COS 423 Challenge Problem 1, Version 2 Due Wednesday March 12
Spring 2014
Tarjan

Collaboration of up to 3 people allowed. List your collaborators, if any. "Collaboration" means you can discuss ideas with your collaborators, but your write-up should be entirely your own. As always, cite any sources (papers, websites, etc.)

Note: I have changed the problem slightly: At the start, the battery has charge 0 rather than B. This change affects only the "A-" level of the problem, in which the battery capacity B is infinite (or sufficiently large): if B is sufficiently large, and the battery starts fully charged, the car can get from s to t if and only if there is *some* path from s to t, independent of the arc costs. It is easy to reduce the original version of the problem, in which the battery starts fully charged, to Version 2. (How?)

(The "electric car" problem) Given is a directed graph, each arc (v, w) of which has a cost that can be positive, zero, or negative. Also given are two vertices, a source vertex s and a destination vertex t. Finally, a non-negative battery capacity B is given. Devise the most efficient algorithm you can to answer the following question: given an electric car starting at vertex s with an initial battery charge of 0, can the car be driven to vertex t? The battery charge b changes as follows: If the car is at vertex v, it can move from v to w along arc (v, w) provided that $b \geq c(v, w)$. On arrival at w, the battery charge is $\min\{B, b - c(v, w)\}$. The battery charge is always between 0 and B (inclusive).

That is, if $c(v, w) = 0$, the car can move from v to w without losing (or gaining) any battery charge. If $b \geq c(v, w) > 0$, the car can move from v to w but its charge drops to $b - c(v, w)$. If $c(v, w) < 0$ and $b - c(v, w) \leq B$, the car can move from v to w while increasing its charge to $b - c(v, w)$. Finally, if $c(v, w) < 0$ and $b - c(v, w) > B$, the car can move from v to w while increasing its charge, but only to the maximum, B.

In particular, if the current charge is 0 the car can move only along an arc of non-positive cost. If the battery charge is B, the charge remains at B if the car moves along an arc of non-positive cost. If $B < c(v, w)$, the car cannot move along (v, w) no matter what charge the battery has at v. Thus one can delete all arcs (v, w) with $B < c(v, w)$ without changing the problem.

For A level credit, produce an algorithm whose worst-case running time is at most a fixed-degree polynomial in n, the number of vertices, and m, the number of arcs, even if B and the arc capacities are arbitrary real numbers (assuming addition, subtraction, and comparison of two such numbers takes constant time), or provide a reason why such an algorithm might not exist.

For A- level credit, consider the special case of the problem in which *B* is large enough that with a fully charged battery the car can reach any reachable vertex, independent of the arc costs. Give a simple upper bound on the necessary battery capacity for this to be the case, as a function of *n*, *m*, and/or *C*, the maximum arc cost. Produce an algorithm for this special case whose worst-case running time is at most a fixed-degree polynomial in *n* and *m*, even if *B* and the arc capacities are arbitrary real numbers (assuming addition, subtraction, and comparison of two such numbers takes constant time).

For B+ level credit, produce an algorithm for the general problem (B arbitrary), whose worst-case running time is polynomial in *n*, *m*, and log*B*, assuming *B* and all the arc costs are integers.

Note that an A level solution is both an A- level solution and a B+ level solution. One may be able to obtain a faster and/or simpler A- level and/or B+ level solution, however, so I encourage you to think about the A- level and the B+ level even if you have an A level solution, and you will get extra credit for an A- and/or B+ level solution that is in some way better than your A level solution.