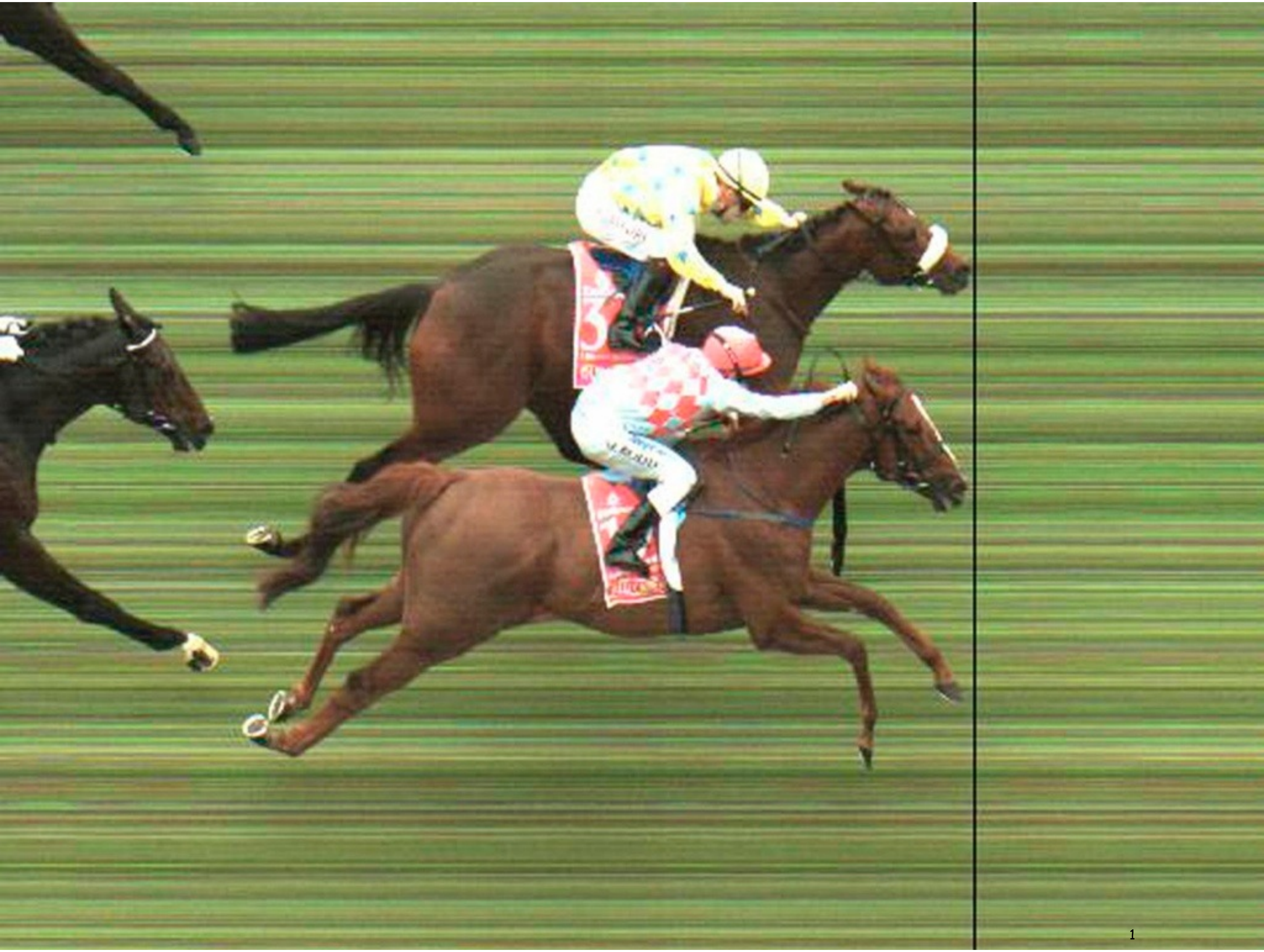
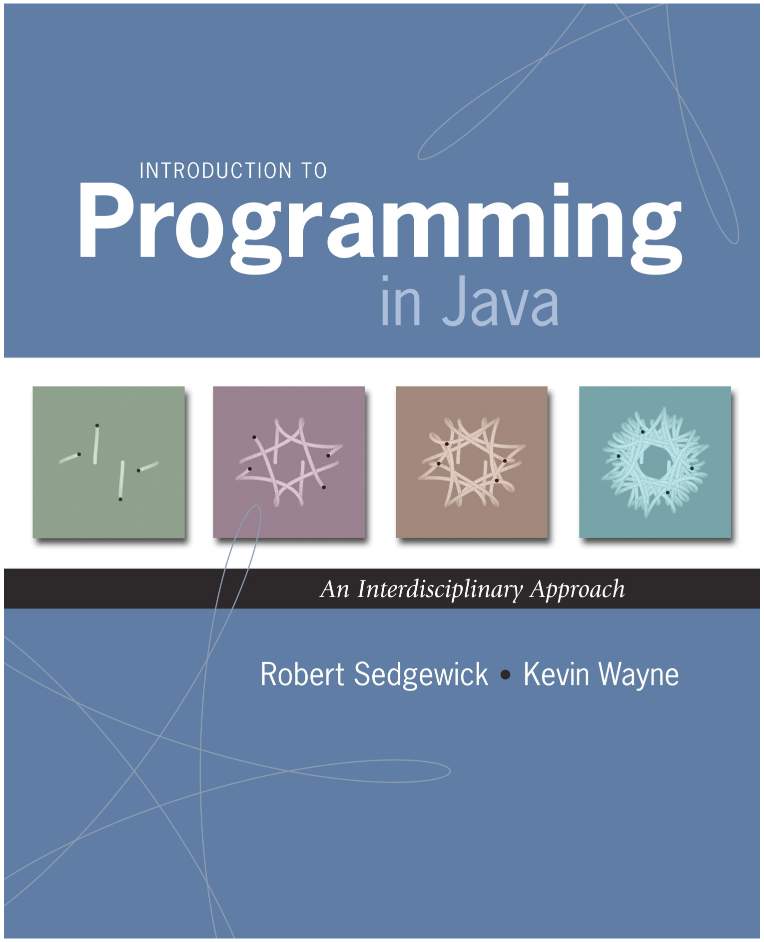
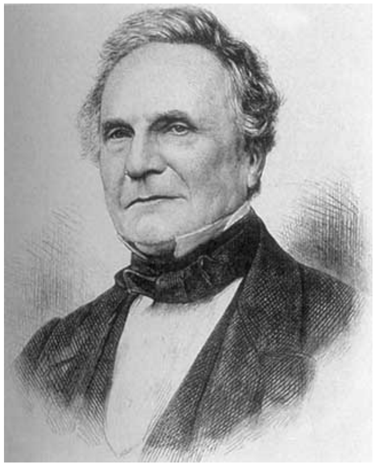


4.1, 4.2 Performance, with Sorting

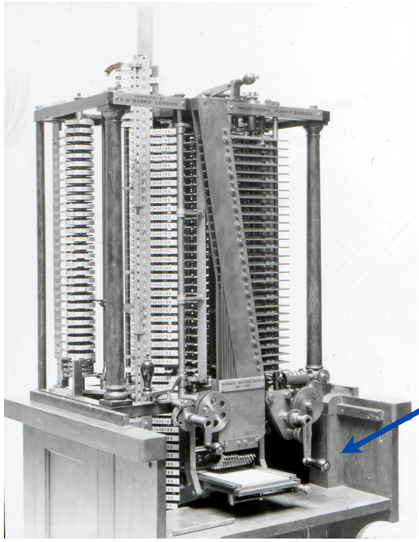


Running Time

"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?" – *Charles Babbage*



Charles Babbage (1864)



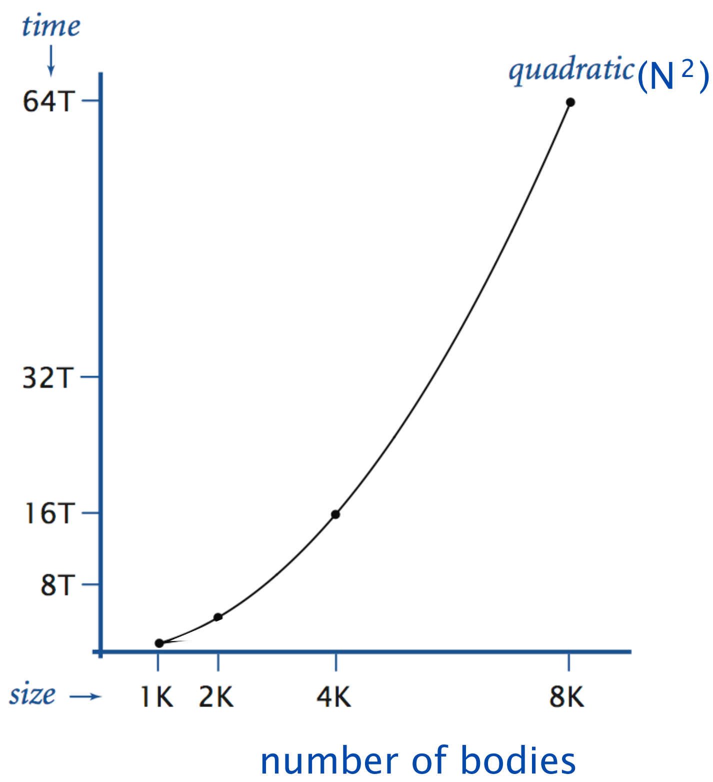
Analytic Engine

how many times do you have to turn the crank?

Algorithmic Successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.



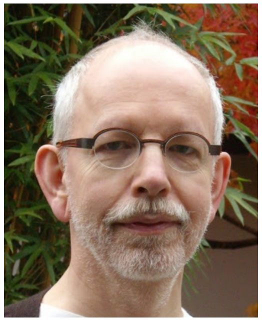
Algorithmic Successes

Andrew Appel
PU '81

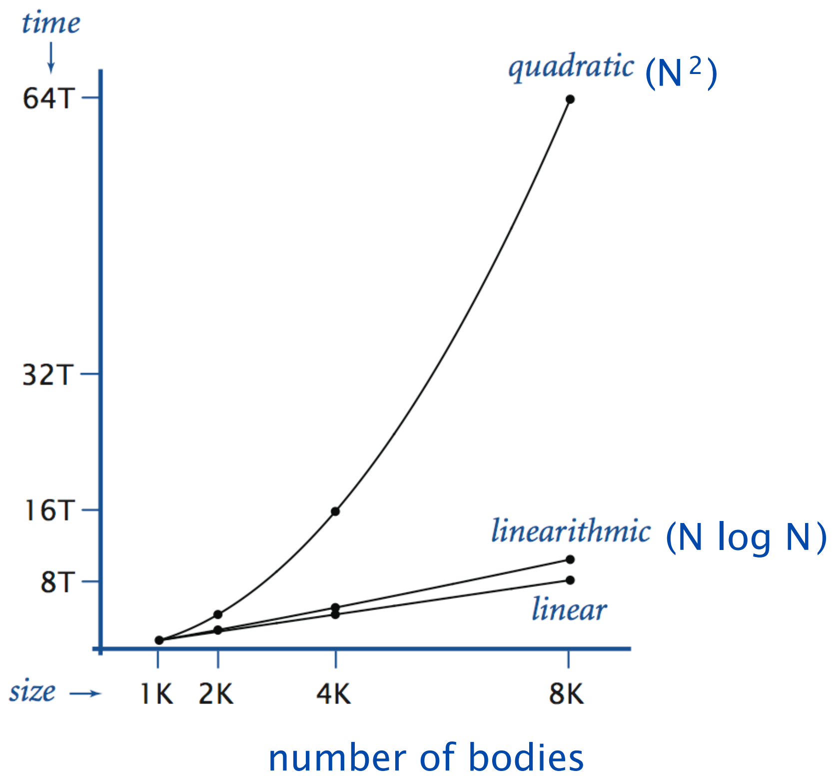
N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: N log N steps, enables new research.

Josh Barnes



Piet Hut

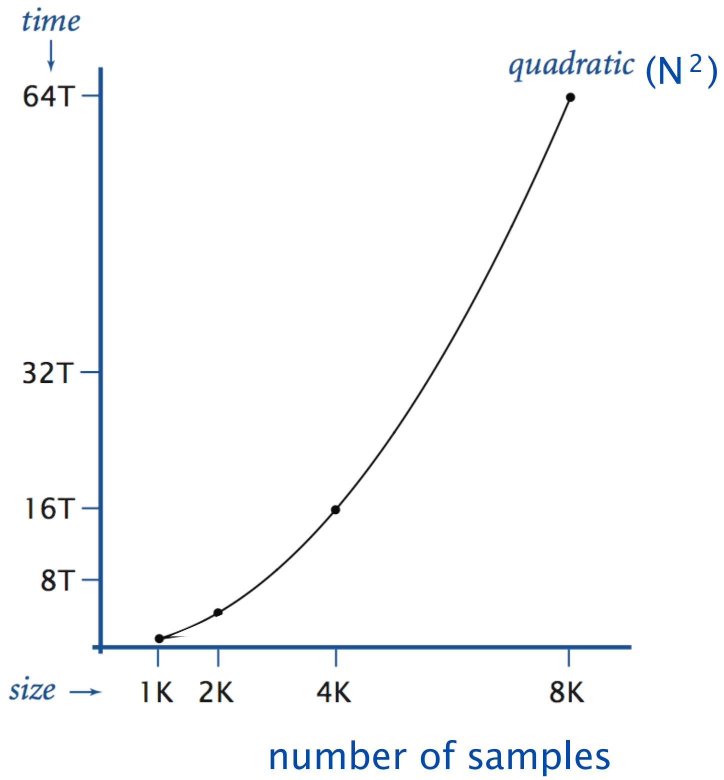


Algorithmic Successes

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ...
- Brute force: N^2 steps.

Freidrich Gauss
1805

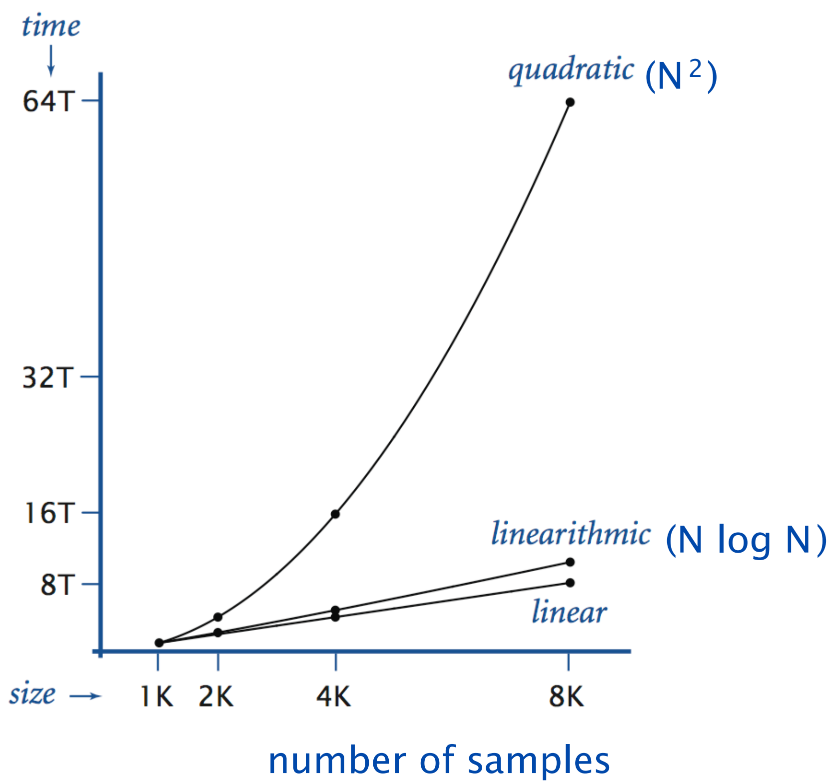


Algorithmic Successes

John Tukey
1965

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ...
- Brute force: N^2 steps.
- FFT algorithm: N log N steps, enables new technology.



Sorting

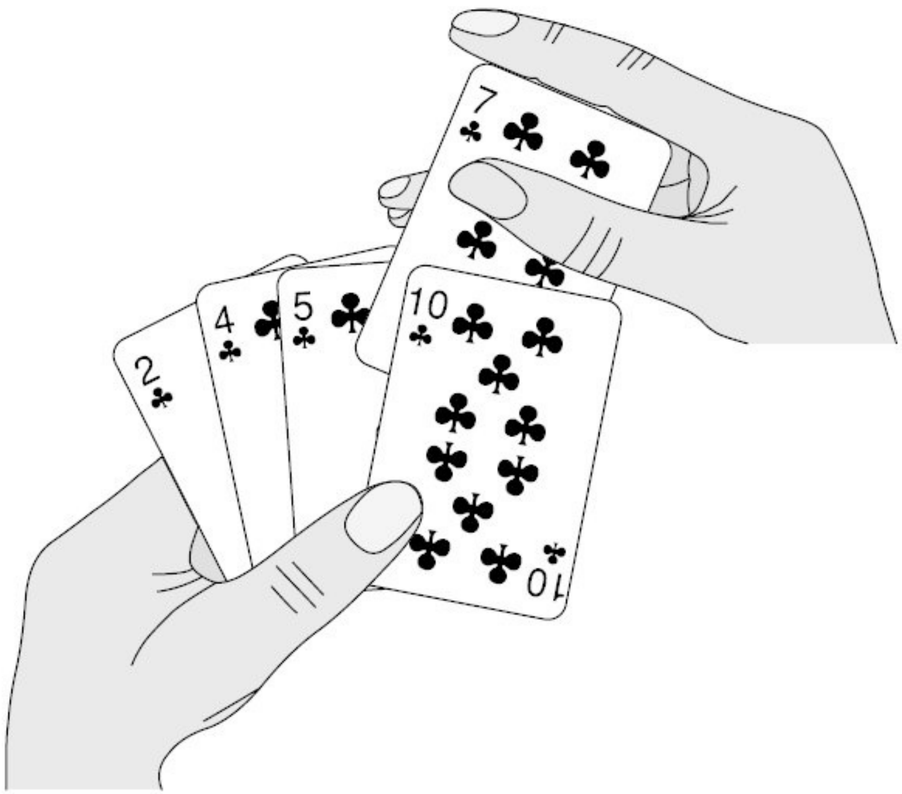
Sorting

Sorting problem. Rearrange N items in ascending order.

Applications. Binary search, statistics, databases, data compression, bioinformatics, computer graphics, scientific computing, (too numerous to list) ...

`Hauser`	`Hanley`
`Hong`	`Haskell`
`Hsu`	`Hauser`
`Hayes`	`Hayes`
`Haskell`	`Hong`
`Hanley`	`Hornet`
`Hornet`	`Hsu`

Insertion Sort



Insertion Sort

Insertion sort.

- Brute-force sorting solution.
- Move left-to-right through array.
- Insert each element into correct position by exchanging it with larger elements to its left, one-by-one.

i	j	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
6	6	and	had	him	his	was	you	the	but
6	5	and	had	him	his	was	the	you	but
6	4	and	had	him	his	the	was	you	but
		and	had	him	his	the	was	you	but

Inserting a[6] *into position by exchanging with larger entries to its left*

Insertion Sort

Insertion sort.

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

i	j	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
		was	had	him	and	you	his	the	but
1	0	had	was	him	and	you	his	the	but
2	1	had	him	was	and	you	his	the	but
3	0	and	had	him	was	you	his	the	but
4	4	and	had	him	was	you	his	the	but
5	3	and	had	him	his	was	you	the	but
6	4	and	had	him	his	the	was	you	but
7	1	and	but	had	him	his	the	was	you
		and	but	had	him	his	the	was	you

Inserting a[1] *through* a[N-1] *into position (insertion sort)*

Insertion Sort Demo

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	2.78	7.42	0.56	1.12	1.17	0.32	6.21	4.42	3.14	7.71

Insertion Sort: Java Implementation

```
public class Insertion
{

   public static void sort(double[] a)
   {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1] > a[j])
               exch(a, j-1, j);
            else break;            // see text p. 70
   }


   private static void exch(double[] a, int i, int j)
   {
      double swap = a[i];
      a[i] = a[j];
      a[j] = swap;
   }
}
```

Insertion Sort: Observation

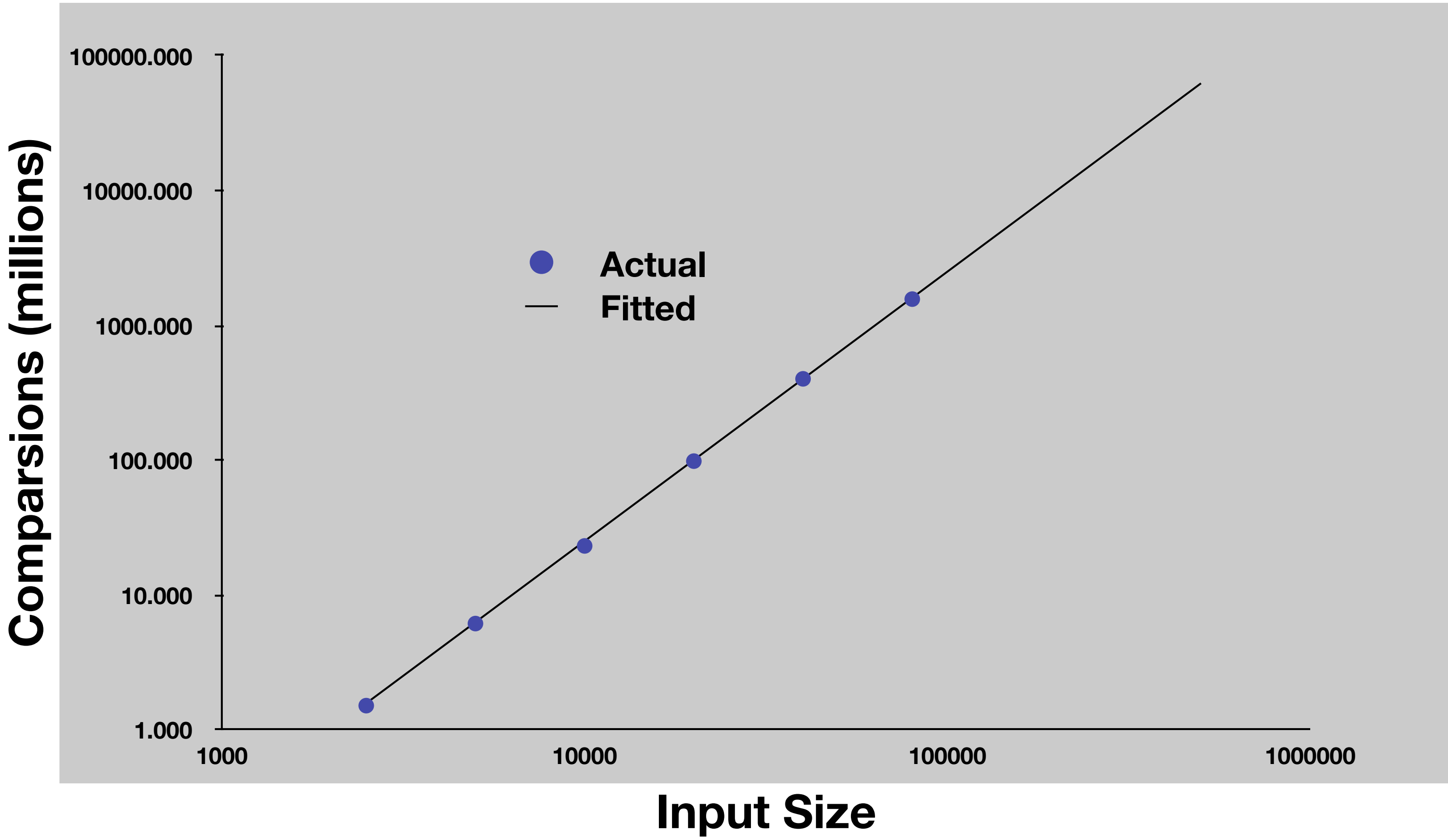
Observe and tabulate running time for various values of N.

- Data source: N random numbers between 0 and 1.
- Machine: Apple Model XXX with lots of memory, running OS X.
- Timing: Skagen wristwatch.

N	Comparisons	Time
5,000	6.2 million	0.13 seconds
10,000	25 million	0.43 seconds
20,000	99 million	1.5 seconds
40,000	400 million	5.6 seconds
80,000	1600 million	23 seconds

Insertion Sort: Empirical Analysis

Data analysis. Plot # comparisons vs. input size on log-log scale.



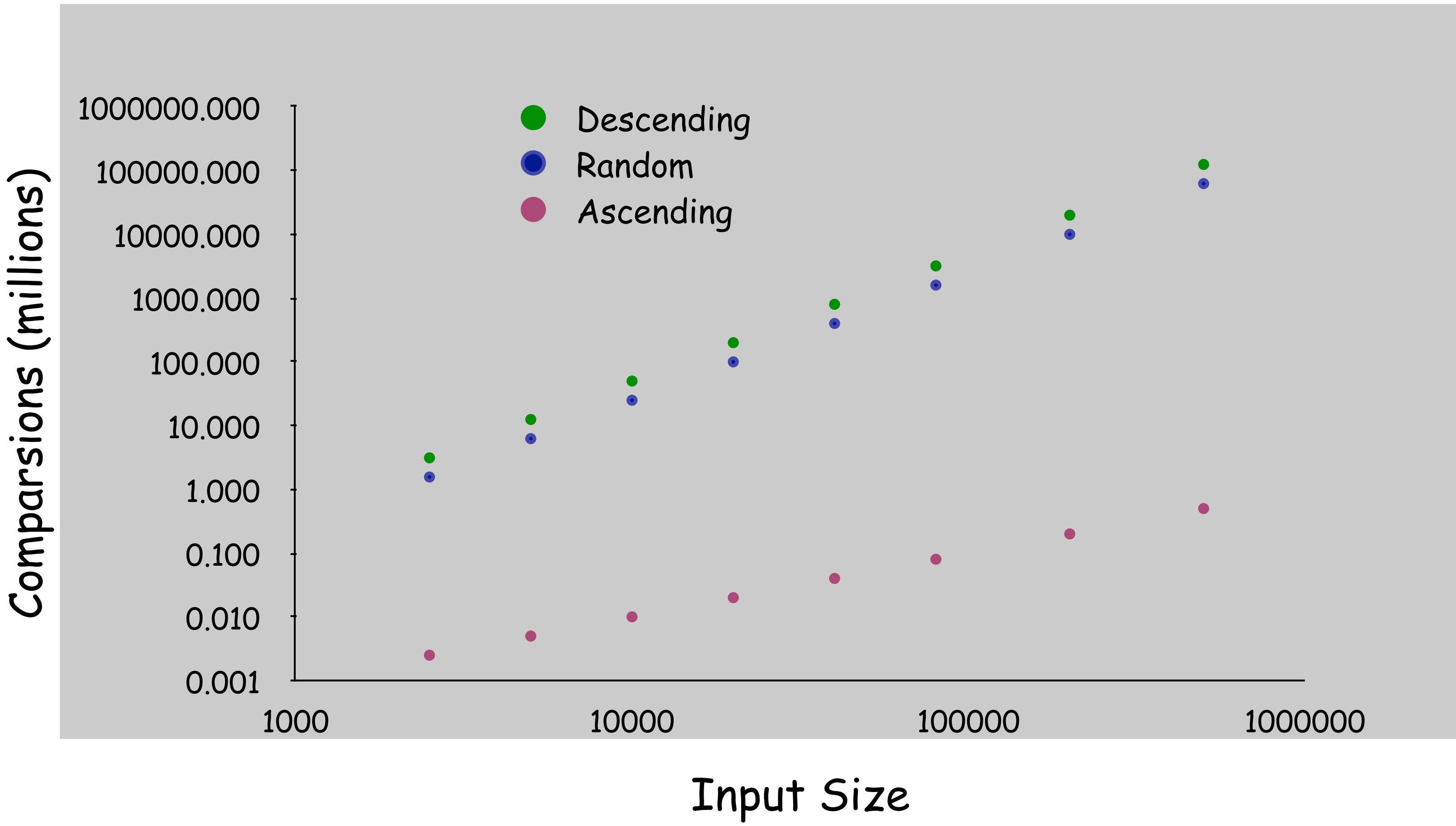
slope

Hypothesis. # comparisons grows quadratically with input size $\sim N^2 / 4$.

Insertion Sort: Empirical Analysis

Observation. Number of compares depends on input family.

- Descending: $\sim N^2 / 2$.
- Random: $\sim N^2 / 4$.
- Ascending: $\sim N$.



Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate number of ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

Insertion Sort: Mathematical Analysis

Worst case. [descending]

- Iteration i requires i comparisons.
- Total = $(0 + 1 + 2 + \ldots + N\text{-}1) \;\sim\; N^2 / 2$ compares.

E	F	G	H	I	J	D	C	B	A
						i			

Average case. [random]

- Iteration i requires $i / 2$ comparisons on average.
- Total = $(0 + 1 + 2 + \ldots + N\text{-}1) / 2 \;\sim\; N^2 / 4$ compares

A	C	D	F	H	J	E	B	I	G
						i			

Insertion Sort: Lesson

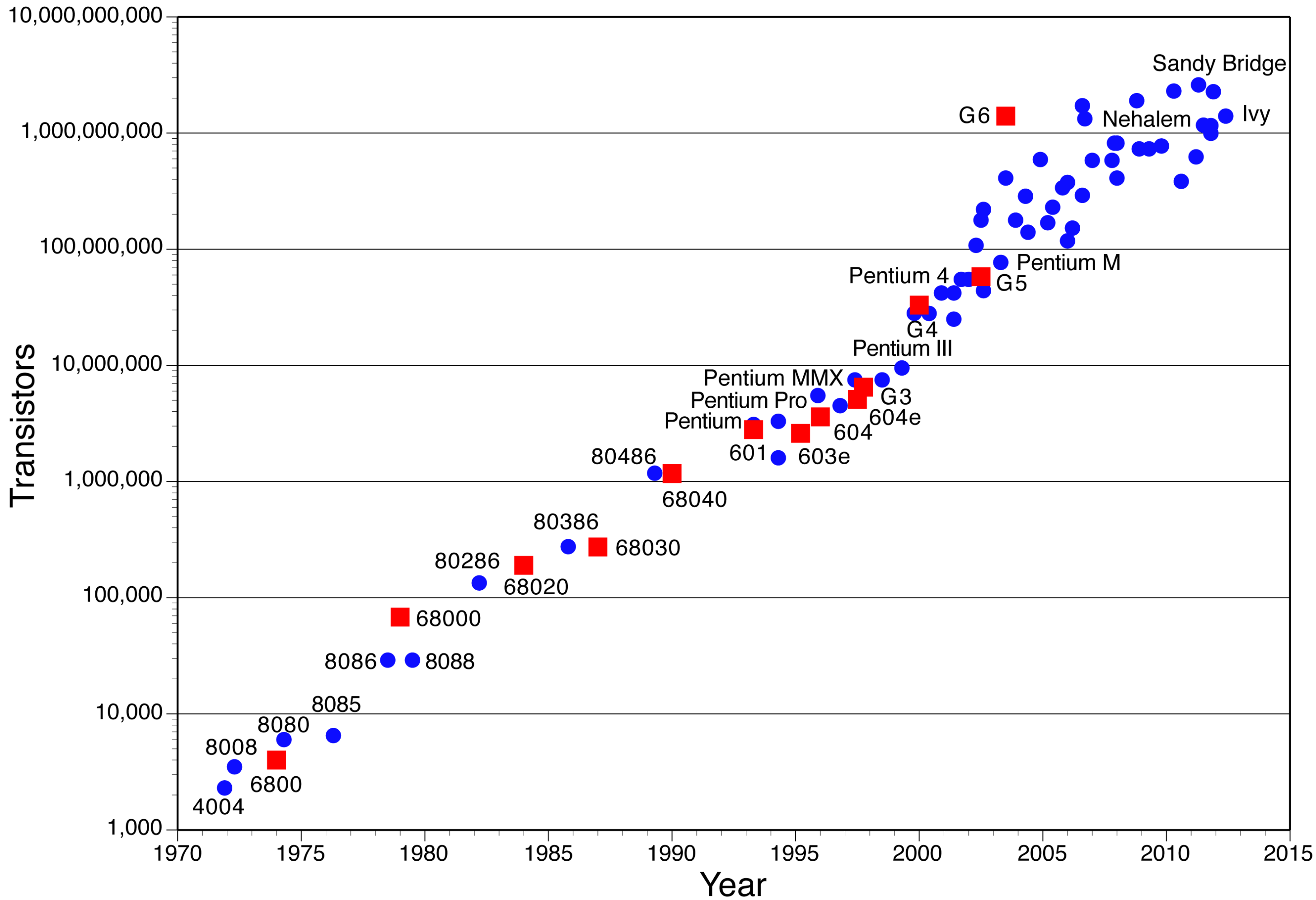
Lesson. Supercomputer can't rescue a bad algorithm.

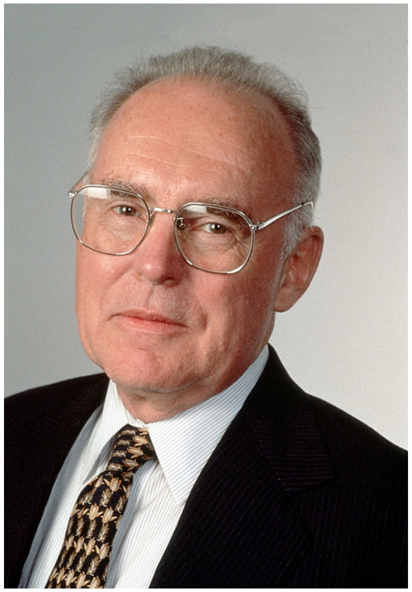
Computer	Comparisons Per Second	Thousand	Million	Billion
laptop	10^7	instant	1 day	3 centuries
super	10^{12}	instant	1 second	2 weeks

Moore's Law

Moore's law. Transistor density on a chip doubles every 2 years.

Variants. Memory, disk space, bandwidth, computing power per $.





Moore's Law and Algorithms

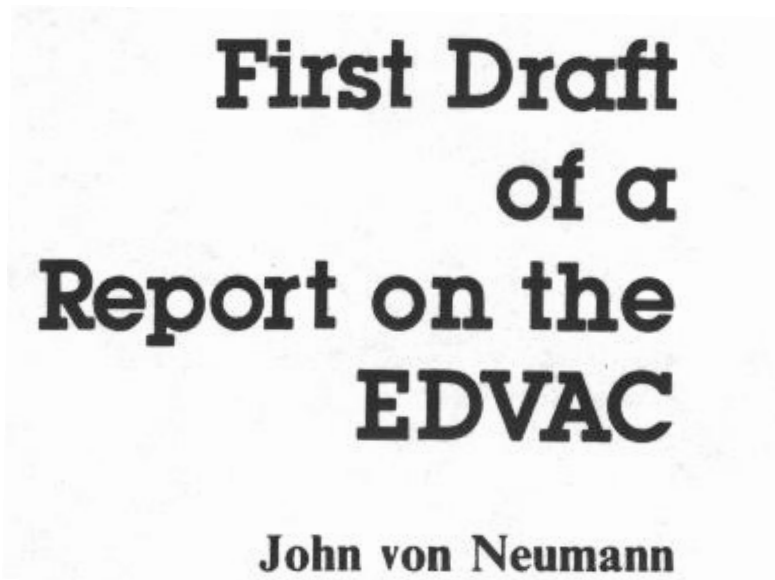
Quadratic algorithms do not scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

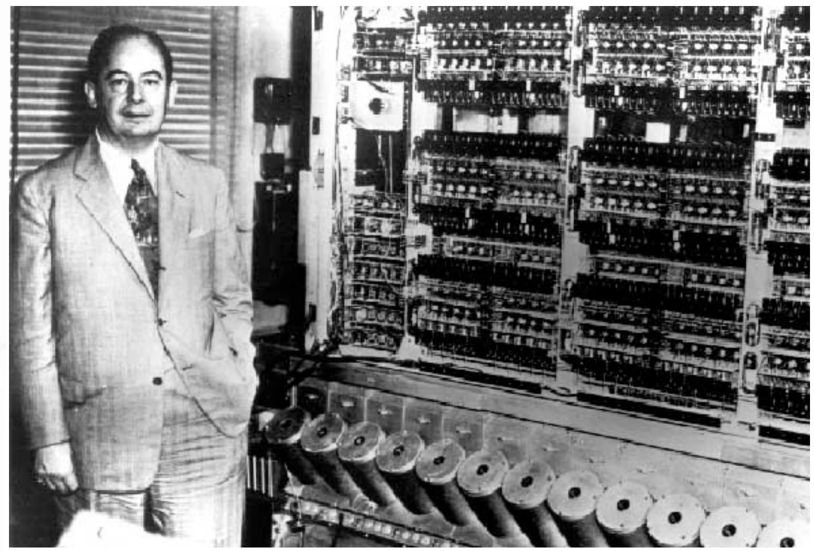
> *"Software inefficiency can always outpace Moore's Law. Moore's Law isn't a match for our bad coding."* – *Jaron Lanier*

Lesson. Need linear (or linearithmic) algorithm to keep pace with Moore's law.

Mergesort


First Draft of a Report on the EDVAC

John von Neumann



Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

input
was had him and you his the but

sort left
and had him was you his the but

sort right
and had him was but his the you

merge
and but had him his the was you

Mergesort: Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Top-down mergesort

Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

i	j	k	aux[k]	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
				and	had	him	was	but	his	the	you
0	4	0	and	and	had	him	was	but	his	the	you
1	4	1	but	and	had	him	was	but	his	the	you
1	5	2	had	and	had	him	was	but	his	the	you
2	5	3	him	and	had	him	was	but	his	the	you
3	5	4	his	and	had	him	was	but	his	the	you
3	6	5	the	and	had	him	was	but	his	the	you
3	7	6	was	and	had	him	was	but	his	the	you
4	7	7	you	and	had	him	was	but	his	the	you

Trace of the merge of the sorted left half with the sorted right half

Merging Demo

Merge.

- Keep track of smallest element in each sorted half.
- Choose smaller of two elements.
- Repeat until done.

↓						↓				
A	G	L	O	R		H	I	M	S	T

Merging

Merge.
- Keep track of smallest element in each sorted half.
- Choose smaller of two elements.
- Repeat until done.

A	G	L	O	R	H	I	M	S	T

A	G	H	I	L	M	O	R	S	T

Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

```
String[] aux = new String[N];
// Merge into auxiliary array.
int i = lo, j = mid;
for (int k = 0; k < N; k++)
{
   if      (i == mid) aux[k] = a[j++];
   else if (j == hi)  aux[k] = a[i++];
   else if (a[j].compareTo(a[i]) < 0) // String compare: text p. 523
        aux[k] = a[j++];
   else aux[k] = a[i++];
}

// Copy back.
for (int k = 0; k < N; k++)
   a[lo + k] = aux[k];
```

Mergesort: Java Implementation

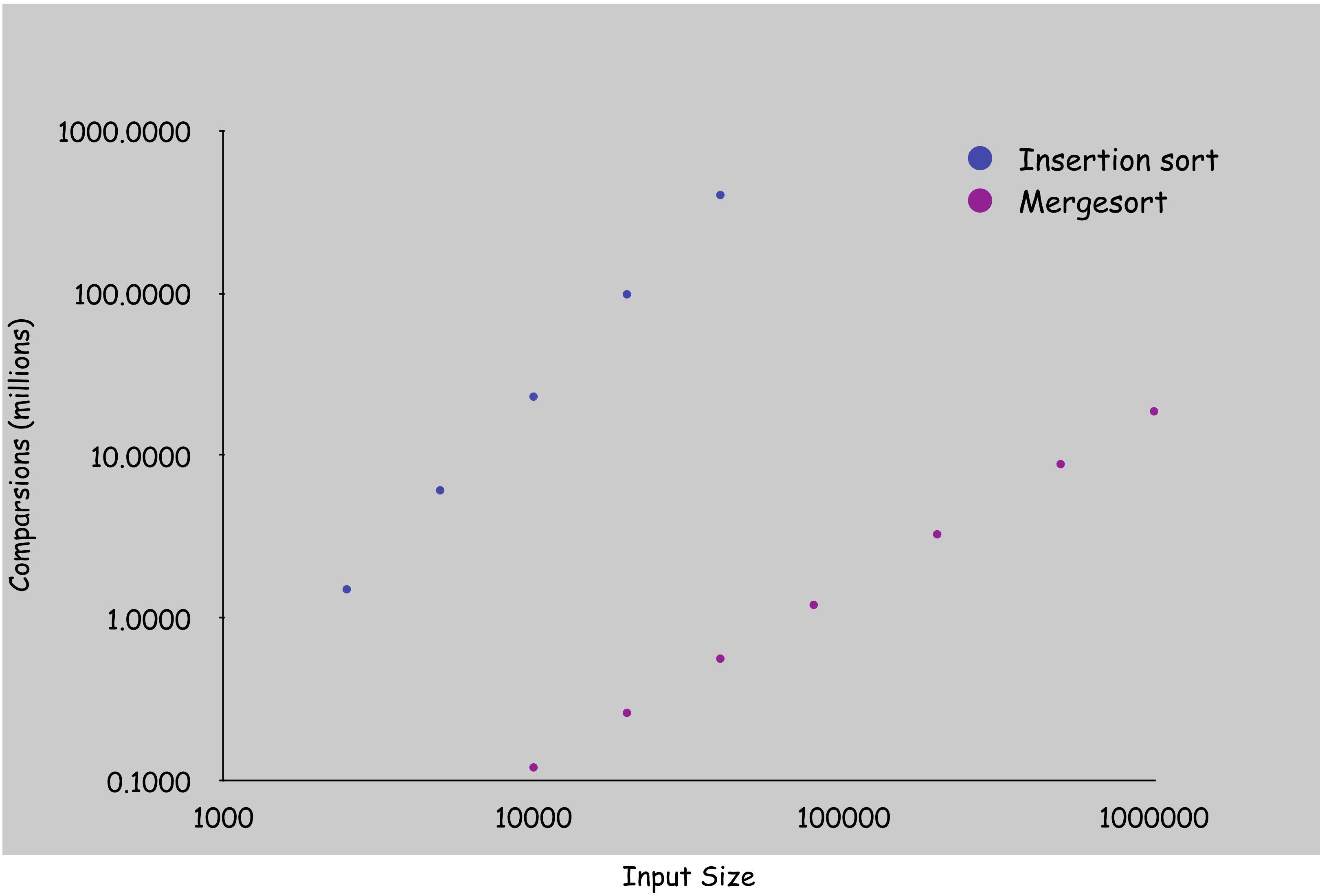
```
public class Merge
{
   public static void sort(String[] a)
   {  sort(a, 0, a.length);  }

   // Sort a[lo, hi).
   public static void sort(String[] a, int lo, int hi)
   {
      int N = hi - lo;
      if (N <= 1) return;

      // Recursively sort left and right halves.
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);

      // Merge sorted halves (see previous slide).
   }

}
```

lo mid hi

10 11 12 13 14 15 16 17 18 19

Mergesort: Empirical Analysis

Experimental hypothesis. Number of comparisons ≈ 20N.



Mergesort: Prediction and Verification

Experimental hypothesis. Number of comparisons ≈ 20N.

Prediction. 80 million comparisons for N = 4 million.

Observations.

N	Comparisons	Time
4 million	82.7 million	3.13 sec
4 million	82.7 million	3.25 sec
4 million	82.7 million	3.22 sec

Agrees.

Prediction. 400 million comparisons for N = 20 million.

Observations.

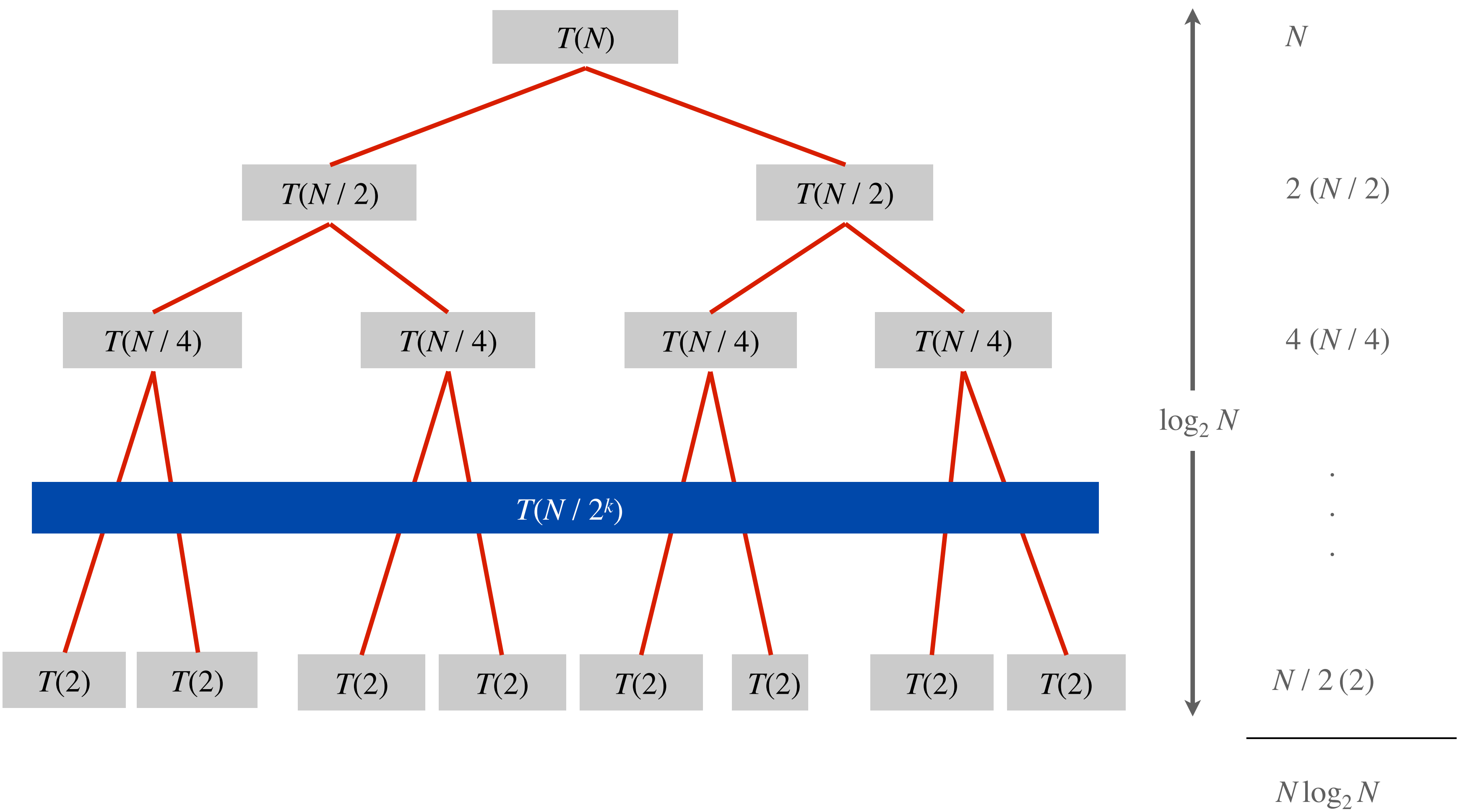
N	Comparisons	Time
20 million	460 million	17.5 sec
50 million	1216 million	45.9 sec

Not quite.

Mergesort: Mathematical Analysis

Analysis. To mergesort array of size N, mergesort two subarrays of size $N / 2$, and merge them together using $\leq N$ comparisons.

we assume N is a power of 2



Mergesort: Mathematical Analysis

Mathematical analysis.

analysis	comparisons
worst	$N \log_2 N$
average	$N \log_2 N$
best	$1/2\ N \log_2 N$

Validation. Theory agrees with observations.

N	actual	predicted
10,000	120 thousand	133 thousand
20 million	460 million	485 million
50 million	1,216 million	1,279 million

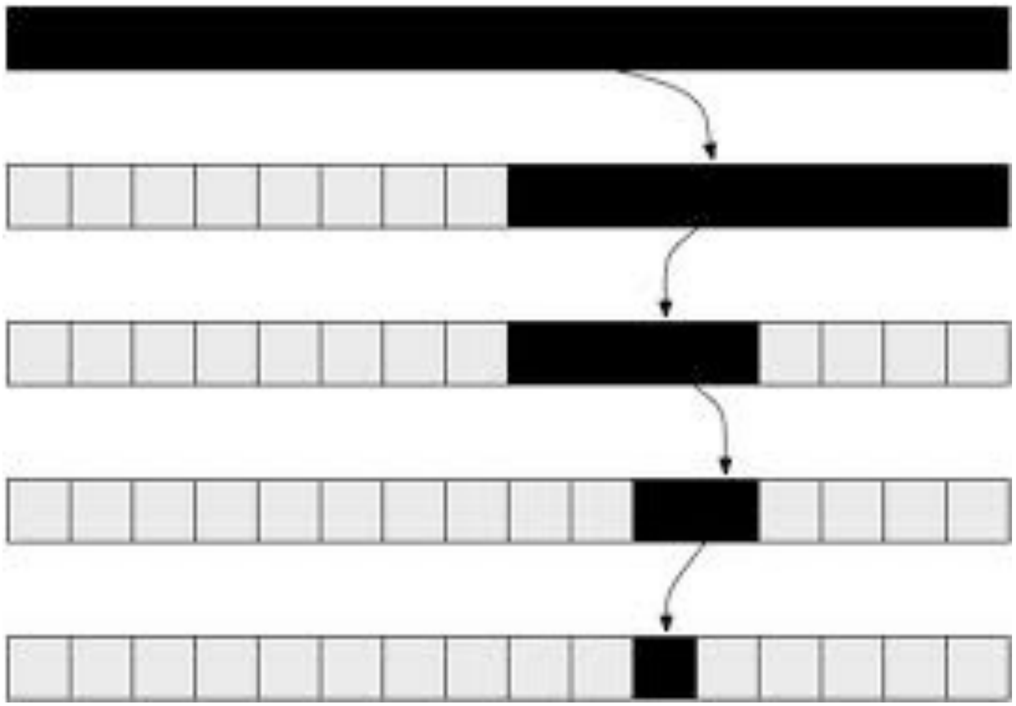
Mergesort: Lesson

Lesson. Great algorithms can be more powerful than supercomputers.

Computer	Comparisons Per Second	Insertion	Mergesort
laptop	10^7	3 centuries	3 hours
super	10^{12}	2 weeks	instant

N = 1 billion

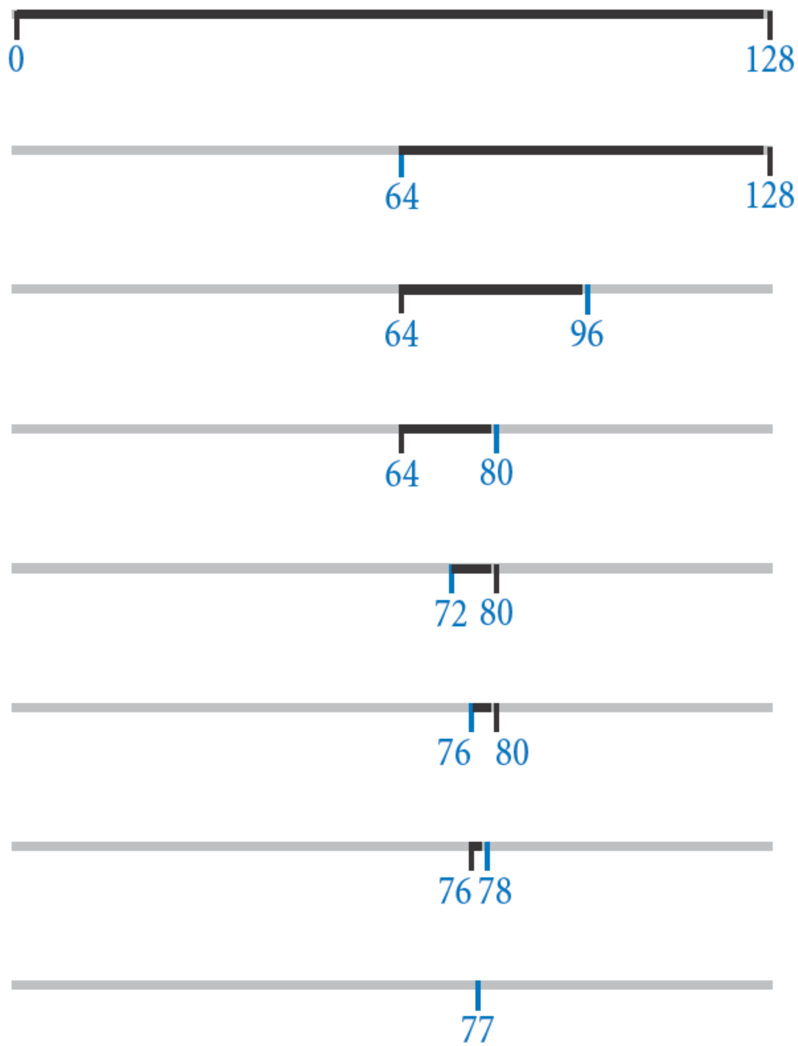
Binary Search



Twenty Questions

Intuition. Find a hidden integer.

interval



size	*Q*	*A*
128	< 64 ?	*no*
64	< 96 ?	*yes*
32	< 80 ?	*yes*
16	< 72 ?	*no*
8	< 76 ?	*no*
4	< 78 ?	*yes*
2	< 77 ?	*no*
1	= 77	

Binary Search

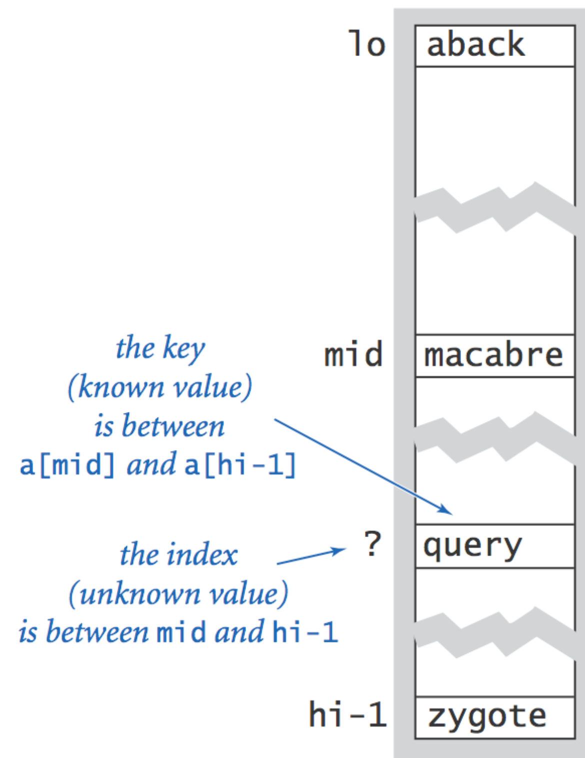
Idea:

- Sort the array
- Play "20 questions" to determine the index associated with a given key.

Ex. Dictionary, phone book, book index, credit card numbers, ...

Binary search.

- Examine the middle key.
- If it matches, return its index.
- Otherwise, search either the left or right half.



Binary search in a sorted array (one step)

Binary Search Demo

Binary search. Given a `key` and sorted array `a[]`, find index `i` such that `a[i] = key`, or report that no such index exists.

Invariant. Algorithm maintains `a[lo]` ≤ `key` ≤ `a[hi-1]`.

Ex. Binary search for 33.

6	13	14	25	33	43	51	53	64	72	84	93	95	96	97	
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
↑ lo															↑ hi

Binary Search: Java Implementation

Invariant. Algorithm maintains `a[lo]` $\leq$ `key` $\leq$ `a[hi-1]`.

```
public static int search(String key, String[] a)
{
   return search(key, a, 0, a.length);
}

public static int search(String key, String[] a, int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);    // String compare: text p. 523
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

Java library implementation: `Arrays.binarySearch()`

Binary Search: Mathematical Analysis

Analysis. To binary search in an array of size N: do one comparison, then binary search in an array of size $N / 2$.

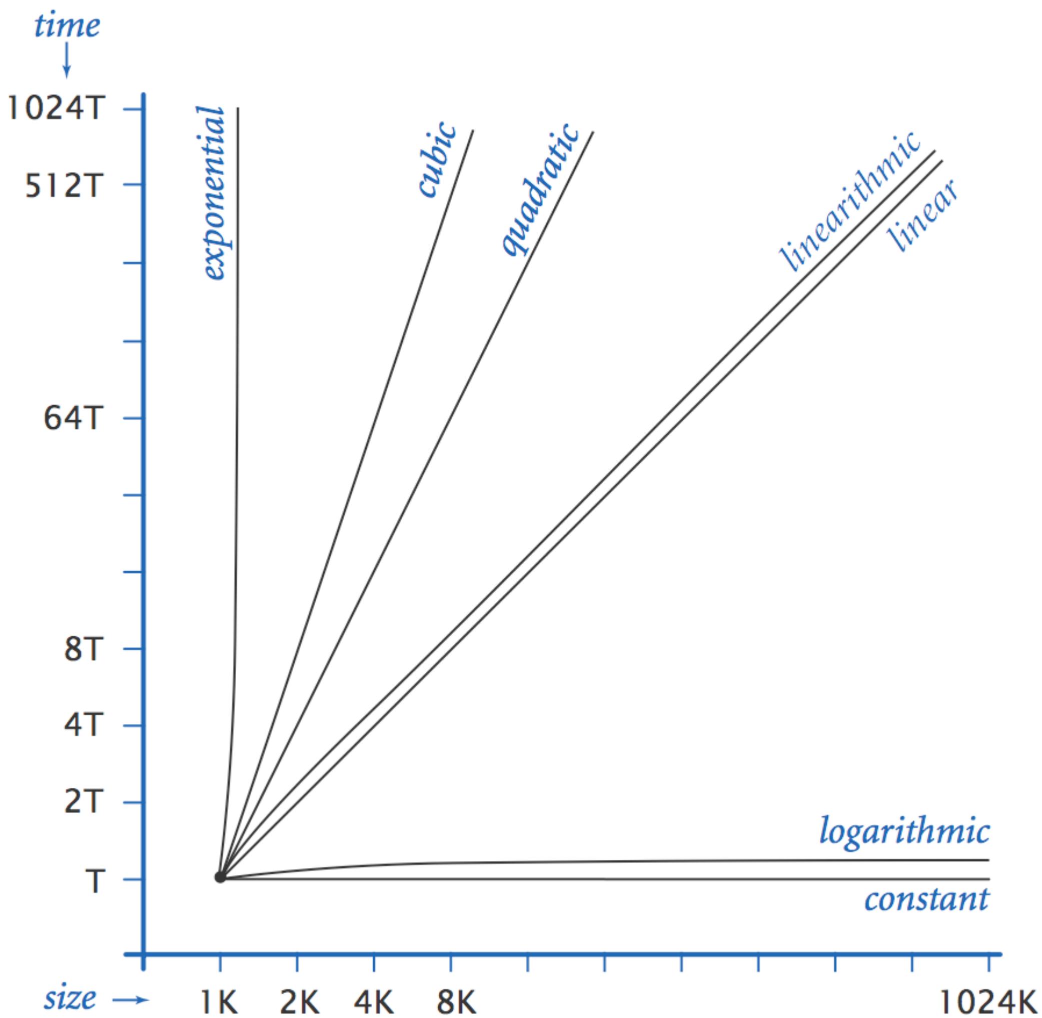
$$N \rightarrow N/2 \rightarrow N/4 \rightarrow N/8 \rightarrow \ldots \rightarrow 1$$

Q. How many times can you divide a number by 2 until you reach 1?

A. $\log_2 N$.

$$1$$
$$2 \rightarrow 1$$
$$4 \rightarrow 2 \rightarrow 1$$
$$8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$512 \rightarrow 256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$
$$1024 \rightarrow 512 \rightarrow 256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

Order of Growth Classifications



Orders of growth (log-log plot)

order of growth: description	order of growth: function	factor for doubling hypothesis
constant	1	1
logarithmic	$\log N$	1
linear	N	2
linearithmic	$N \log N$	2
quadratic	N^2	4
cubic	N^3	8
exponential	2^N	2^N

Commonly encountered growth functions

Order of Growth Classification

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

lg N

lg N = $\log_2$ N

```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

N lg N

```
for (int i = 0; i < N; i++)
   ...
```

N

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

2^N

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

N^2

Summary

Q. How can I evaluate the performance of my program?

A. Computational experiments, mathematical analysis

Q. What if it's not fast enough? Not enough memory?

- Understand why.
- Buy a faster computer.
- Learn a better algorithm (COS 226, COS 423).
- Discover a new algorithm.

attribute	better machine	better algorithm
cost	$$$ or more.	$ or less.
applicability	makes "everything" run faster	does not apply to some problems
improvement	incremental quantitative improvements expected	dramatic qualitative improvements possible

Typical Job Interview Question on Sorting (2008)

Q. What's the fastest way to sort 1 million 32-bit integers?

Eric Schmidt '76 (then CEO)