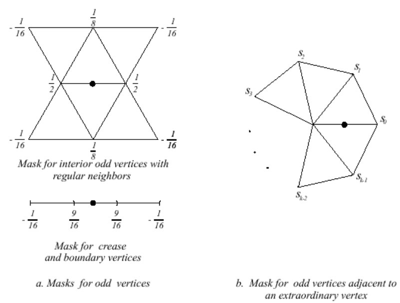
$$S_{i-1} = \frac{1}{8}\begin{pmatrix} 6 & 1 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$

Loop Scheme

- C2 continuous on regular meshes
- C1 continuous on extraordinary vertices

Modified Butterfly Scheme

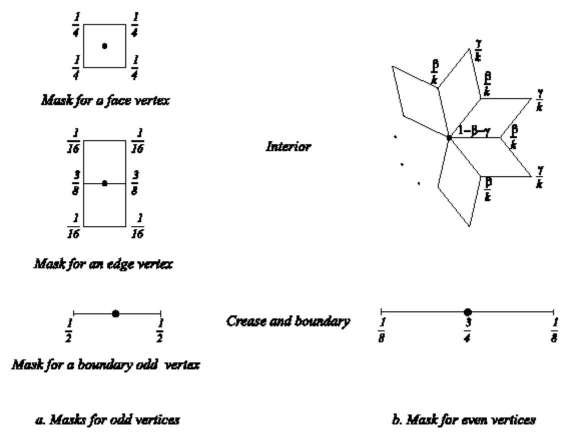
- Interpolating
- C1 continuous
- Not C1 on extraordinary vertices k=3, k>7



Catmull-Clark Scheme

- Quadrilateral Meshes
- Approximating
- C2 continuous on regular vertices
- C1 continuous on extraordinary vertices

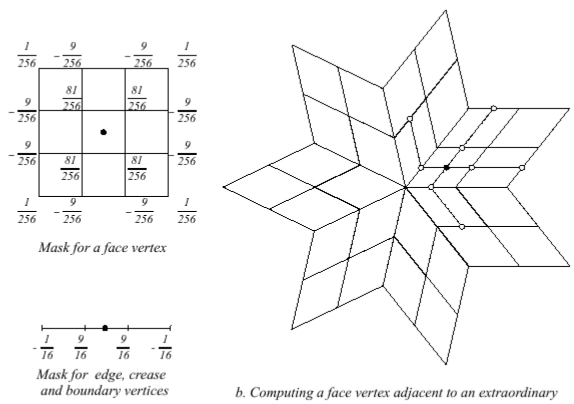
Catumull-Clark Scheme (cont)



Kobbelt Scheme

- Quadrilateral, approximating
- C1 continuous
- Two step subdivision

Kobbelt Scheme (cont)



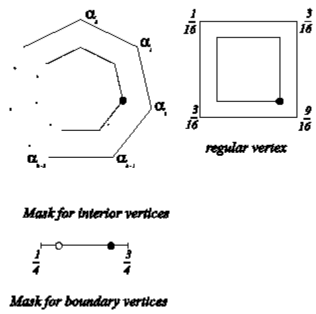
a. Regular masks

b. Computing a face vertex adjacent to an extraordinary vertex

Doo-Sabin and Midedge Schemes

- Single mask for the scheme
- C1 continuous
- Disadvantage: no vertex correspondence between meshes

Doo-Sabin and Midedge (cont)



Limitation of Stationary Subdivision

- Problems with curvature continuity
- Decrease of smoothness with valence
- Ripples
- Uneven mesh structure

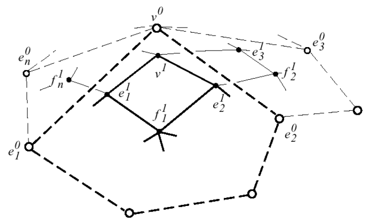
Subdivision Surfaces in Geri's Game

- Catmull-Clark
  - Easy to use in existing systems
  - Quads capture symmetries of natural and man-made objects
- Modified to allow sharp edges
  - Parametrize subdivision weights
  - Hybrid subdivision

Hybrid Subdivision

- Infinitely sharp rules applied ~s times
  - s an integer
  - Linear interpolation between $\lfloor s \rfloor$ and $\lceil s \rceil$ subdivided surfaces
- Followed by smooth rules

Smooth Rules



$$e_j^{i+1} = \frac{v^i + e_j^i + f_{j-1}^{i+1} + f_j^{i+1}}{4}$$

$$v^{i+1} = \frac{n-2}{n} v^i + \frac{1}{n^2} \sum_j e_j^i + \frac{1}{n^2} \sum_j f_j^{i+1}$$

Infinitely Sharp Creases

- Sharp edge
- Vertex
  - 1 sharp edge, smooth rule

    $$e_j^{i+1} = \frac{v^i + e_j^i}{2}$$

  - 2 sharp edges, crease

    $$v^{i+1} = \frac{e_j^i + 6v^i + e_k^i}{8}$$

  - >3 sharp edges, corner

    $$v^{i+1} = v^i$$

Cloth Simulation

- Spring-mass energy functional
- Use quad grid for warp and weft directions of fabric

Energy Functional

- Minimize warp/weft stretch

$$E_s(p_1, p_2) = \frac{1}{2}\left(\frac{|p_1 - p_2|}{|p_1^* - p_2^*|} - 1\right)^2$$

- Minimize skew

$$E_d(p_1, p_2, p_3, p_4) = E_s(p_1, p_2)E_s(p_3, p_4)$$

- Minimize bending along virtual threads

$$E_p(p_1, p_2, p_3) = \frac{1}{2}[C(p_1, p_2, p_3) - C(p_1^*, p_2^*, p_3^*)]^2$$

$$C(p_1, p_2, p_3) = \left|\frac{p_3 - p_2}{p_3^* - p_2^*} - \frac{p_2 - p_1}{p_3^* - p_2^*}\right|$$

Collision

- N^2 too slow
- Use subdivision hierarchy
- Unsubdivide the mesh
  – Mark all non-boundary level l edges for merging
  – Merge faces f1, f2 into f*
  – Remove all the edges of f* until all level l edges have been merged

Building the Hierarchy

- Preprocessing step
- When a vertex moves bounding boxes are updated bottom up
    - Each leaf points to it’s vertices
- Test each vertex against object hierarchy

Texture Mapping

- Assign smoothly varying texture coordinates (s,t) to all vertices of the original mesh
- Apply subdivision rules to (x, y, z, s, t)

As applicable to mesh recognition

- Hmm….
- Inherent simplification algorithm