6.4 Maximum Flow

- introduction
- Ford–Fulkerson algorithm
- maxflow–mincut theorem
- analysis of running time
- Java implementation
- applications
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**Mincut problem**

**Input.** An edge-weighted digraph, source vertex $s$, and target vertex $t$.

Each edge has a positive capacity.
Min-cut problem

**Def.** A \textit{st-cut (cut)} is a partition of the vertices into two disjoint sets, with \( s \) in one set \( A \) and \( t \) in the other set \( B \).

**Def.** Its \textit{capacity} is the sum of the capacities of the edges from \( A \) to \( B \).

capacity = 10 + 5 + 15 = 30
Min-cut problem

**Def.** A *st-cut (cut)* is a partition of the vertices into two disjoint sets, with $s$ in one set $A$ and $t$ in the other set $B$.

**Def.** Its **capacity** is the sum of the capacities of the edges from $A$ to $B$.

capacity $= 10 + 8 + 16 = 34$
Mincut problem

**Def.** A \textit{st-cut (cut)} is a partition of the vertices into two disjoint sets, with \( s \) in one set \( A \) and \( t \) in the other set \( B \).

**Def.** Its \textit{capacity} is the sum of the capacities of the edges from \( A \) to \( B \).

**Minimum st-cut (mincut) problem.** Find a cut of minimum capacity.

capacity = 10 + 8 + 10 = \( 28 \)
What is the capacity of the *st*-cut \{ \text{A, E, F, G} \}?

A. \(34 \ (8 + 11 + 9 + 6)\)

B. \(45 \ (20 + 25)\)

C. \(78 \ (20 + 8 + 11 + 9 + 6 + 24)\)

D. *I don't know.*
Min-cut application (RAND 1950s)

"Free world" goal. Cut supplies (if cold war turns into real war).

rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)
Potential mincut application (2010s)

Government-in-power’s goal. Cut off communication to set of people.
Though maximum flow algorithms have a long history, revolutionary progress is still being made.

BY ANDREW V. GOLDBERG AND ROBERT E. TARJAN

Efficient Maximum Flow Algorithms

Efficient Maximum Flow Algorithms by Andrew Goldberg and Bob Tarjan
http://vimeo.com/100774435
Input. An edge-weighted digraph, source vertex $s$, and target vertex $t$. Each edge has a positive capacity.
Maxflow problem

Def. An *st-flow (flow)* is an assignment of values to the edges such that:

- Capacity constraint: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except $s$ and $t$).
Maxflow problem

**Def.** An *st*-flow (flow) is an assignment of values to the edges such that:
- Capacity constraint: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except $s$ and $t$).

**Def.** The value of a flow is the inflow at $t$.

we assume no edges point to $s$ or from $t$

value = $5 + 10 + 10 = 25$
Maxflow problem

**Def.** An *st-flow (flow)* is an assignment of values to the edges such that:
- Capacity constraint: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except $s$ and $t$).

**Def.** The *value* of a flow is the inflow at $t$.

**Maximum st-flow (maxflow) problem.** Find a flow of maximum value.

![Graph with labels and capacities](image)

value $= 8 + 10 + 10 = \boxed{28}$
Maxflow application (Tolstoj 1930s)

Soviet Union goal. Maximize flow of supplies to Eastern Europe.

rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)
"Free world" goal. Maximize flow of information to specified set of people.
Summary

Input. A weighted digraph, source vertex $s$, and target vertex $t$.

Min-cut problem. Find a cut of minimum capacity.

Maxflow problem. Find a flow of maximum value.

Remarkable fact. These two problems are dual!
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Ford–Fulkerson algorithm

**Initialization.** Start with 0 flow.
Idea: increase flow along augmenting paths

**Augmenting path.** Find an undirected path from \( s \) to \( t \) such that:
- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

**1\textsuperscript{st} augmenting path**

![Graph showing the first augmenting path with bottleneck capacity and flow calculations](image)
Idea: increase flow along augmenting paths

**Augmenting path.** Find an undirected path from \( s \) to \( t \) such that:
- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

**2nd augmenting path**
Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from \( s \) to \( t \) such that:
- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

3\textsuperscript{rd} augmenting path
Idea: increase flow along augmenting paths

**Augmenting path.** Find an undirected path from $s$ to $t$ such that:
- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

4th augmenting path
Idea: increase flow along augmenting paths

Termination. All paths from $s$ to $t$ are blocked by either a
- Full forward edge.
- Empty backward edge.

no more augmenting paths
Which is the augmenting path of highest bottleneck capacity?

A. $A \rightarrow F \rightarrow G \rightarrow H$

B. $A \rightarrow F \rightarrow B \rightarrow G \rightarrow H$

C. $A \rightarrow F \rightarrow B \rightarrow G \rightarrow D \rightarrow H$

D. *I don't know.*
**Ford–Fulkerson algorithm**

<table>
<thead>
<tr>
<th>Start with 0 flow.</th>
</tr>
</thead>
<tbody>
<tr>
<td>While there exists an augmenting path:</td>
</tr>
<tr>
<td>- find an augmenting path</td>
</tr>
<tr>
<td>- compute bottleneck capacity</td>
</tr>
<tr>
<td>- increase flow on that path by bottleneck capacity</td>
</tr>
</tbody>
</table>

**Fundamental questions.**

- How to compute a mincut?
- How to find an augmenting path?
- If FF terminates, does it always compute a maxflow?
- Does FF always terminate? If so, after how many augmentations?
6.4 Maximum Flow

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**Relationship between flows and cuts**

**Def.** The net flow across a cut \((A, B)\) is the sum of the flows on its edges from \(A\) to \(B\) minus the sum of the flows on its edges from \(B\) to \(A\).

![Graph with labels showing flow values](image)

**net flow across cut** = \(5 + 10 + 10 = 25\)

**value of flow** = \(25\)
Def. The net flow across a cut \((A, B)\) is the sum of the flows on its edges from \(A\) to \(B\) minus the sum of the flows on its edges from \(B\) to \(A\).

\[
\text{net flow across cut} = 10 + 5 + 10 = 25
\]
Relationship between flows and cuts

**Def.** The **net flow across** a cut \((A, B)\) is the sum of the flows on its edges from \(A\) to \(B\) minus the sum of the flows on its edges from \(B\) to \(A\).

\[
\text{net flow across cut } = (10 + 10 + 5 + 10 + 0 + 0) - (5 + 5 + 0 + 0) = 25
\]

![Graph showing the relationship between flows and cuts](image-url)
Maxflow: quiz 3

Which is the net flow across the st-cut \( \{ A, E, F, G \} \)?

A. 11 \((20 + 25 - 8 - 11 - 9 - 6)\)

B. 26 \((20 + 22 - 8 - 4 - 4)\)

C. 42 \((20 + 22)\)

D. 45 \((20 + 25)\)

E. I don't know.
Relationship between flows and cuts

Flow-value lemma. Let $f$ be any flow and let $(A, B)$ be any cut. Then, the net flow across $(A, B)$ equals the value of $f$.

Intuition. Conservation of flow.

Pf. By induction on the size of $B$.
- Base case: $B = \{ t \}$.
- Induction step: remains true by local equilibrium when moving any vertex from $A$ to $B$.

Corollary. Outflow from $s = \text{inflow}$ to $t = \text{value of flow}$. 
Relationship between flows and cuts

**Weak duality.** Let \( f \) be any flow and let \((A, B)\) be any cut. Then, the value of the flow \( \leq \) the capacity of the cut.

**Pf.** Value of flow \( f = \) net flow across cut \((A, B) \leq \) capacity of cut \((A, B)\).
Maxflow–mincut theorem

**Maxflow-mincut theorem.** Value of the maxflow = capacity of mincut.

**Augmenting path theorem.** A flow $f$ is a maxflow iff no augmenting paths.

**Pf.** The following three conditions are equivalent for any flow $f$:

i. There exists a cut whose capacity equals the value of the flow $f$.

ii. $f$ is a maxflow.

iii. There is no augmenting path with respect to $f$.

\[ i \Rightarrow ii \]

- Suppose that $(A, B)$ is a cut with capacity equal to the value of $f$.
- Then, the value of any flow $f' \leq$ capacity of $(A, B) = \text{value of } f$.
- Thus, $f$ is a maxflow.

↑ weak duality

↑ by assumption
**Maxflow–mincut theorem**

**Maxflow–mincut theorem.** Value of the maxflow = capacity of mincut.

**Augmenting path theorem.** A flow $f$ is a maxflow iff no augmenting paths.

**Pf.** The following three conditions are equivalent for any flow $f$:

i. There exists a cut whose capacity equals the value of the flow $f$.

ii. $f$ is a maxflow.

iii. There is no augmenting path with respect to $f$.

[ ii $\Rightarrow$ iii ] We prove contrapositive: $\sim$iii $\Rightarrow$ $\sim$ii.

- Suppose that there is an augmenting path with respect to $f$.
- Can improve flow $f$ by sending flow along this path.
- Thus, $f$ is not a maxflow.
Maxflow-mincut theorem

[ iii ⇒ i ]

- Let $f$ be a flow with no augmenting paths.
- Let $A$ be set of vertices connected to $s$ by an undirected path with no full forward or empty backward edges.
- By definition of cut $A$, $s$ is in $A$.
- By definition of cut $A$ and flow $f$, $t$ is in $B$.
- Capacity of cut = net flow across cut = value of flow $f$. 

![Diagram of a graph with sets A and B, vertices s and t, and edges with flow and capacity values.](image)
Computing a mincut from a maxflow

To compute mincut \((A, B)\) from maxflow \(f\):

- By augmenting path theorem, no augmenting paths with respect to \(f\).
- Compute \(A = \) set of vertices connected to \(s\) by an undirected path with no full forward or empty backward edges.

![Graph](image)
6.4 Maximum Flow

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Ford–Fulkerson algorithm

Start with 0 flow.
While there exists an augmenting path:
- find an augmenting path
- compute bottleneck capacity
- increase flow on that path by bottleneck capacity

Fundamental questions.

• How to compute a mincut? Easy. ✔
• How to find an augmenting path? BFS works well.
• If FF terminates, does it always compute a maxflow? Yes. ✔
• Does FF always terminate? If so, after how many augmentations?

yes, provided edge capacities are integers (or augmenting paths are chosen carefully)
requires clever analysis
Ford–Fulkerson algorithm with integer capacities

**Important special case.** Edge capacities are integers between 1 and $U$.

**Invariant.** The flow is integral throughout Ford–Fulkerson.

**Pf.** [by induction]
- Bottleneck capacity is an integer.
- Flow on an edge increases/decreases by bottleneck capacity.

**Proposition.** Number of augmentations $\leq$ the value of the maxflow.

**Pf.** Each augmentation increases the value by at least 1.

**Integrality theorem.** There exists an integral maxflow.

**Pf.** Ford–Fulkerson terminates and maxflow that it finds is integer-valued.
**Bad case for Ford–Fulkerson**

**Bad news.** Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.
Bad case for Ford–Fulkerson

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**Bad news.** Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

4th iteration

![Graph diagram](attachment:image.png)
Bad case for Ford–Fulkerson

**Bad news.** Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.
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Bad case for Ford–Fulkerson

**Bad news.** Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow. 

*can be exponential in input size*

**Good news.** This case is easily avoided.  [ use shortest/fattest path ]
How to choose augmenting paths?

Choose augmenting paths with:

- Shortest path: fewest number of edges.
- Fattest path: max bottleneck capacity.

Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems

JACK EDMONDS
University of Waterloo, Waterloo, Ontario, Canada

AND

RICHARD M. KARP
University of California, Berkeley, California

Abstract. This paper presents new algorithms for the maximum flow problem, the Hitchcock transportation problem, and the general minimum-cost flow problem. Upper bounds on the numbers of steps in these algorithms are derived, and are shown to compare favorably with upper bounds on the numbers of steps required by earlier algorithms.

Edmonds–Karp 1972 (USA)

Dokl. Akad. Nauk SSSR
Tom 154 (1970), No. 4

ALGORITHM FOR SOLUTION OF A PROBLEM OF MAXIMUM FLOW IN A NETWORK WITH POWER ESTIMATION

UDC 518.5

E. A. DINIC

Vol. 11 (1970), No. 5

Different variants of the formulation of the problem of maximal stationary flow in a network and its many applications are given in [1]. There also is given an algorithm solving the problem in the case where the initial data are integers (or, what is equivalent, commensurable). In the general case this algorithm requires preliminary rounding off of the initial data, i.e. only an approximate solution of the problem is possible. In this connection the rapidity of convergence of the algorithm is inversely proportional to the relative precision.

Dinic 1970 (Soviet Union)
How to choose augmenting paths?

Use care when selecting augmenting paths.
- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

<table>
<thead>
<tr>
<th>augmenting path</th>
<th>number of paths</th>
<th>implementation</th>
</tr>
</thead>
<tbody>
<tr>
<td>random path</td>
<td>$\leq EU$</td>
<td>randomized queue</td>
</tr>
<tr>
<td>DFS path</td>
<td>$\leq EU$</td>
<td>stack (DFS)</td>
</tr>
<tr>
<td>shortest path</td>
<td>$\leq \frac{1}{2} EV$</td>
<td>queue (BFS)</td>
</tr>
<tr>
<td>fattest path</td>
<td>$\leq E \ln(EU)$</td>
<td>priority queue</td>
</tr>
</tbody>
</table>

flow network with V vertices, E edges, and integer capacities between 1 and U
6.4 **Maximum Flow**

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Flow network representation

**Flow edge data type.** Associate flow $f_e$ and capacity $c_e$ with edge $e = v \rightarrow w$.

![Flow edge diagram](image)

**Flow network data type.** Must be able to process edge $e = v \rightarrow w$ in either direction: include $e$ in adjacency lists of both $v$ and $w$.

**Residual (spare) capacity.**
- Forward edge: residual capacity $= c_e - f_e$.
- Backward edge: residual capacity $= f_e$.

**Augment flow.**
- Forward edge: add $\Delta$.
- Backward edge: subtract $\Delta$.  

![Augment flow diagram](image)
Flow network representation

**Residual network.** A useful view of a flow network.

Original network includes all edges with positive residual capacity.

Residual network includes all edges with positive residual capacity.

Key point. Augmenting paths in original network are in one-to-one correspondence with directed paths in residual network.
Flow edge API

```java
public class FlowEdge {
    FlowEdge(int v, int w, double capacity) {
        // create a flow edge v→w
    }
    int from() {
        // vertex this edge points from
    }
    int to() {
        // vertex this edge points to
    }
    int other(int v) {
        // other endpoint
    }
    double capacity() {
        // capacity of this edge
    }
    double flow() {
        // flow in this edge
    }
    double residualCapacityTo(int v) {
        // residual capacity toward v
    }
    void addResidualFlowTo(int v, double delta) {
        // add delta flow toward v
    }
}
```

**Diagram:**
- **Flow $f_e$** and **Capacity $c_e$**
  - Edge v → w with flow 7/9

**Residual Capacity and Edges:**
- **Residual Capacity**
  - Edge v → w with residual capacity of 7
- **Forward Edge**
  - Edge v → w with flow 2
- **Backward Edge**
  - Edge w → v with flow 7
public class FlowEdge
{
    private final int v, w;       // from and to
    private final double capacity; // capacity
    private double flow;          // flow

    public FlowEdge(int v, int w, double capacity)
    {
        this.v = v;
        this.w = w;
        this.capacity = capacity;
    }

    public int from() { return v; }
    public int to() { return w; }
    public double capacity() { return capacity; }
    public double flow() { return flow; }

    public int other(int vertex)
    {
        if (vertex == v) return w;
        else if (vertex == w) return v;
        else throw new IllegalArgumentException();
    }

    public double residualCapacityTo(int vertex) { ... }
    public void addResidualFlowTo(int vertex, double delta) { ... }
}
public double residualCapacityTo(int vertex) {
    if (vertex == v) return flow;
    else if (vertex == w) return capacity - flow;
    else throw new IllegalArgumentException();
}

public void addResidualFlowTo(int vertex, double delta) {
    if (vertex == v) flow -= delta;
    else if (vertex == w) flow += delta;
    else throw new IllegalArgumentException();
}
Flow network API

public class FlowNetwork

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>FlowNetwork(int V)</td>
<td>create an empty flow network with V vertices</td>
</tr>
<tr>
<td>FlowNetwork(In in)</td>
<td>construct flow network input stream</td>
</tr>
<tr>
<td>void addEdge(FlowEdge e)</td>
<td>add flow edge e to this flow network</td>
</tr>
<tr>
<td>Iterable&lt;FlowEdge&gt; adj(int v)</td>
<td>forward and backward edges incident to/from v</td>
</tr>
<tr>
<td>Iterable&lt;FlowEdge&gt; edges()</td>
<td>all edges in this flow network</td>
</tr>
<tr>
<td>int V()</td>
<td>number of vertices</td>
</tr>
<tr>
<td>int E()</td>
<td>number of edges</td>
</tr>
<tr>
<td>String toString()</td>
<td>string representation</td>
</tr>
</tbody>
</table>

Conventions. Allow self-loops and parallel edges.
Flow network: Java implementation

```java
public class FlowNetwork {
    private final int V;
    private Bag<FlowEdge>[] adj;

    public FlowNetwork(int V) {
        this.V = V;
        adj = (Bag<FlowEdge>[])(new Bag[V]);
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<FlowEdge>();
    }

    public void addEdge(FlowEdge e) {
        int v = e.from();
        int w = e.to();
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<FlowEdge> adj(int v) {
        return adj[v];
    }
}
```

same as `EdgeWeightedGraph`, but adjacency lists of `FlowEdges` instead of `Edges`
Flow network: adjacency-lists representation

Maintain vertex-indexed array of FlowEdge lists (use Bag abstraction).

Note. Adjacency list includes edges with 0 residual capacity.
(residual network is represented implicitly)
private boolean hasAugmentingPath(FlowNetwork G, int s, int t) {
    edgeTo = new FlowEdge[G.V()];
    marked = new boolean[G.V()];

    Queue<Integer> queue = new Queue<Integer>();
    queue.enqueue(s);
    marked[s] = true;
    while (!queue.isEmpty() && !marked[t]) {
        int v = queue.dequeue();

        for (FlowEdge e : G.adj(v)) {
            int w = e.other(v);
            if (!marked[w] && (e.residualCapacityTo(w) > 0)) {
                edgeTo[w] = e;
                marked[w] = true;
                queue.enqueue(w);
            }
        }
    }

    return marked[t];
}
Ford–Fulkerson: Java implementation

```java
public class FordFulkerson {
    private boolean[] marked;  // true if s->v path in residual network
    private FlowEdge[] edgeTo;  // last edge on s->v path
    private double value;       // value of flow

    public FordFulkerson(FlowNetwork G, int s, int t) {
        value = 0.0;
        while (hasAugmentingPath(G, s, t)) {
            double bottle = Double.POSITIVE_INFINITY;
            for (int v = t; v != s; v = edgeTo[v].other(v))
                bottle = Math.min(bottle, edgeTo[v].residualCapacityTo(v));

            for (int v = t; v != s; v = edgeTo[v].other(v))
                edgeTo[v].addResidualFlowTo(v, bottle);

            value += bottle;
        }
    }

    private boolean hasAugmentingPath(FlowNetwork G, int s, int t) {
        /* See previous slide. */
    }

    public double value() {
        return value;
    }

    public boolean inCut(int v) {  // is v reachable from s in residual network?
        return marked[v];
    }
}
```
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Maxflow and mincut applications

Maxflow/mincut is a widely applicable problem-solving model.

- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.
Bipartite matching problem

Problem. Given $N$ people and $N$ tasks, assign the tasks to people so that:

- Every task is assigned to a qualified person.
- Every person is assigned to exactly one task.

Bipartite matching problem
Problem. Given a bipartite graph, find a perfect matching.
Network flow formulation of bipartite matching

- Create $s$, $t$, one vertex for each task, and one vertex for each person.
- Add edge from $s$ to each task (of capacity 1).
- Add edge from each person to $t$ (of capacity 1).
- Add edge from task to qualified person (of infinite capacity).
Network flow formulation of bipartite matching

1–1 correspondence between perfect matchings in bipartite graph and integer-valued maxflows of value $N$ in flow network.
What the mincut tells us

**Goal.** When no perfect matching, explain why.

$$S = \{ 2, 4, 5 \}$$
$$T = \{ 7, 10 \}$$

tasks in $S$ can be matched only to people in $T$

$|S| > |T|$
What the mincut tells us

**Min cut.** Consider mincut \((A, B)\).

- Let \(S = \text{tasks on } s\text{ side of cut.}\)
- Let \(T = \text{people on } s\text{ side of cut.}\)
- Fact: \(|S| > |T|\); tasks in \(S\) can be matched only to people in \(T\).

\[
S = \{2, 4, 5\} \\
T = \{7, 10\} \\
\text{tasks in } S \text{ can be matched only to people in } T \\
|S| > |T|
\]

Bottom line. When no perfect matching, mincut explains why.
Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

<table>
<thead>
<tr>
<th>i</th>
<th>team</th>
<th>wins</th>
<th>losses</th>
<th>to play</th>
<th>ATL</th>
<th>PHI</th>
<th>NYM</th>
<th>WAS</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>Atlanta</td>
<td>83</td>
<td>71</td>
<td>8</td>
<td>–</td>
<td>1</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>Philly</td>
<td>80</td>
<td>79</td>
<td>3</td>
<td>1</td>
<td>–</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>New York</td>
<td>78</td>
<td>78</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>–</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Washington</td>
<td>77</td>
<td>82</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>–</td>
</tr>
</tbody>
</table>

Washington is mathematically eliminated.

- Washington finishes with ≤ 80 wins.
- Atlanta already has 83 wins.
Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

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<td>83</td>
<td>71</td>
<td>8</td>
<td>–</td>
<td>1</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>Philly</td>
<td>80</td>
<td>79</td>
<td>3</td>
<td>1</td>
<td>–</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>New York</td>
<td>78</td>
<td>78</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>–</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Washington</td>
<td>77</td>
<td>82</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>–</td>
</tr>
</tbody>
</table>

Philadelphia is mathematically eliminated.
- Philadelphia finishes with $\leq 83$ wins.
- Either New York or Atlanta will finish with $\geq 84$ wins.

Observation. Answer depends not only on how many games already won and left to play, but on whom they're against.
Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

<table>
<thead>
<tr>
<th>i</th>
<th>team</th>
<th>wins</th>
<th>losses</th>
<th>to play</th>
<th>NYY</th>
<th>BAL</th>
<th>BOS</th>
<th>TOR</th>
<th>DET</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>New York</td>
<td>75</td>
<td>59</td>
<td>28</td>
<td>–</td>
<td>3</td>
<td>8</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>Baltimore</td>
<td>71</td>
<td>63</td>
<td>28</td>
<td>3</td>
<td>–</td>
<td>2</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Boston</td>
<td>69</td>
<td>66</td>
<td>27</td>
<td>8</td>
<td>2</td>
<td>–</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Toronto</td>
<td>63</td>
<td>72</td>
<td>27</td>
<td>7</td>
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</tr>
<tr>
<td>4</td>
<td>Detroit</td>
<td>49</td>
<td>86</td>
<td>27</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>–</td>
</tr>
</tbody>
</table>

AL East (August 30, 1996)

Detroit is mathematically eliminated.

- Detroit finishes with ≤ 76 wins.
- Wins for $R = \{ \text{NYY, BAL, BOS, TOR} \} = 278$.
- Remaining games among $\{ \text{NYY, BAL, BOS, TOR} \} = 3 + 8 + 7 + 2 + 7 = 27$.
- Average team in $R$ wins $305/4 = 76.25$ games.
Intuition. Remaining games flow from $s$ to $t$.

Fact. Team 4 not eliminated iff all edges pointing from $s$ are full in maxflow.
Maximum flow algorithms: theory

(Yet another) holy grail for theoretical computer scientists.

<table>
<thead>
<tr>
<th>year</th>
<th>method</th>
<th>worst case</th>
<th>discovered by</th>
</tr>
</thead>
<tbody>
<tr>
<td>1951</td>
<td>simplex</td>
<td>$E^3 U$</td>
<td>Dantzig</td>
</tr>
<tr>
<td>1955</td>
<td>augmenting path</td>
<td>$E^2 U$</td>
<td>Ford–Fulkerson</td>
</tr>
<tr>
<td>1970</td>
<td>shortest augmenting path</td>
<td>$E^3$</td>
<td>Dinitz, Edmonds-Karp</td>
</tr>
<tr>
<td>1970</td>
<td>fattest augmenting path</td>
<td>$E^2 \log E \log(EU)$</td>
<td>Dinitz, Edmonds-Karp</td>
</tr>
<tr>
<td>1977</td>
<td>blocking flow</td>
<td>$E^{5/2}$</td>
<td>Cherkasky</td>
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<tr>
<td>1978</td>
<td>blocking flow</td>
<td>$E^{7/3}$</td>
<td>Galil</td>
</tr>
<tr>
<td>1983</td>
<td>dynamic trees</td>
<td>$E^2 \log E$</td>
<td>Sleator-Tarjan</td>
</tr>
<tr>
<td>1985</td>
<td>capacity scaling</td>
<td>$E^2 \log U$</td>
<td>Gabow</td>
</tr>
<tr>
<td>1997</td>
<td>length function</td>
<td>$E^{3/2} \log E \log U$</td>
<td>Goldberg-Rao</td>
</tr>
<tr>
<td>2012</td>
<td>compact network</td>
<td>$E^2 / \log E$</td>
<td>Orlin</td>
</tr>
<tr>
<td></td>
<td>?</td>
<td>$E$</td>
<td>?</td>
</tr>
</tbody>
</table>

maxflow algorithms for sparse networks with E edges, integer capacities between 1 and U
Maximum flow algorithms: practice

Warning. Worst-case order-of-growth is generally not useful for predicting or comparing maxflow algorithm performance in practice.


Computer vision. Specialized algorithms for problems with special structure.

On Implementing Push-Relabel Method for the Maximum Flow Problem

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Abstract. We study efficient implementations of the push-relabel method for the maximum flow problem. The resulting codes are faster than the previous codes, and much faster on some problem families. The speedup is due to the combination of heuristics used in our implementations. We also exhibit a family of problems for which the running time of all known methods seem to have a roughly quadratic growth rate.

Theory and Methodology

Computational investigations of maximum flow algorithms

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Received 30 August 1995; accepted 27 June 1996
Summary

**Mincut problem.** Find an $st$-cut of minimum capacity.

**Maxflow problem.** Find an $st$-flow of maximum value.

**Duality.** Value of the maxflow = capacity of mincut.

**Proven successful approaches.**
- Ford–Fulkerson (various augmenting-path strategies).
- Preflow–push (various versions).

**Open research challenges.**
- Practice: solve real-world maxflow/mincut problems in linear time.
- Theory: prove it for worst-case inputs.
- Still much to be learned!