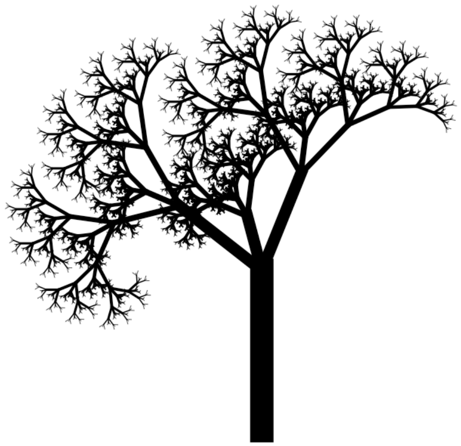
2.3 Recursion

Overview

What is recursion? When one function calls itself directly or indirectly.



Why learn recursion?

- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.

- Binary search, mergesort, FFT, GCD.
- Linked data structures.
- A folder contains files and other folders.

M. C. Escher, 1956

Closely related to mathematical induction.

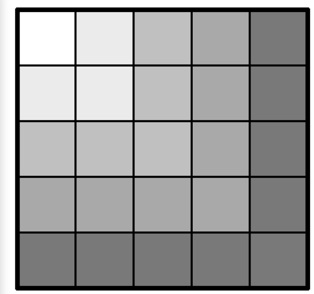
Mathematical Induction

Mathematical induction. Prove a statement involving an integer N by

- base case: Prove it for some specific N (usually 0 or 1).
- induction step: Assume it to be true for all positive integers less than N, use that fact to prove it for N.

Ex. Sum of the first N odd integers is N^2.

1	= 1
1 + 3	= 4
1 + 3 + 5	= 9
1 + 3 + 5 + 7	= 16
1 + 3 + 5 + 7 + 9	= 25
...	...



Base case: True for N = 1.

Induction step:

- Let T(N) be the sum of the first N odd integers: 1 + 3 + 5 + ... + (2N - 1).
- Assume that $T(N-1) = (N-1)^2$.
- $T(N) = T(N-1) + (2N - 1)$

$$= (N-1)^2 + (2N - 1)$$
$$= N^2 - 2N + 1 + (2N - 1)$$
$$= N^2$$

Recursive Program

Recursive Program. Implement a function having integer arguments by

- base case: Do something specific in response to “base” argument values.
- reduction step: Assume the function works for all smaller argument values, and use the function to implement itself for general argument values.

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}
```

automatic cast to `String` (either `"0"` or `"1"`)

Ex 1. Convert positive `int` to binary `String`.

Base case: return “1” for x = 1.

Reduction step:

- convert x/2 to binary
- append “0” if x even
- append “1” if x odd

```
    37         18
"100101" = "10010" + "1"
```

Recursive Program

Recursive Program. Implement a function having integer arguments by

- base case: Implementing it for some specific values of the arguments.
- reduction step: Assume the function works for smaller values of its arguments and use it to implement the function for the given values.

```
public class Binary
{
   public static String convert(int x)
   {
      if (x == 1) return "1";
      return convert(x/2) + (x % 2);
   }

   public static void main(String[] args)
   {
      int x = Integer.parseInt(args[0]);
      System.out.println(convert(x));

   }

}
```

```
% java Binary 6
110
% java Binary 37
100101
% java Binary  999999
11110100001000111111
```

x = 6

environment

convert(6)

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x / 2) + (x % 2);
}
```

"110"

```
public class Binary
{
   public static String convert(int x)
   {
      if (x == 1) return "1";
      return convert(x/2) + (x % 2);
   }

   public static void main(String[] args)
   {
      int x = Integer.parseInt(args[0]);
      System.out.println(convert(x));
   }
}
```

"110"

```
% java Binary 6
110
```

Recursion vs. Iteration

Every program with 1 recursive call corresponds to a loop.

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}
```

```
public static String convertNR(int x)
{
   String s = "1";
   while (x > 1)
   {
      s = (x % 2) + s;
      x = x/2;
   }
   return s;
}
```

Reasons to use recursion:

- code more compact
- easier to understand
- easier to reason about correctness
- easy to add multiple recursive calls (stay tuned)

Reasons not to use recursion: (stay tuned)

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$$
\begin{aligned}
4032 &= 2^6 \times 3^2 \times 7^1 \\
1272 &= 2^3 \times 3^1 \times 53^1 \\
\gcd(4032, 1272) &= 2^3 \times 3^1 = 24
\end{aligned}
$$

Applications.

- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.

Greatest Common Divisor

GCD. Find largest integer that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \% \ q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

```
gcd(4032, 1272)    = gcd(1272, 216)
                   = gcd(216, 192)
                   = gcd(192, 24)
                   = gcd(24, 0)
                   = 24.
```

4032 = 3 × 1272 + 216

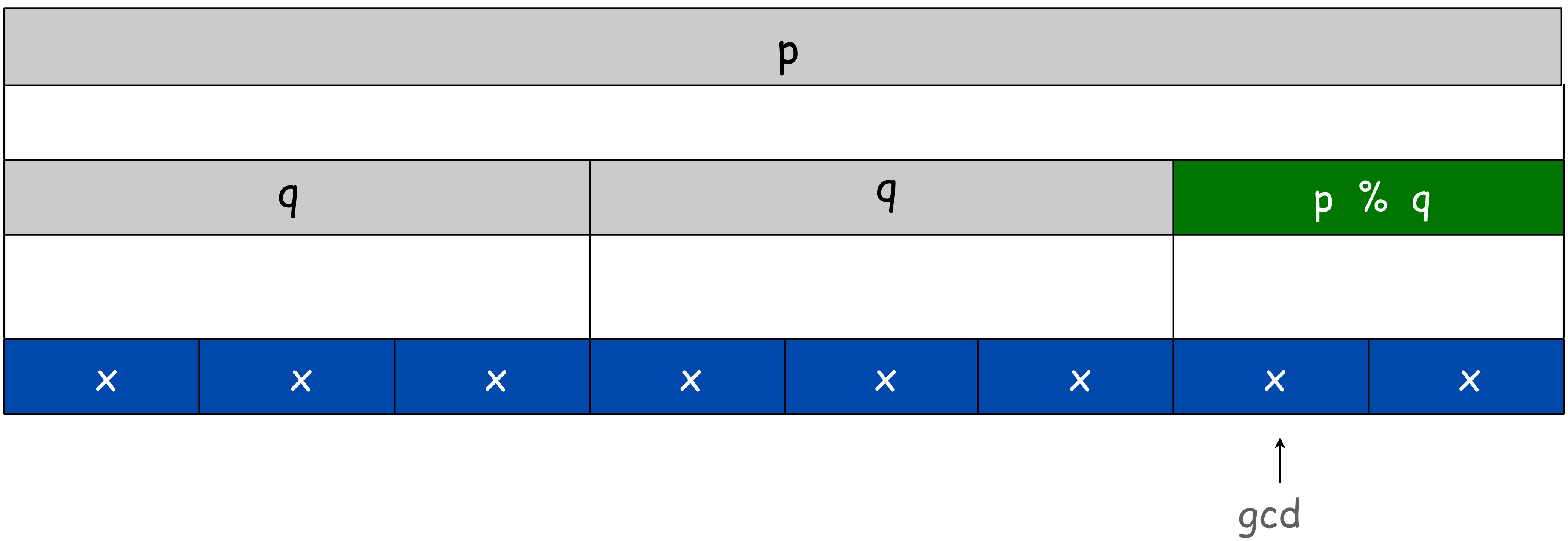
Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \%\ q) & \text{otherwise} \end{cases}$$

$\longleftarrow$ base case

$\longleftarrow$ reduction step, converges to base case



p = 8x
q = 3x

gcd(p, q) = gcd(3x, 2x) = x

Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \% \ q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

Recursive program

```
public static int gcd(int p, int q)
{
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

← base case

← reduction step

p = 1272, q = 216

environment

gcd(1272, 216)

```
static int gcd(int p, int q)
{
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

24

```
public class Euclid
{
   public static int gcd(int p, int q)
   {
      if (q == 0) return p;
      else return gcd(q, p % q);
   }

   public static void main(String[] args)
   {
      int p = Integer.parseInt(args[0]);
      int q = Integer.parseInt(args[1]);
      System.out.println(gcd(p, q));
   }
}
```

24

```
% java Euclid 1272 216
24
```

Possible debugging challenges with recursion

Missing base case.

```
public static double BAD(int N)
{
   return BAD(N-1) + 1.0/N;
}
```

No convergence guarantee.

```
public static double BAD(int N)
{
   if (N == 1) return 1.0;
   return BAD(1 + N/2) + 1.0/N;
}
```

Both lead to INFINITE RECURSIVE LOOP (bad news).

Try it!

so that you can recognize and deal with it if it later happens to you

Collatz Sequence

Collatz sequence.

- If n is 1, stop.
- If n is even, divide by 2.
- If n is odd, multiply by 3 and add 1.

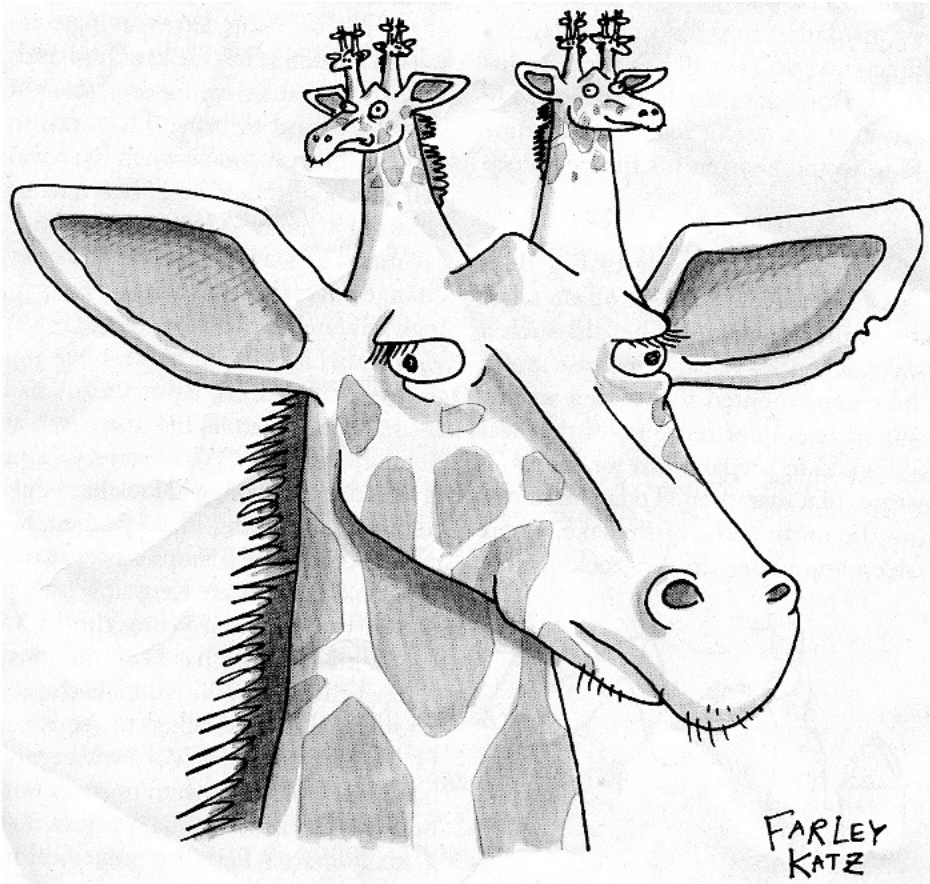
Ex. 35 106 53 160 80 40 20 10 5 16 8 4 2 1.

```
public static void collatz(int N)
{
    StdOut.print(N + " ");
    if (N == 1) return;
    if (N % 2 == 0) collatz(N / 2);
    else collatz(3 * N + 1);
 }
```

No one knows whether or not this function terminates for all N (!)

[usually we decrease N for all recursive calls]

Recursive Graphics



New Yorker Magazine, August 11, 2008

WEEKEND **Arts** FINE ARTS LEISURE

N E41

FRIDAY, DECEMBER 15, 2006

The New York Times

Design Life Now *at the Cooper-Hewitt National Design Museum includes this toy from the New York company Kidrobot.*

Fruits of Design, Certified Organic

ROBERTA SMITH

ART REVIEW

It's Triennial time at the Cooper-Hewitt National Design Museum. This means that the former Andrew Carnegie mansion is up to its neck in mostly American design from the last three years. Like its predecessors, "Design Life Now," the museum's third National Design Triennial, is a crazed affair that illuminates a volatile, contradictory, ever-expanding field but fails to call it to order.

The exhibition has been organized by the Cooper-Hewitt curators Barbara Bloemink, Ellen Lupton and Matilda McQuaid and a guest, Brooke Hodge, a curator at the Museum of Contemporary Art, Los Angeles.

Once again the Triennial answers the question "What's design?" with the evasive catchall "What's not?" Covering so many bases so equivocally, it never gets around to tackling the weightier questions of "What is good design?" or, more to the point, "What is design good for?" It refuses to take sides on the issue of whether design should aim for social or environmental benefit or serve a relatively decorative purpose. Still, the show's benefits are many, even if you have to work for them.

The displays here range from genius to schlock, delightful to dispiriting. They cover life-extending innovations, completely frivolous reiterations of received ideas (far too many of which trace to Surrealism) and more varieties of recycling than you can easily count. Fashion, building materials, furniture, toys, theatrical sets, jewelry and textiles, medical and military hardware, all qualify as design according to this exhibition.

The main point comes across loud and clear: design permeates every aspect of contemporary life. Everything that exists is designed, whether natural or cultural. And while all of nature's designs are intelligent, whether you go by Darwin or the Bible, the human kind are much

Continued on Page 51

Lars Klove for The New York Times

From "The Yale Book of Quotations" to "Postcards From Mars," a selection of the best holiday books.

The Gifts to Open Again and Again

WILLIAM GRIMES

BOOKS

I've made my list, and I'm checking it twice. It's a list of the qualities that make the ideal holiday book, and after carefully considering the books of Christmas past, I have come up with some guidelines. A gift book should either be no surprise or a big surprise: the one you always wanted or the one you never knew you wanted. It should either be expensive and large, or cheap and small. It should be high-minded or totally frivolous. And no matter what, it should not require sustained attention, which is impossible during the yuletide season. My gift selections, chosen entirely at random but with exquisite taste, satisfy at least two of these requirements.

Let's open the big presents first. The season's whopper, in every way, is "New York 2000," the fifth installment in Robert A. M. Stern's architectural history of New York. The series starts in 1880, when 10 stories qualified as a skyscraper, and has now caught up to the new millennium. Taken together, the volumes make an enormous, endlessly fascinating family scrapbook for New Yorkers, who can coo over baby pictures of the Flatiron Building and leaf forward, through many hundreds of pages and thousands of photographs, to the big, grown-up New York of the Lipstick Building, countless Trump projects and the new Tweed Courthouse.

At 1,520 pages and 10 pounds 12 ounces, "New York

Continued on Page 46

Serkan Ozkaya

Black, White and Read All Over Over

By RANDY KENNEDY

In one of Jorge Luis Borges's best-known short stories, "Pierre Menard, Author of the Quixote," a 20th-century French writer sets out to compose a verbatim copy of Cervantes's 17th-century masterpiece simply because he thinks he can, originality perhaps not being all it's cracked up to be.

He manages two chapters word for word, a spontaneous duplicate that Borges's narrator finds to be "infinitely richer" than the original because it contains all manner of new meanings and inflections, wrenched as it is from its proper time and context.

When a young Turkish artist named Serkan Ozkaya set out recently to practice his skills as a copyist — a scrivener, as he says — his goals were a little less ambitious than channeling Cervantes. He simply wanted to draw and see printed a faithful copy of all the type and pictures planned for a broadsheet page of this newspaper: this very page you are reading right now, which shows his version of the page you are reading right now, which shows his version of the page you are reading right now, which. . . .

Do not be alarmed: There has been no break in the

Serkan Ozkaya's drawing of the page you are reading right now, showing his drawing of the page you are reading right now, showing. . . .

space-time-newsprint continuum.

Mr. Ozkaya, a 33-year-old artist who lives and works in Istanbul, did not propose this exercise in handmade surrealism because of a particular love of calligraphy or newspapers or, for that matter, even drawing, which he admits he is not very good at. The

Continued on Page 51

Divine and Devotee Meet Across Hinges

HOLLAND COTTER

ART REVIEW

WASHINGTON — For toothache, dial St. Apollonia ASAP. She'll bring relief in a flash. Keep St. Matthew, ex-banker, in mind in April; he'll help get your taxes in shape. Everyone knows that a prayer to St. Roch, protector from plague, is as good as a flu shot, and that lightning will never strike when St. Barbara's on the job.

Most important, for dire and intangible problems — moral confusion, inconsolable grief, sickness of soul — there's the Virgin. Day and night she's on the toll-free hot line offering gentle attention and prudent advice.

To European Christians half a millennium ago, the Virgin and a raft of familiar saints were the exalted personnel in a kind of celestial welfare system, available to all believers. And one quick way to access its benefits was through devotional painting of the kind found in "Prayers and Portraits: Unfolding the Netherlandish Diptych" at the National Gallery of Art.

Probably nothing in Western art comes closer to formal perfection than these pictures, produced by the likes of Jan van Eyck, Rogier van der Weyden and Hugo van der Goes across an area that now encompasses the Netherlands, Belgium, Luxembourg and parts of France. These painters were pictorial magicians, creating visual worlds, cosmically abstract and microscopically realistic, of peerless breadth.

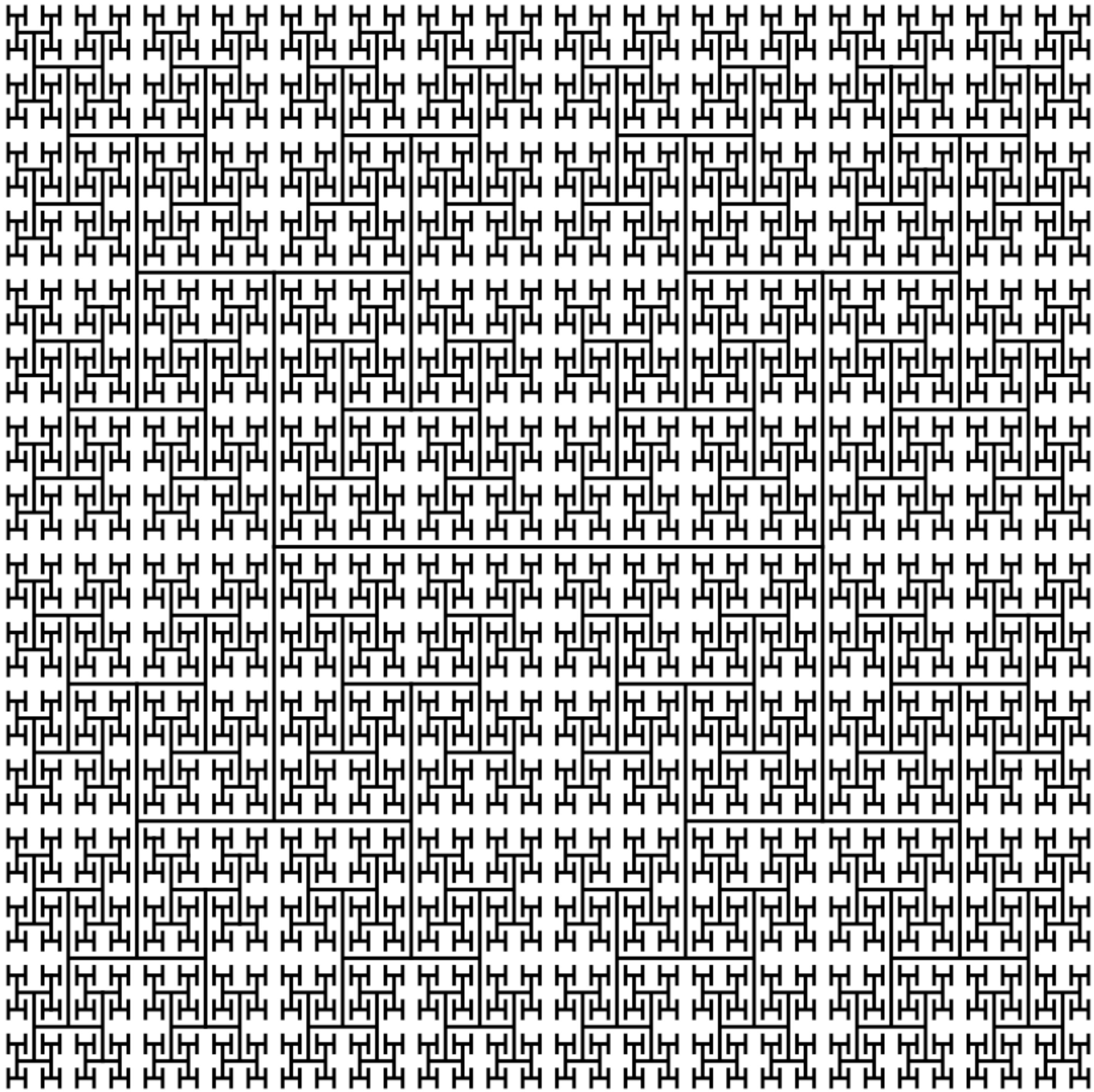
You see all of this in one glance at the 40 double-panel paintings, or diptychs, here. Then you learn gradually as you move through the show how diptych paintings have been unmade and remade, broken up and reconfigured, over the centuries, with the result that few survive in their intended form.

"Prayers and Portraits" is an attempt to restore that form, at least to a few of them. It brings art historians and art

Continued on Page 44

Left, Gemäldegalerie, Berlin; National Gallery of Art, Washington

Prayers and Portraits: *Unfolding the Netherlandish Diptych.* *Two panels of an early 16th-century diptych by Michel Sittow, left, are reunited in an exhibition at the National Gallery of Art in Washington through Feb. 4.*

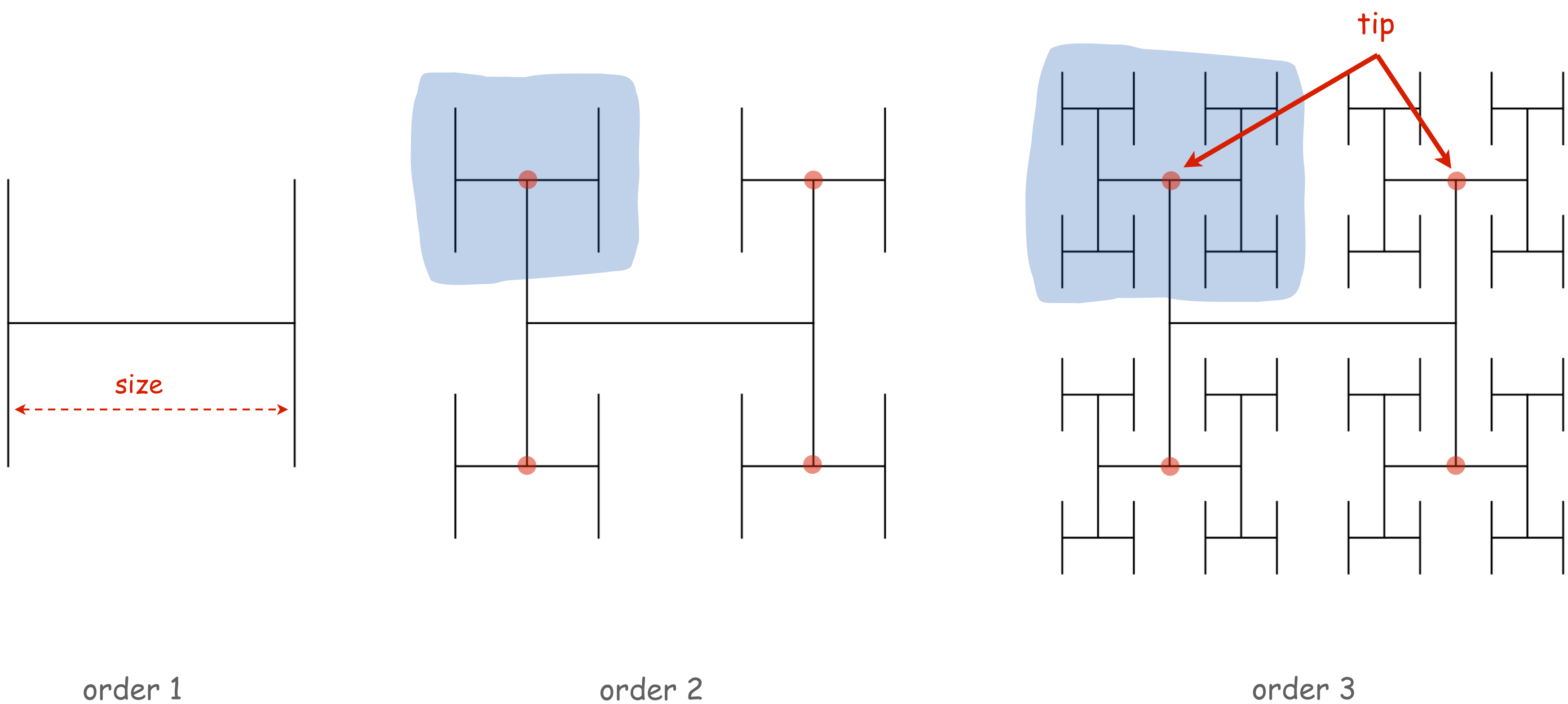


Htree

H-tree of order n.

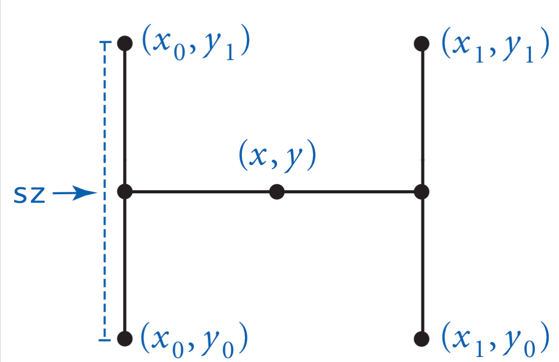
- Return if n is 0
- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip.

and half the size



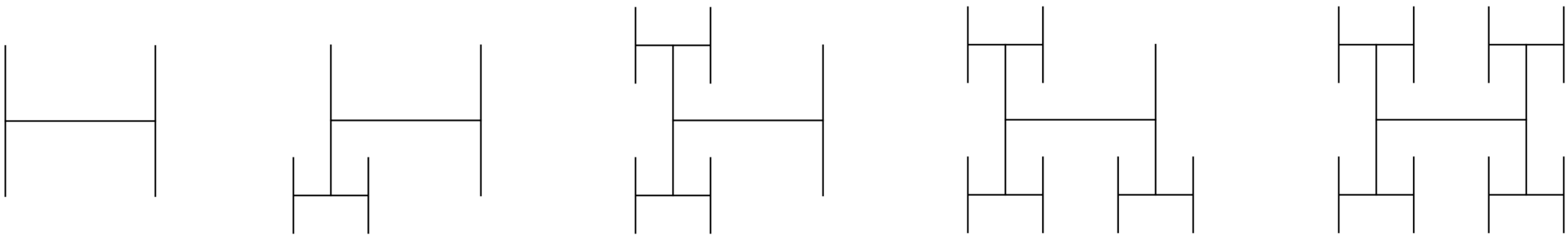
Htree in Java

```
public class Htree
{
   public static void draw(int n, double sz, double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;

      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);     ←  draw the H, centered on (x, y)
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);          ←  recursively draw 4 half-size Hs
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }

   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

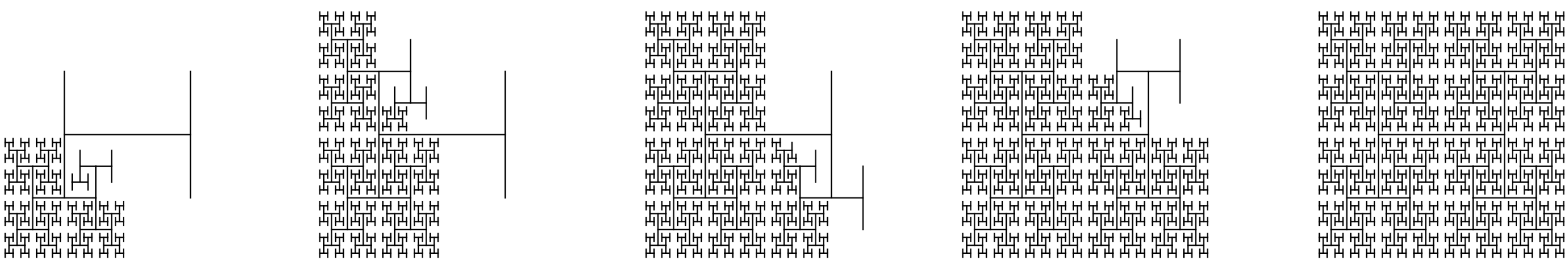


Animated H-tree

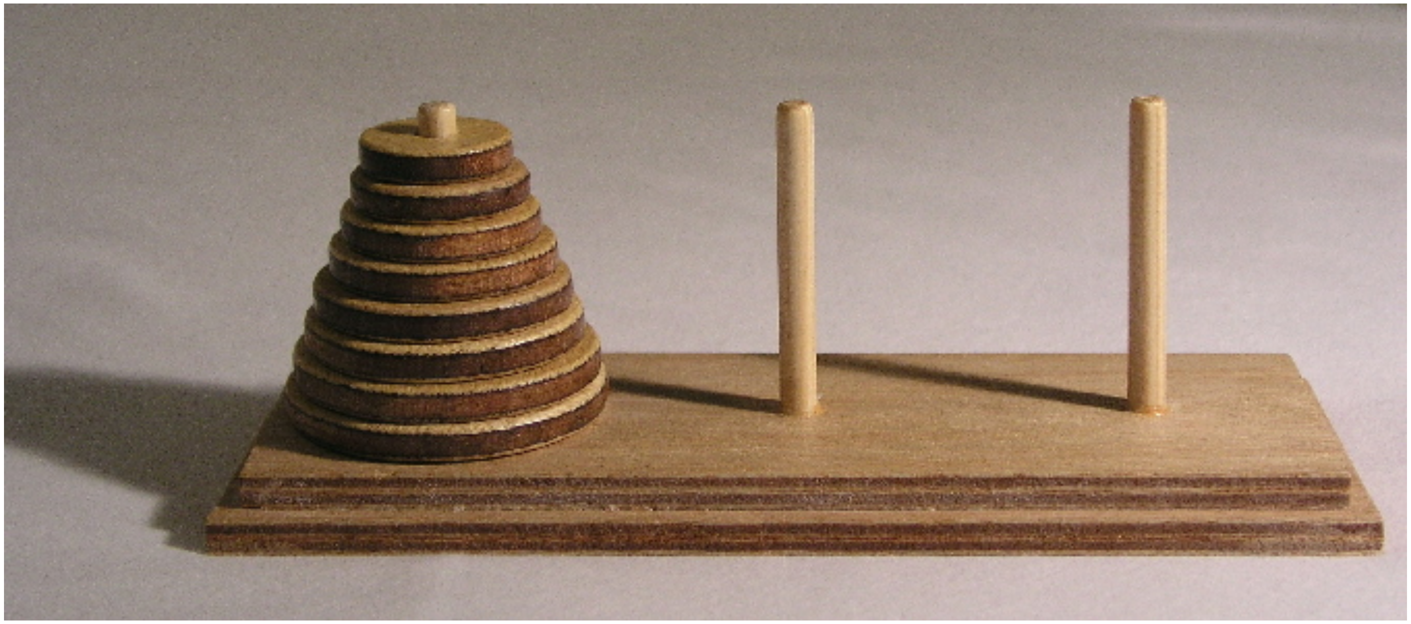
Animated H-tree. Pause after drawing each H.



20% 40% 60% 80% 100%



Towers of Hanoi

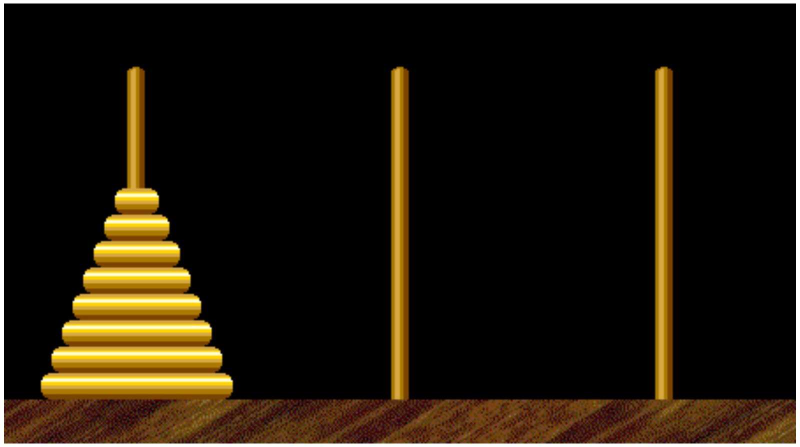


http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

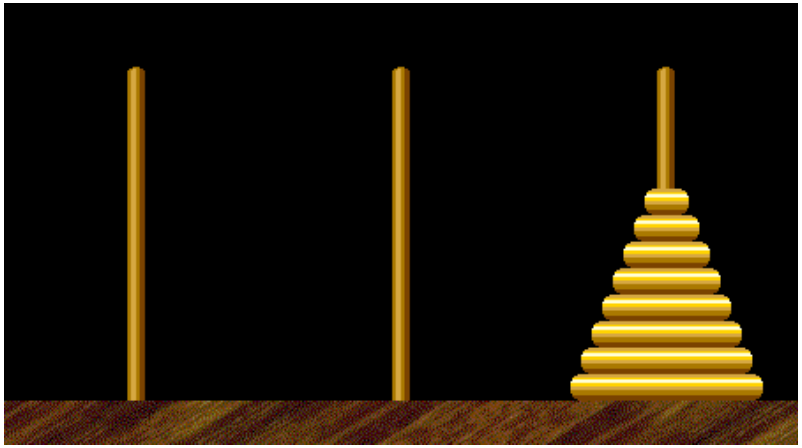
Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



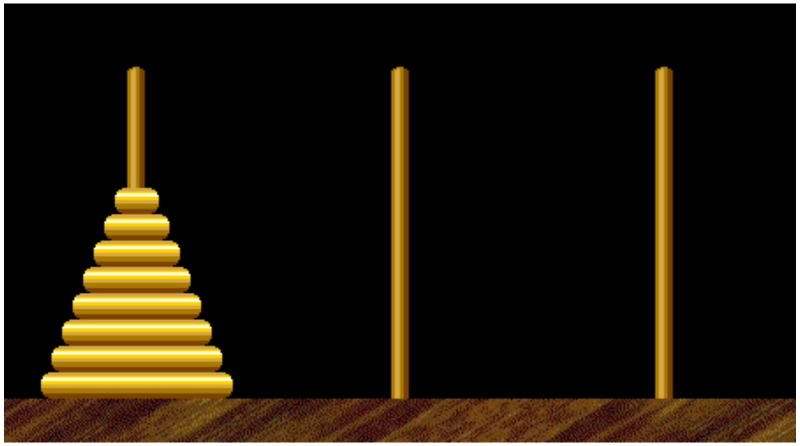
start



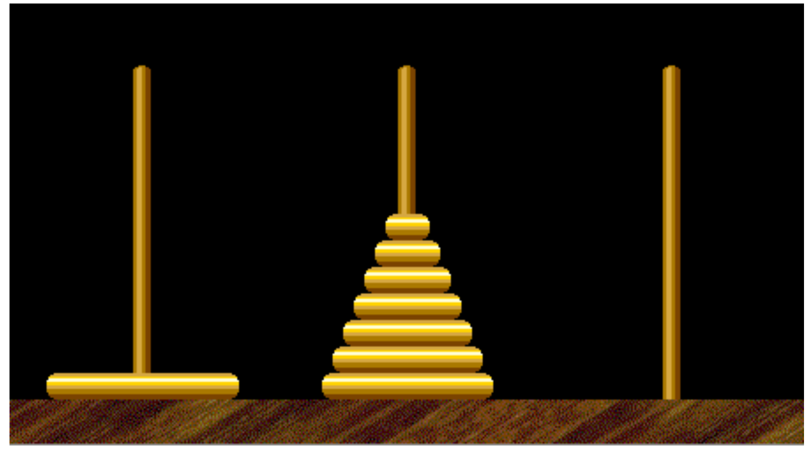
finish

Edouard Lucas (1883)

Towers of Hanoi: Recursive Solution

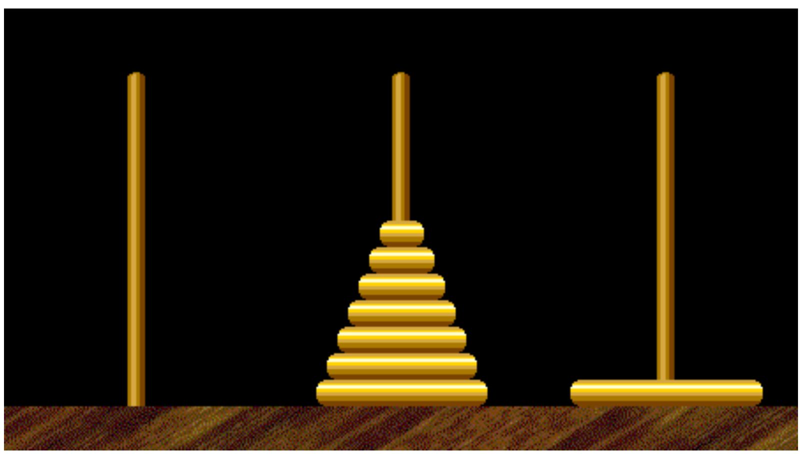


Move n-1 smallest discs right.

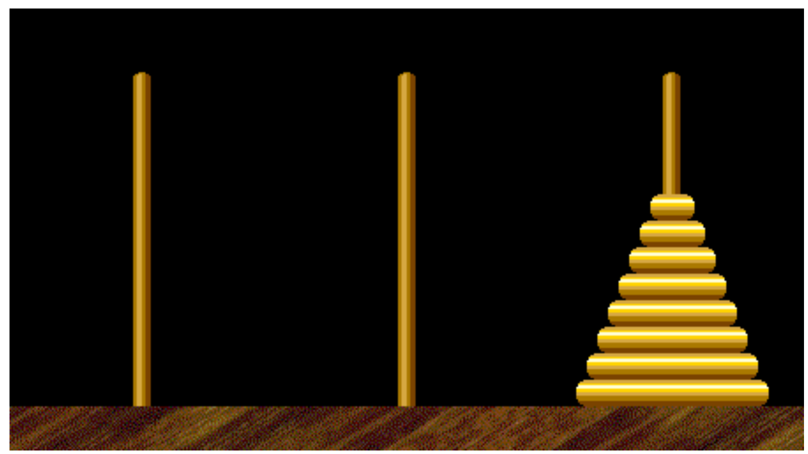


Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.



Towers of Hanoi Legend

Q. Is world going to end (according to legend)?

- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi
{
   public static void moves(int n, boolean left)
   {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

`moves(n, true)` : move discs 1 to n one pole to the left

`moves(n, false)`: move discs 1 to n one pole to the right

smallest disc

Towers of Hanoi: Recursive Solution

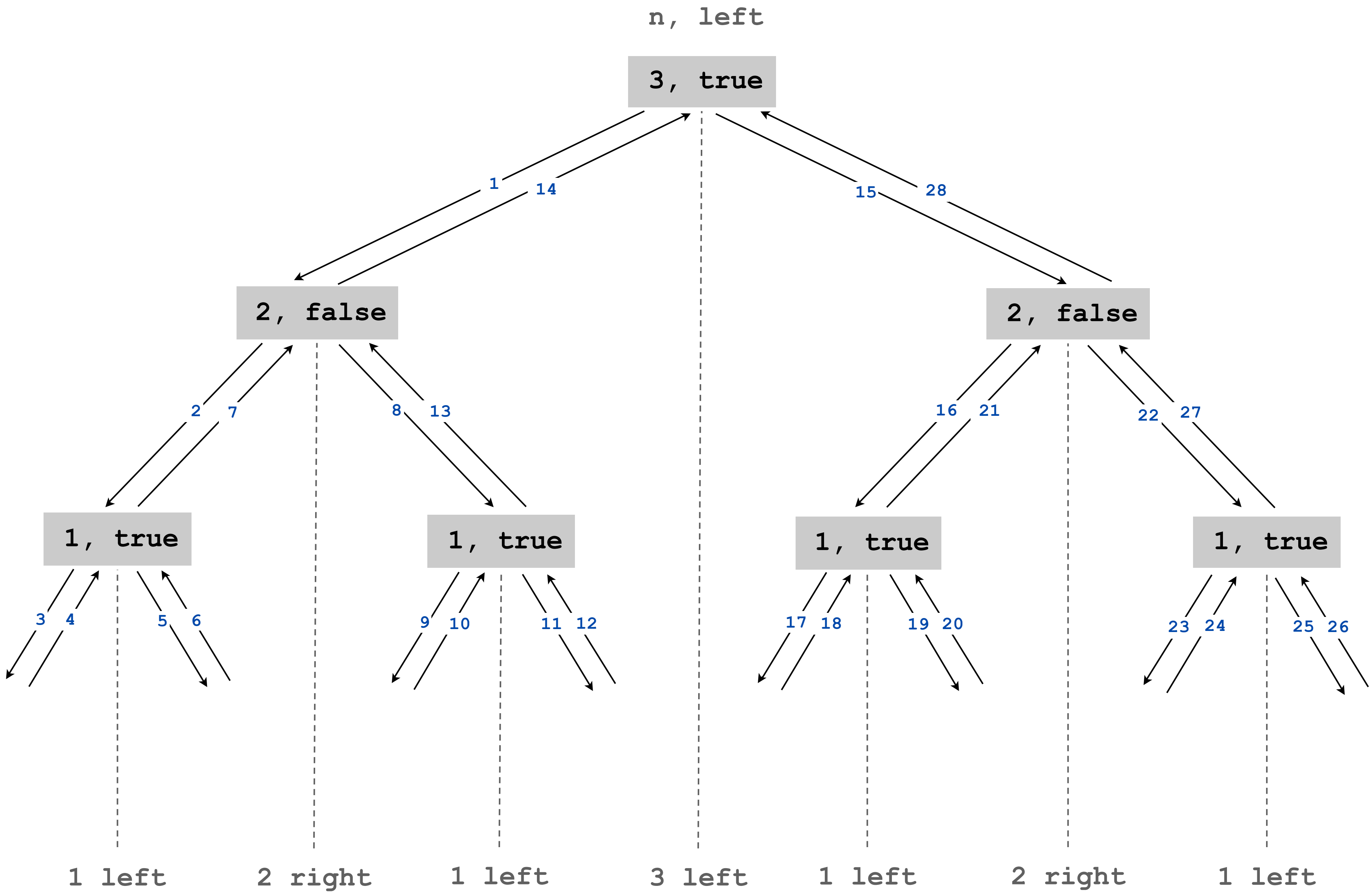
```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions
of
ruler

Towers of Hanoi: Recursion Tree



Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.

- Takes $2^n - 1$ moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!

- Alternate between two moves:
  - move smallest disc to right if n is even
  - make only legal move not involving smallest disc

to left if n is odd

Recursive algorithm may reveal fate of world.

- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Divide-and-Conquer

Divide-and-conquer paradigm.

- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

Many important problems succumb to divide-and-conquer.

- Midpoint displacement method for fractional Brownian motion.
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.

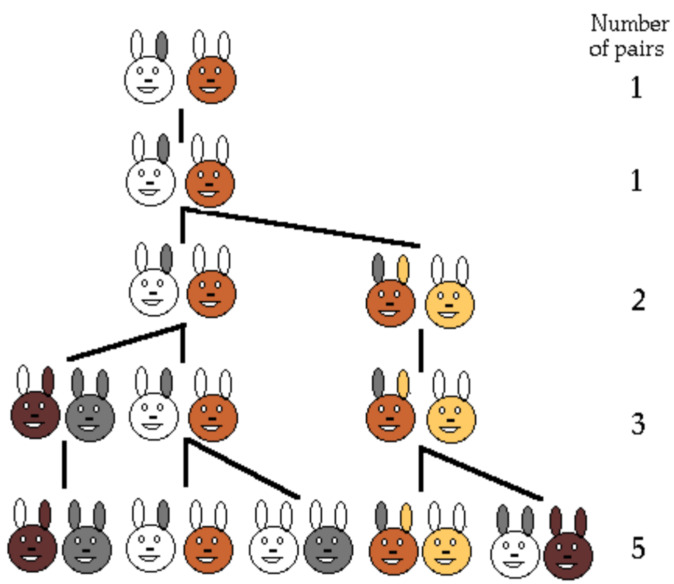
Divide et impera. Veni, vidi, vici. - Julius Caesar

Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

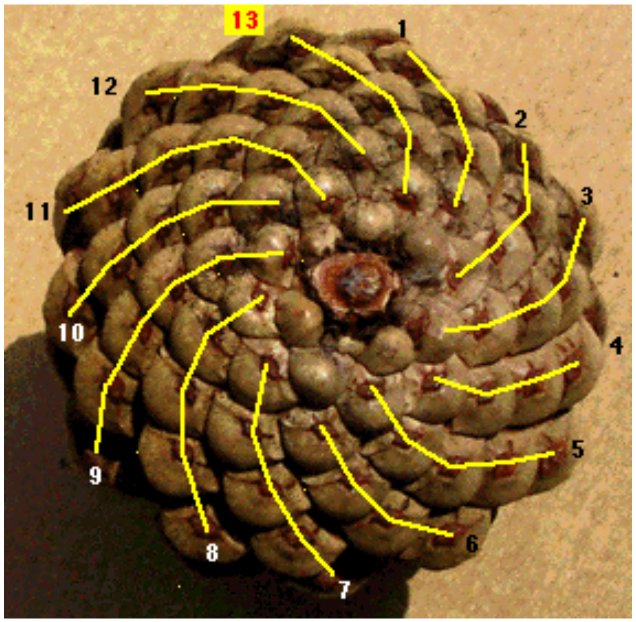
$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$



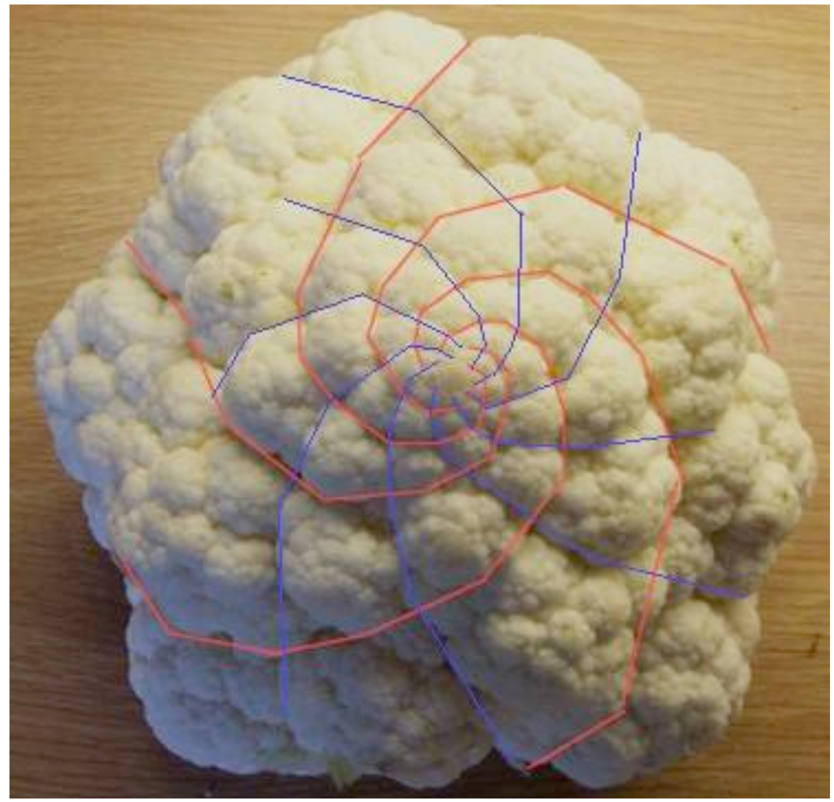
Fibonacci rabbits

L. P. Fibonacci
(1170 - 1250)

Fibonacci Numbers



pinecone



cauliflower

see much, much more at www.youtube.com/user/Vihart

A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

FYI (classical math):

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rceil$$

ϕ = golden ratio $\approx$ 1.618

Ex: F(50) ≈ 1.2×10^{10}

A natural for recursion?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

Recursion Challenge 1 (difficult but important)

Is this an efficient way to compute F(50)?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

Recursion Challenge 2 (easy and also important)

Is this an efficient way to compute F(50)?

```
long[] F = new long[51];
F[0] = 0; F[1] = 1;
if (n == 1) return 1;
for (int i = 2; i <= 50; i++)
   F[i] = F[i-1] + F[i-2];
```

Summary

How to write simple recursive programs?

- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Why learn recursion?

- New mode of thinking.
- Powerful programming tool.

Divide-and-conquer. Elegant solution to many important problems.

Exponential time.

- Easy to specify recursive program that takes exponential time.
- Don't do it unless you plan to (and are working on a small problem).