



Scene Graphs & Modeling Transformations

COS 426

3D Object Representations

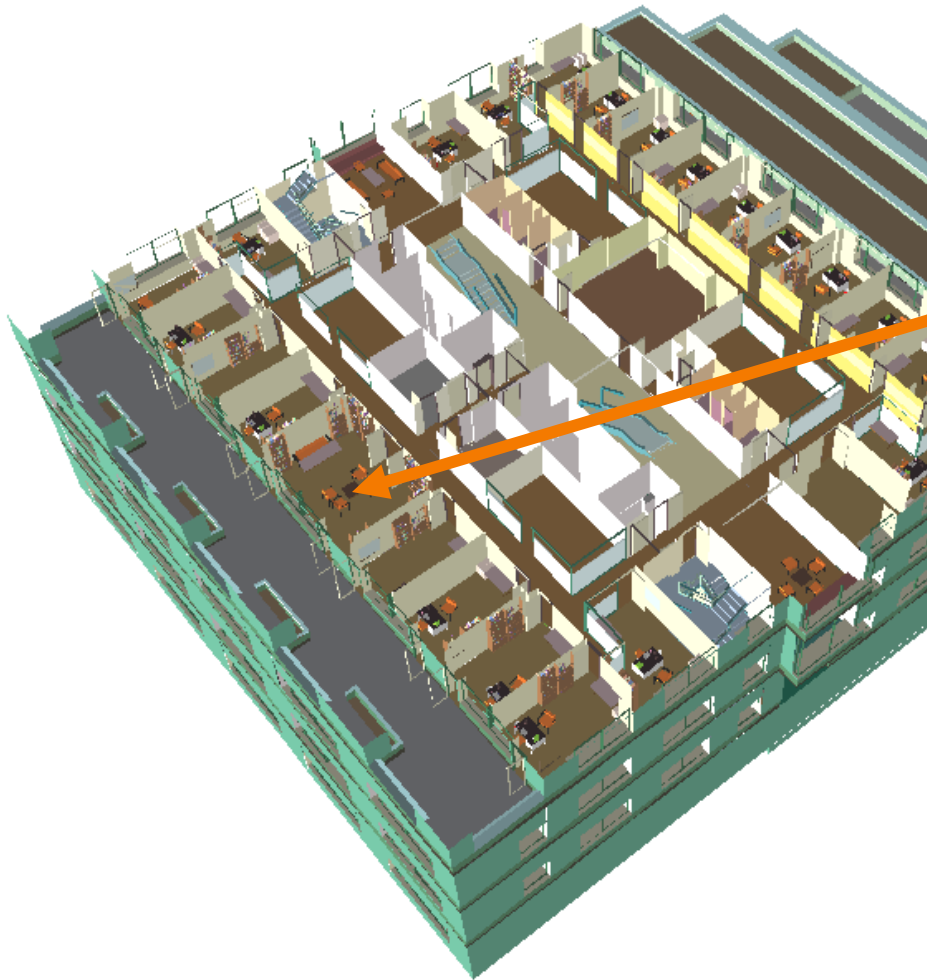


- Points
 - Range image
 - Point cloud
- Surfaces
 - Polygonal mesh
 - Subdivision
 - Parametric
 - Implicit
- Solids
 - Voxels
 - BSP tree
 - CSG
 - Sweep
- High-level structures
 - Scene graph
 - Application specific

3D Object Representations



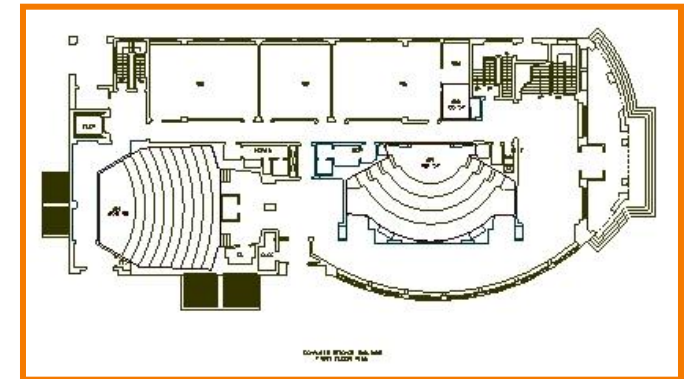
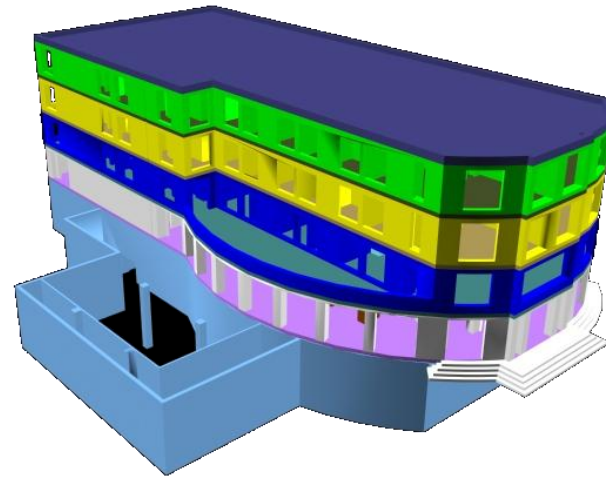
- What object representation is best for this?



3D Object Representations



- Desirable properties of an object representation
 - Easy to acquire
 - Accurate
 - Concise
 - Intuitive editing
 - Efficient editing
 - Efficient display
 - Efficient intersections
 - Guaranteed validity
 - Guaranteed smoothness
 - etc.



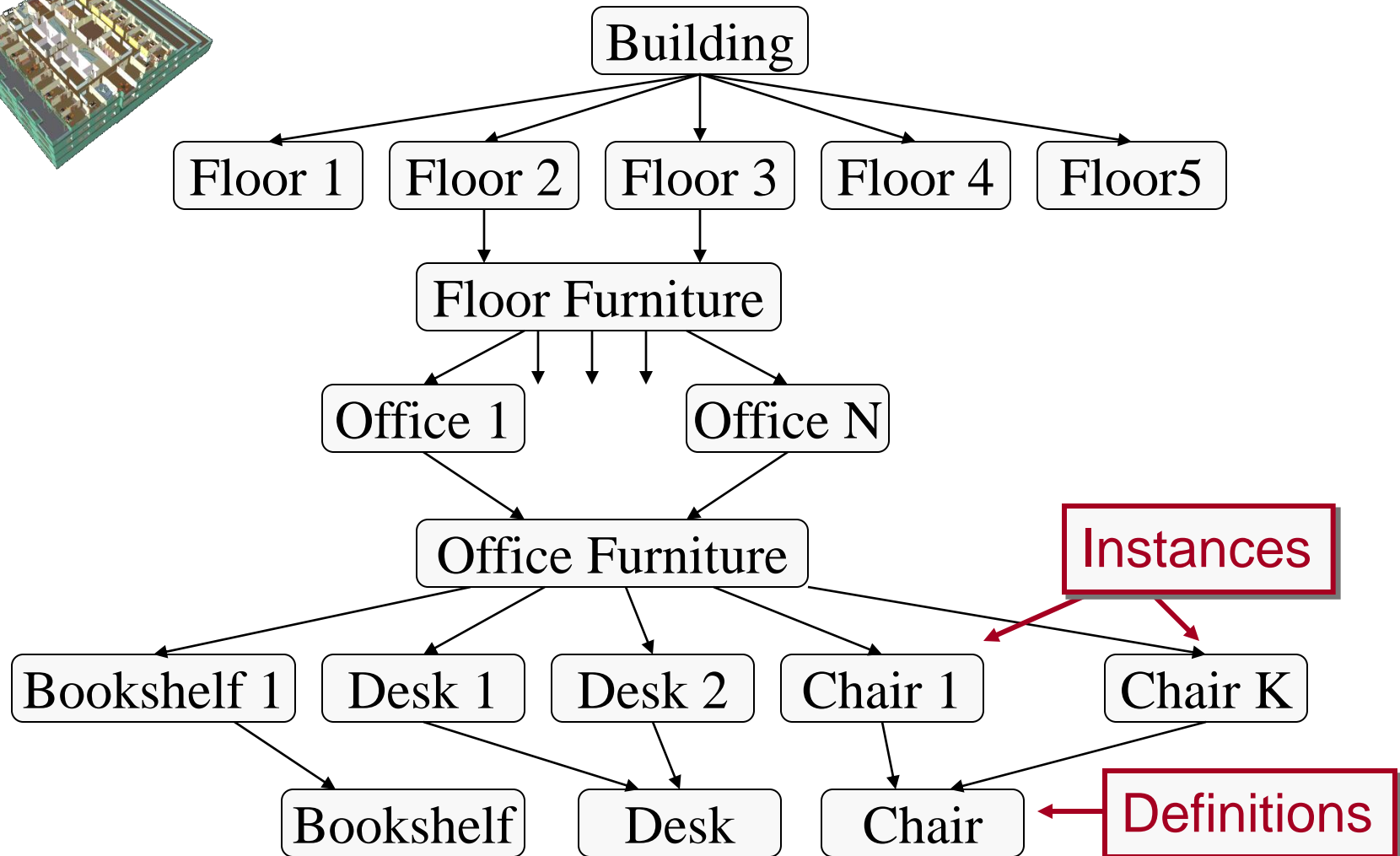
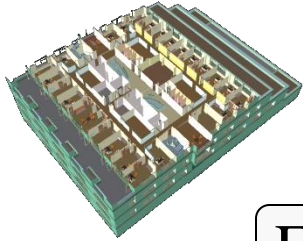
(CS Building, Princeton University)

Overview



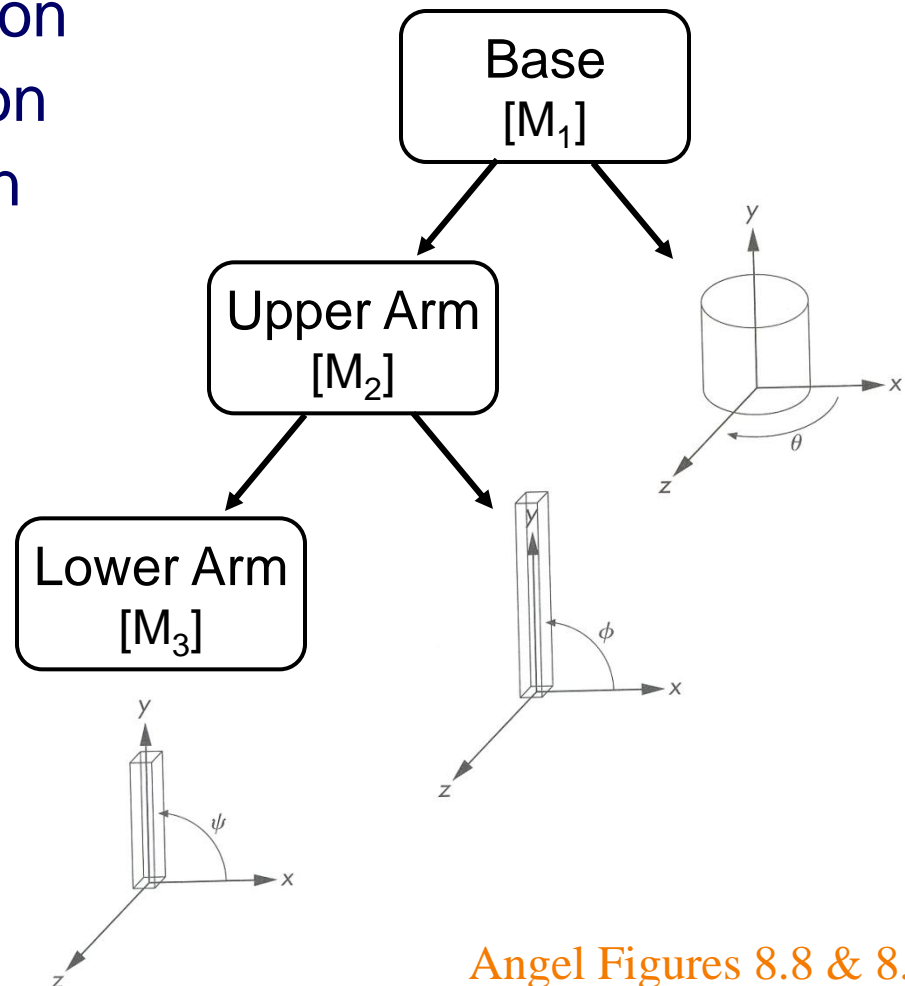
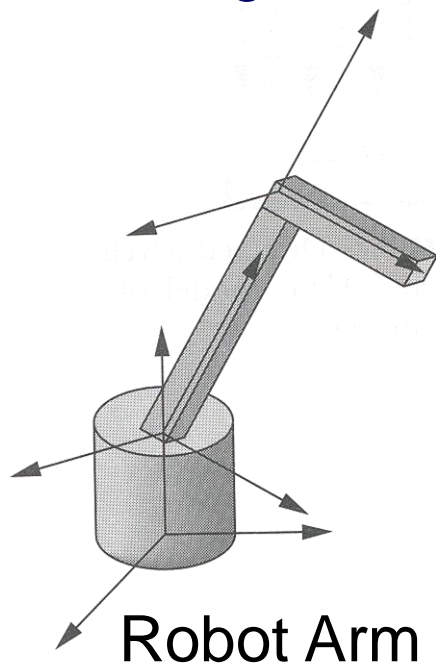
- Scene graphs
 - Geometry & attributes
 - Transformations
 - Bounding volumes
- Transformations
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
 - 3D transformations

Scene Graphs



Scene Graphs

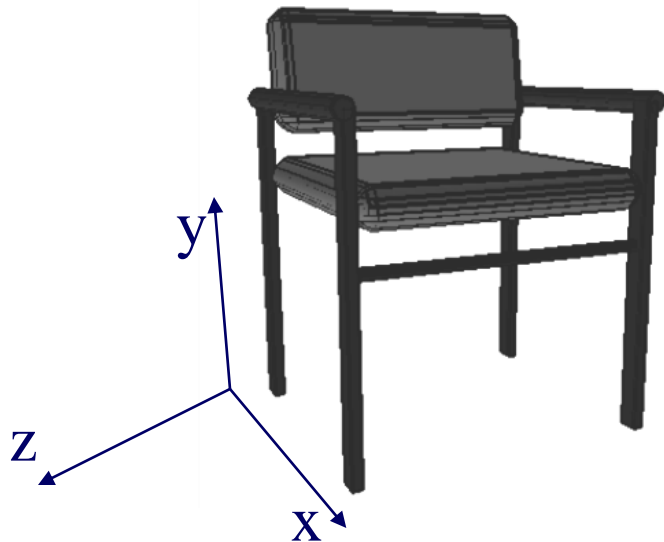
- Hierarchy (DAG) of nodes, where each may have:
 - Geometry representation
 - Modeling transformation
 - Parents and/or children
 - Bounding volume



Scene Graphs



- Advantages
 - Allows definitions of objects in own coordinate systems

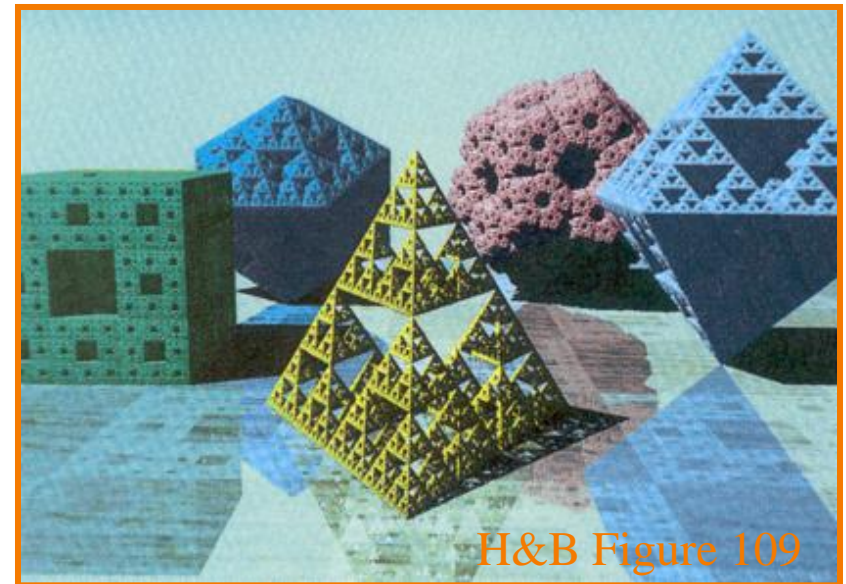
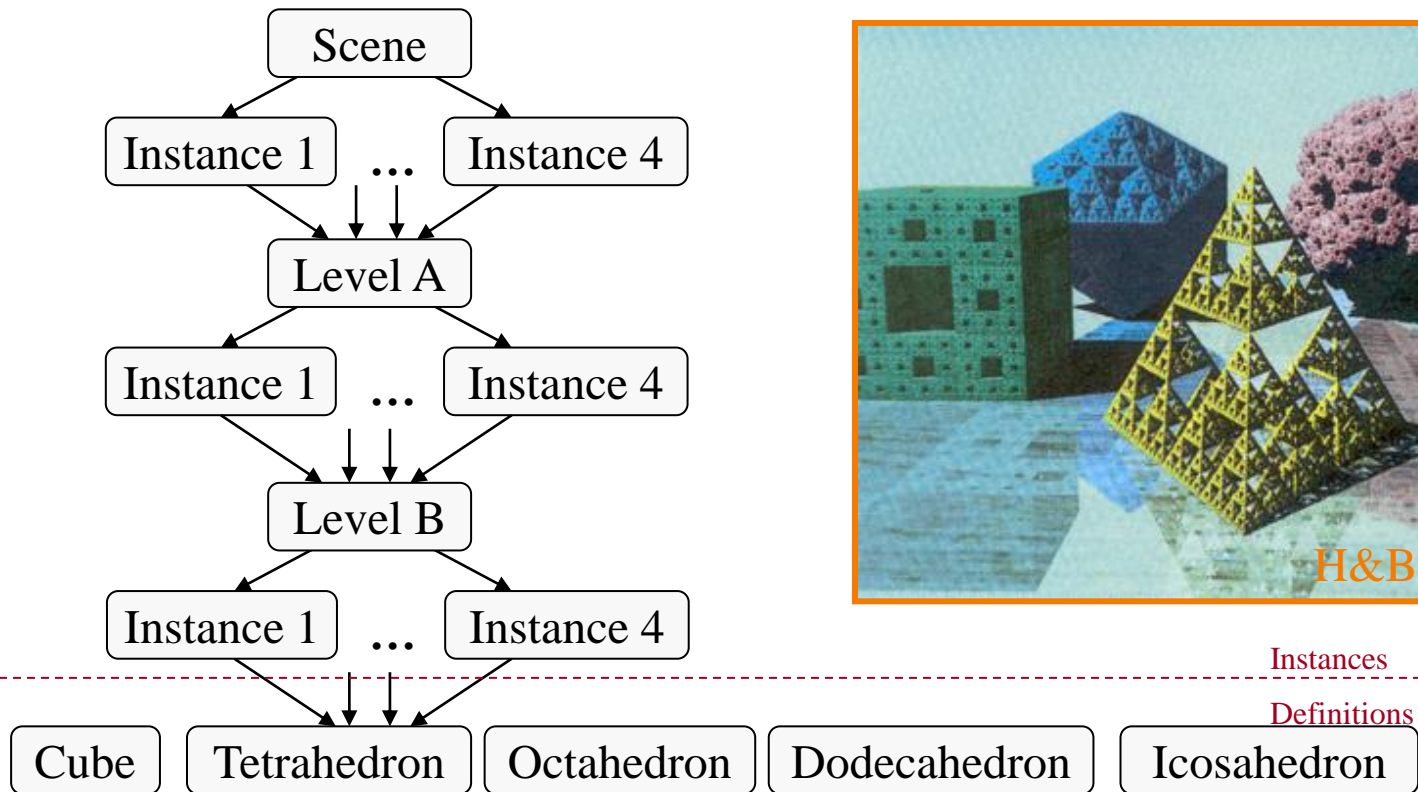


Scene Graphs



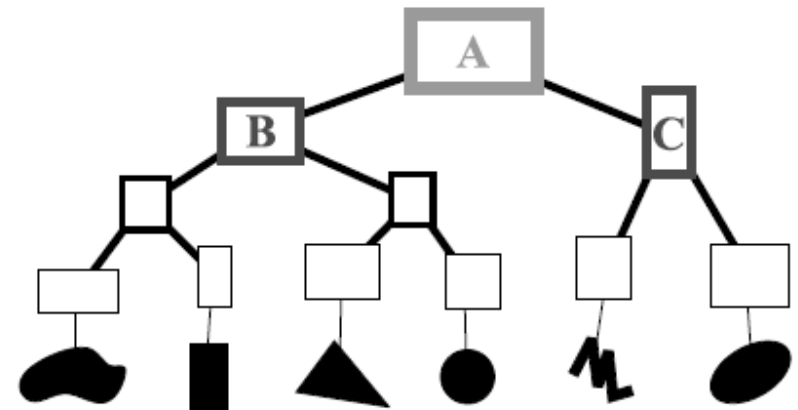
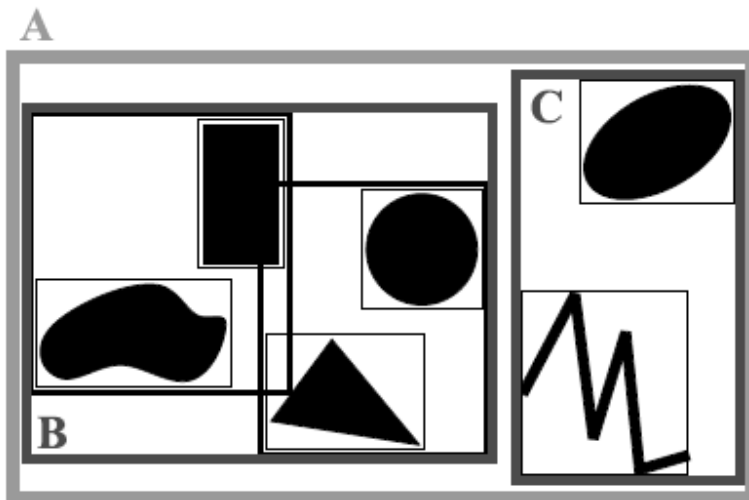
- Advantages

- Allows definitions of objects in own coordinate systems
- Allows use of object definition multiple times in a scene



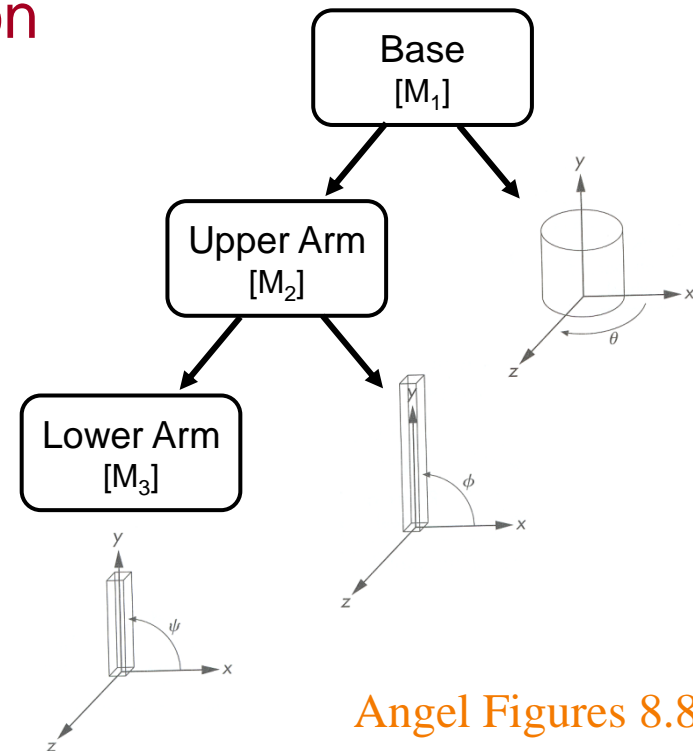
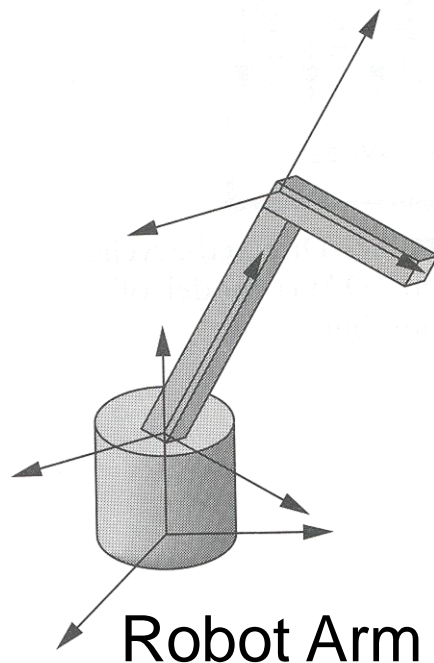
Scene Graphs

- Advantages
 - Allows definitions of objects in own coordinate systems
 - Allows use of object definition multiple times in a scene
 - Allows hierarchical processing (e.g., intersections)



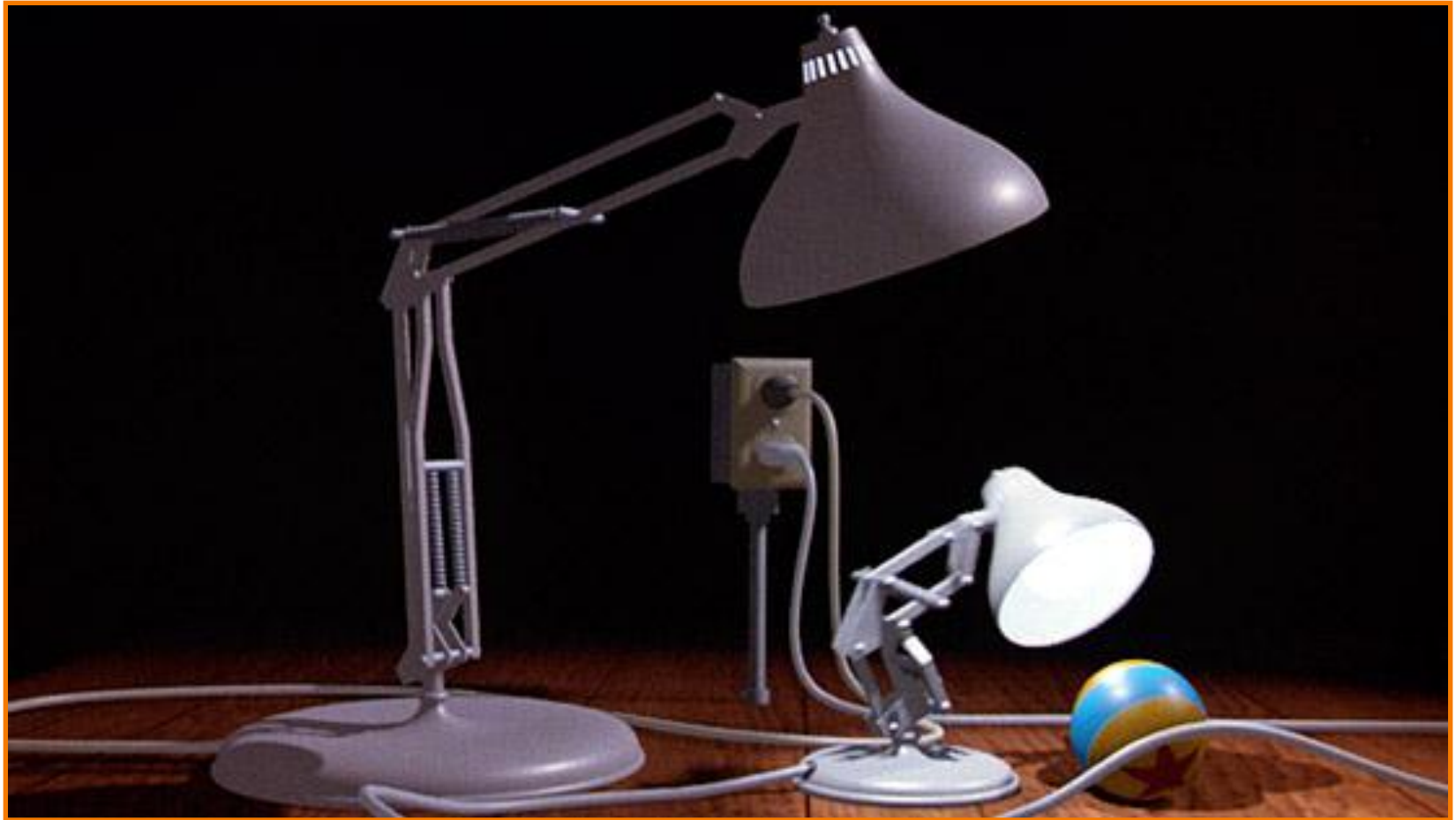
Scene Graphs

- Advantages
 - Allows definitions of objects in own coordinate systems
 - Allows use of object definition multiple times in a scene
 - Allows hierarchical processing (e.g., intersections)
 - **Allows articulated animation**



Angel Figures 8.8 & 8.9

Scene Graph Example

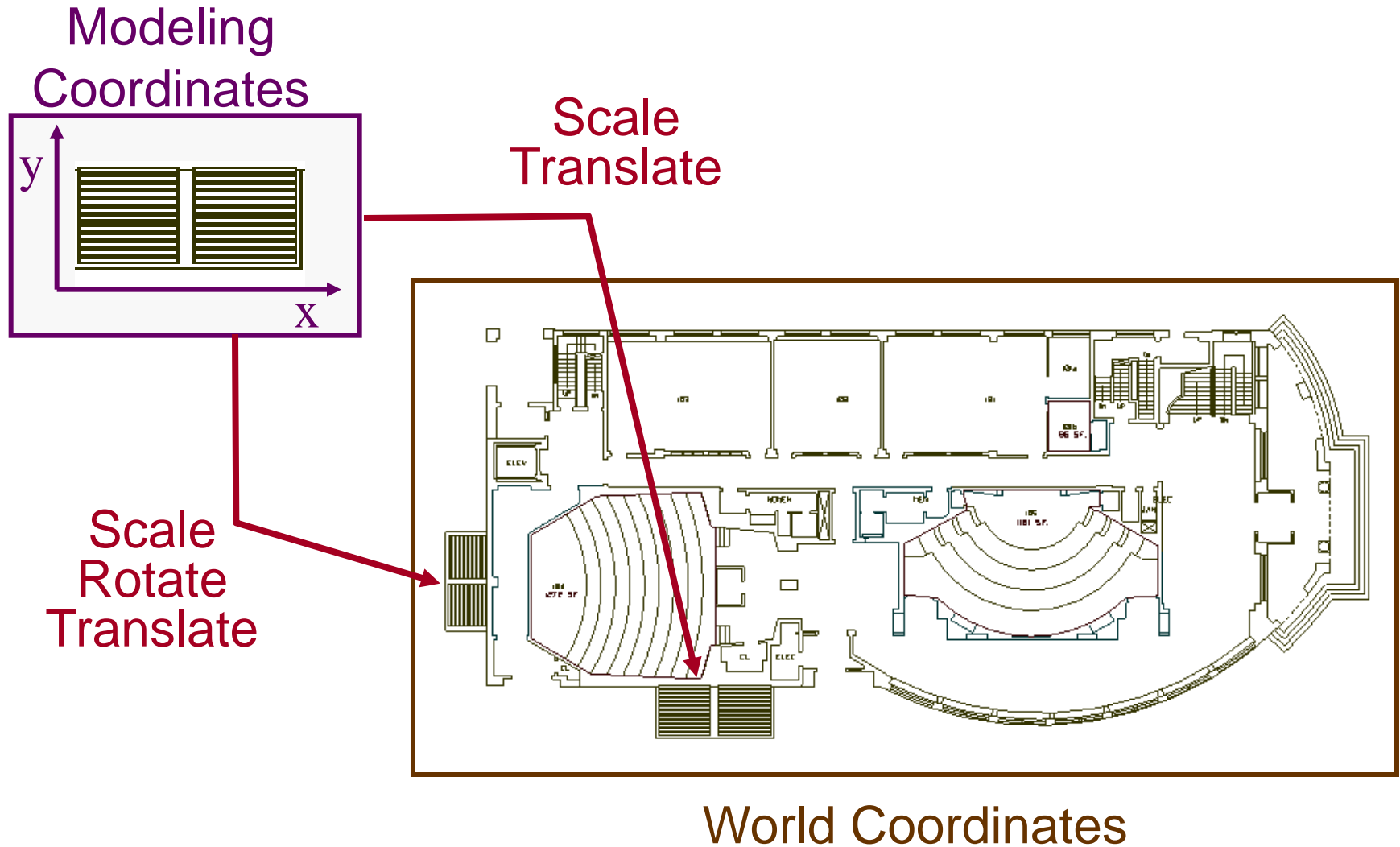


Overview



- Scene graphs
 - Geometry & attributes
 - Transformations
 - Bounding volumes
- Transformations
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
 - 3D transformations

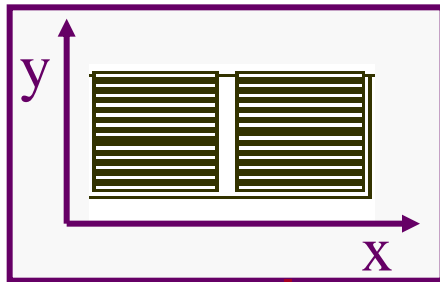
2D Modeling Transformations



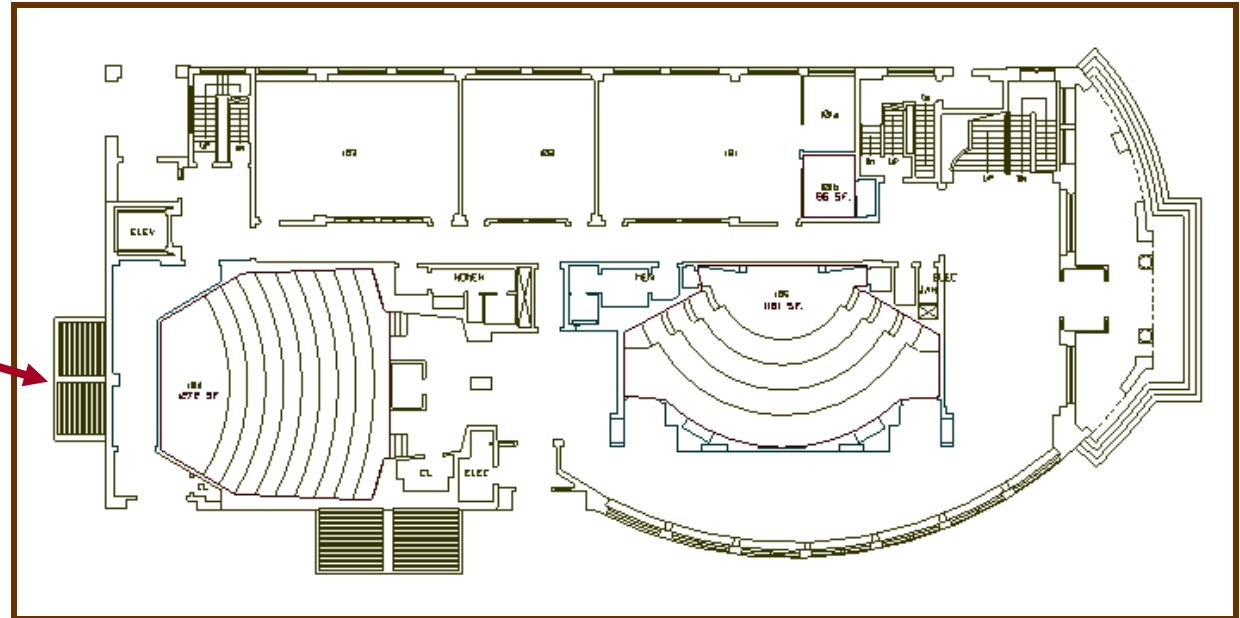
2D Modeling Transformations



Modeling
Coordinates



Let's look
at this in
detail...

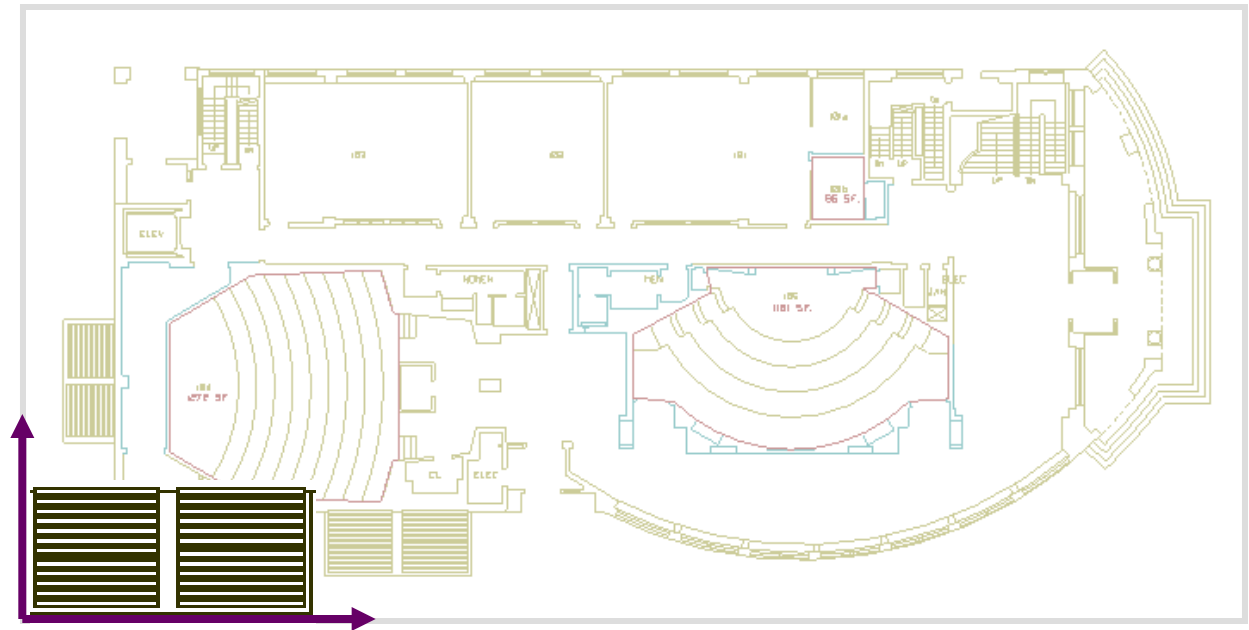
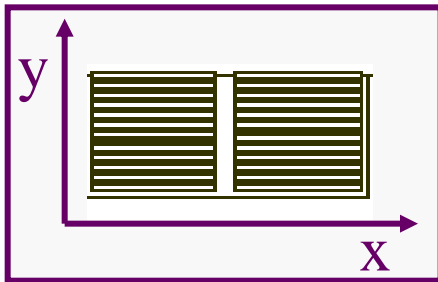


World Coordinates

2D Modeling Transformations



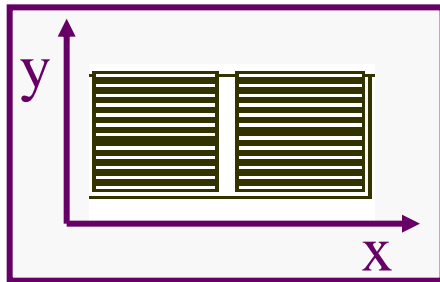
Modeling
Coordinates



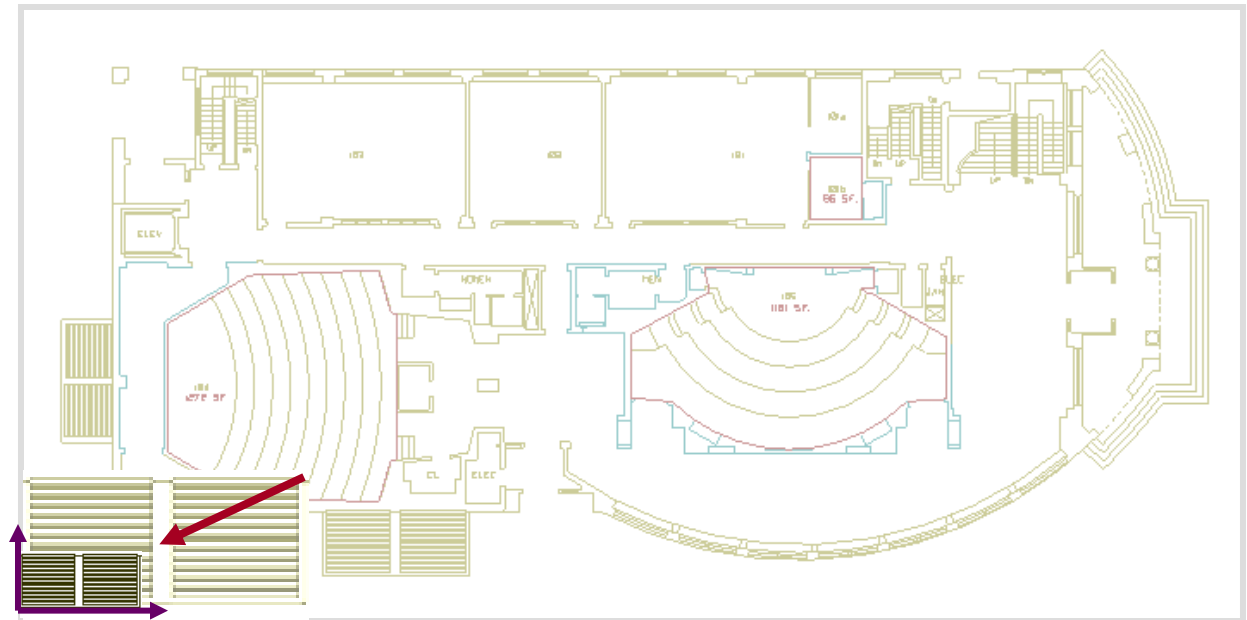
2D Modeling Transformations



Modeling
Coordinates



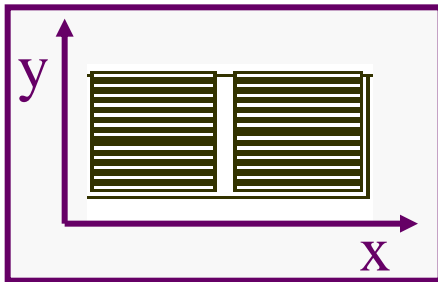
Scale .3, .3



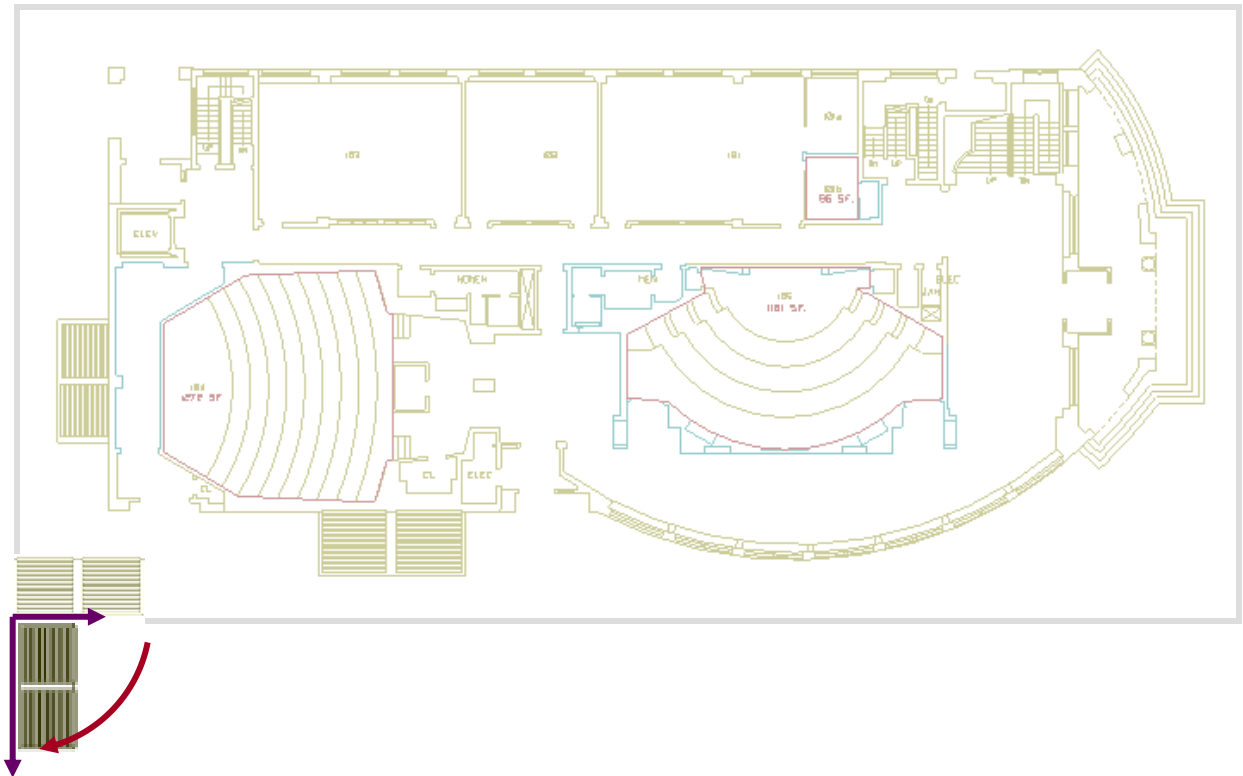
2D Modeling Transformations



Modeling
Coordinates



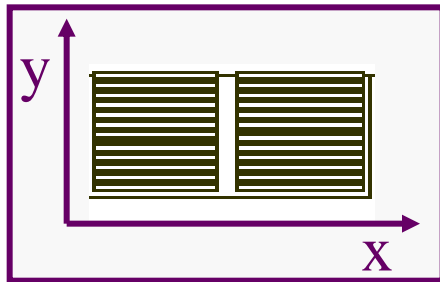
Scale .3, .3
Rotate -90



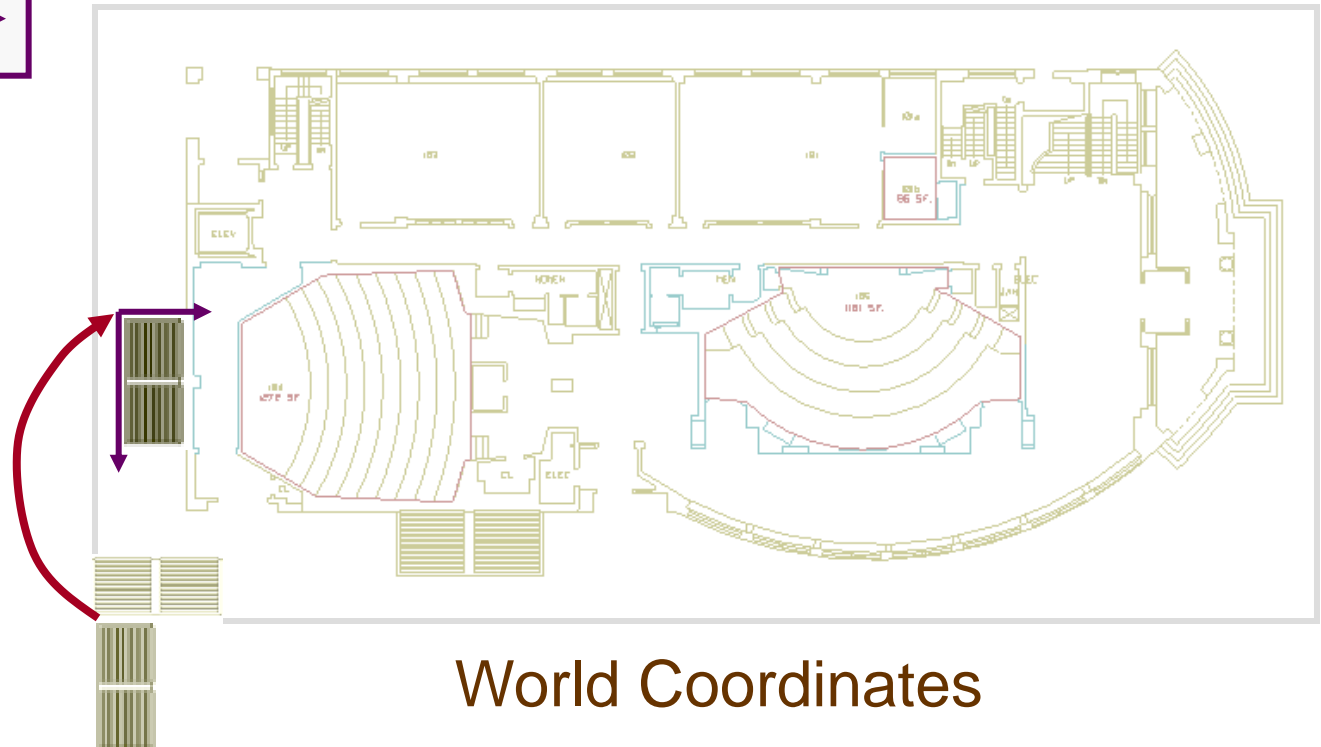
2D Modeling Transformations



Modeling
Coordinates



Scale .3, .3
Rotate -90
Translate 5, 3



World Coordinates

Basic 2D Transformations

- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

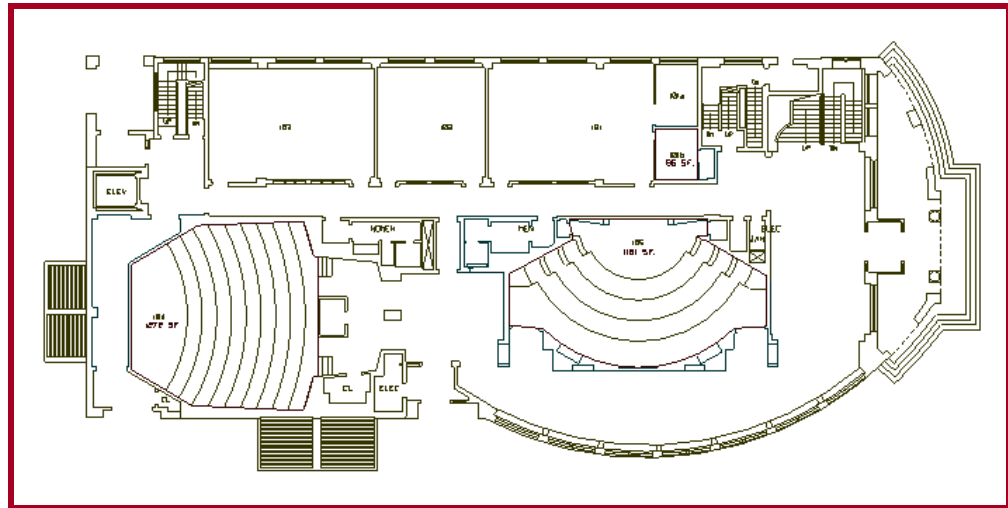
- $x' = x * sx$
- $y' = y * sy$

- Shear:

- $x' = x + hx*y$
- $y' = y + hy*x$

- Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



Transformations
can be combined
(with simple algebra)

Basic 2D Transformations

- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

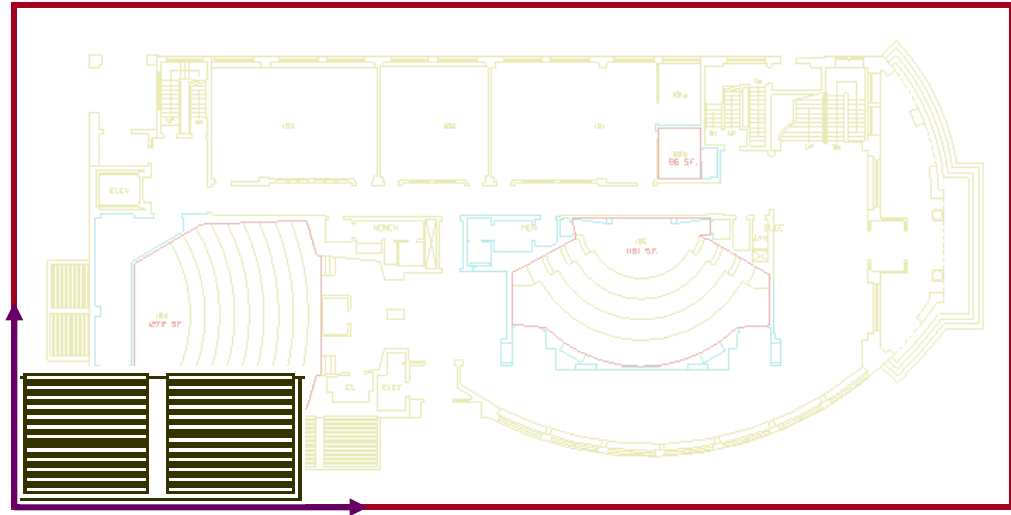
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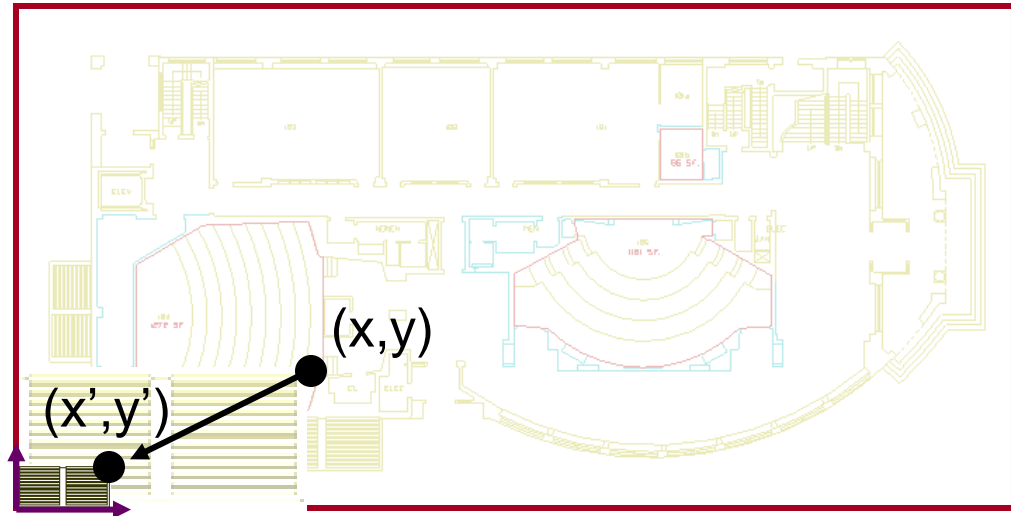
Basic 2D Transformations

- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

- $x' = x * sx$
- $y' = y * sy$



- Shear:

- $x' = x + hx*y$
- $y' = y + hy*x$

$$\begin{aligned} x' &= x * sx \\ y' &= y * sy \end{aligned}$$

- Rotation:

- $x' = x * \cos\Theta - y * \sin\Theta$
- $y' = x * \sin\Theta + y * \cos\Theta$

Basic 2D Transformations

- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

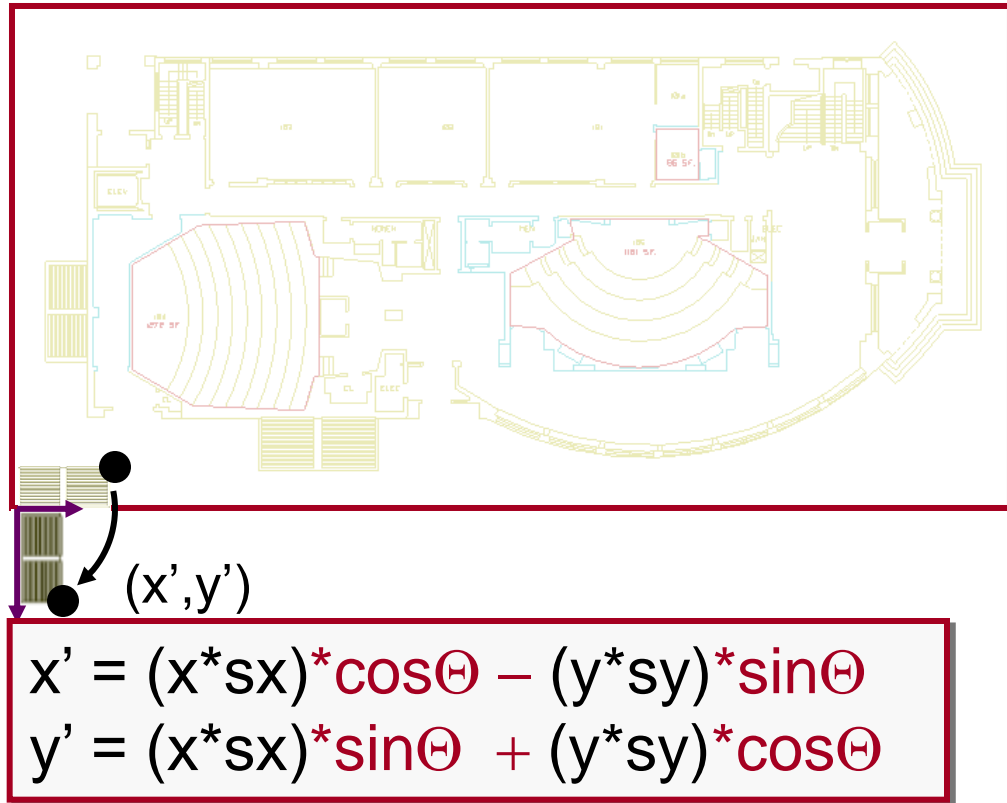
- $x' = x * sx$
- $y' = y * sy$

- Shear:

- $x' = x + hx*y$
- $y' = y + hy*x$

- Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



Basic 2D Transformations

- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

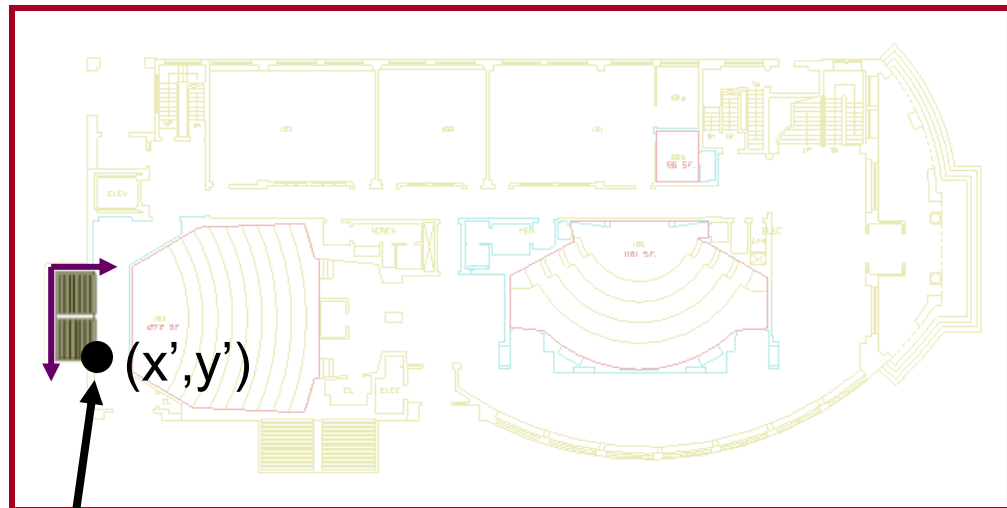
- $x' = x * sx$
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- Shear:

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- $y' = y + hy*x$

- Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



$$\begin{aligned}x' &= ((x*sx)*\cos\Theta - (y*sy)*\sin\Theta) + tx \\y' &= ((x*sx)*\sin\Theta + (y*sy)*\cos\Theta) + ty\end{aligned}$$

Basic 2D Transformations

- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

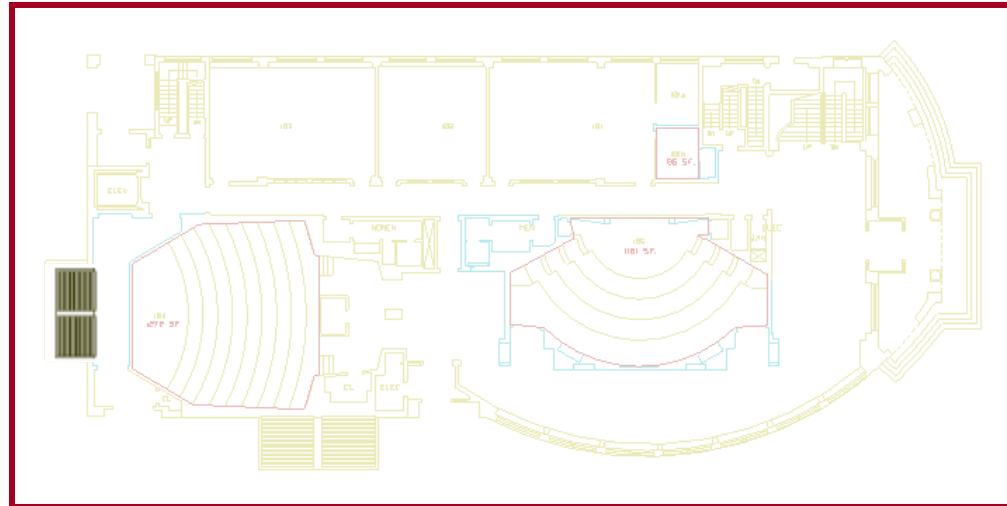
- $x' = x * sx$
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- Shear:

- $x' = x + hx*y$
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- Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



$$x' = ((x*sx)*\cos\Theta - (y*sy)*\sin\Theta) + tx$$
$$y' = ((x*sx)*\sin\Theta + (y*sy)*\cos\Theta) + ty$$

Overview



- Scene graphs
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- Transformations
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
 - 3D transformations



Matrix Representation

- Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector
 \Leftrightarrow apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + by$$

$$y' = cx + dy$$



Matrix Representation

- Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations



2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$\begin{aligned}x' &= x \\ y' &= y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$\begin{aligned}x' &= sx * x \\ y' &= sy * y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$\begin{aligned}x' &= \cos \Theta * x - \sin \Theta * y \\y' &= \sin \Theta * x + \cos \Theta * y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$\begin{aligned}x' &= x + shx * y \\y' &= shy * x + y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & shx \\ shy & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

2D Mirror over Y axis?

$$\begin{aligned}x' &= -x \\ y' &= y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$\begin{aligned}x' &= -x \\ y' &= -y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + tx$$

$$y' = y + ty$$

NO.

Only *linear* 2D transformations can be represented with a 2x2 matrix



Linear Transformations


- 2D linear transformations are combinations of ...
 - Scale,
 - Rotation,
 - Shear, and
 - Mirror
- $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
- Properties of linear transformations:
 - Satisfies: $T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1T(\mathbf{p}_1) + s_2T(\mathbf{p}_2)$
 - Origin maps to origin
 - Points at infinity stay at infinity
 - Lines map to lines
 - Parallel lines remain parallel
 - Ratios are preserved
 - Closed under composition

2D Translation



- 2D translation represented by a 3x3 matrix
 - Point represented with *homogeneous coordinates*

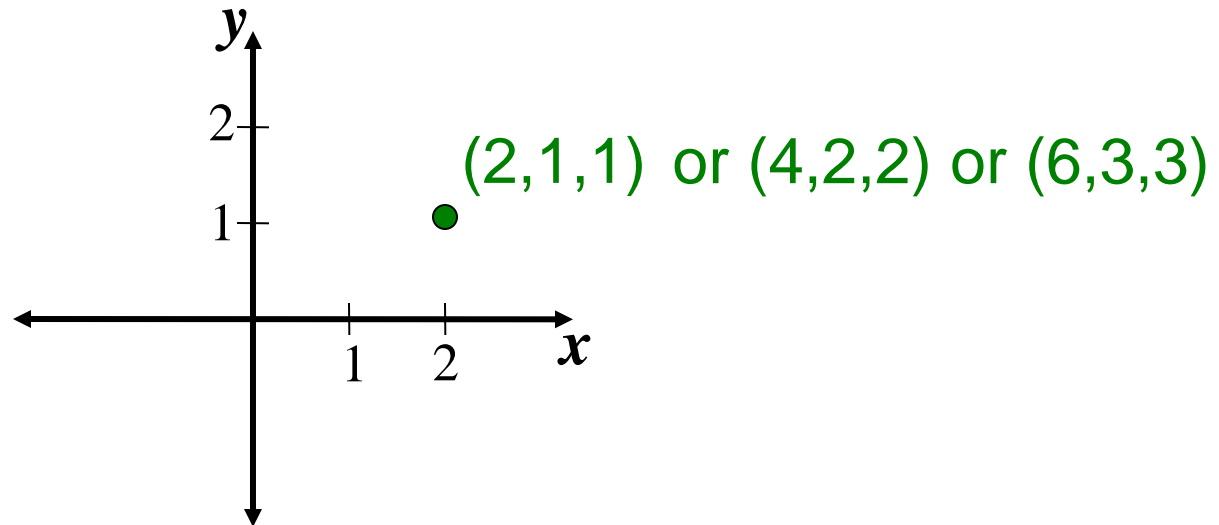
$$\begin{aligned}x' &= x + tx \\ y' &= y + ty\end{aligned}$$


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Homogeneous Coordinates

- Add a 3rd coordinate to every 2D point
 - (x, y, w) represents a point at location $(x/w, y/w)$
 - $(x, y, 0)$ represents a point at infinity
 - $(0, 0, 0)$ is not allowed



Convenient coordinate system to represent many useful transformations



Basic 2D Transformations

- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & shx & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear



Affine Transformations

- Affine transformations are combinations of ...
 - Linear transformations, and
 - Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations:
 - Origin does not necessarily map to origin
 - Points at infinity remain at infinity
 - Lines map to lines
 - Parallel lines remain parallel
 - Ratios are preserved
 - Closed under composition



Projective Transformations

- Projective transformations (homographies):
 - Affine transformations, and
 - Projective warps

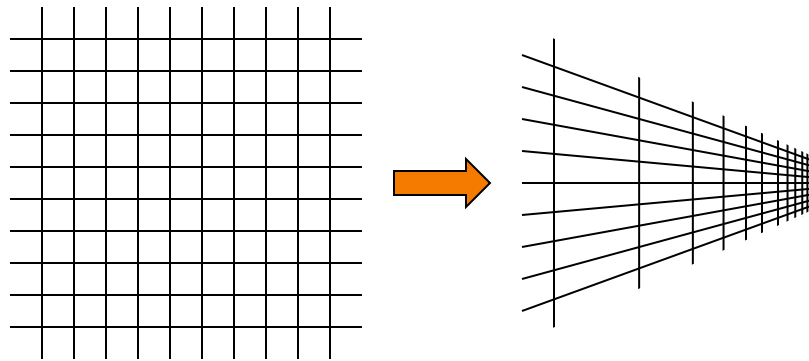
$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:
 - Origin does not necessarily map to origin
 - Point at infinity may map to finite point
 - Lines map to lines
 - Parallel lines do not necessarily remain parallel
 - Ratios are not preserved (but “cross-ratios” are)
 - Closed under composition

Projective Transformations



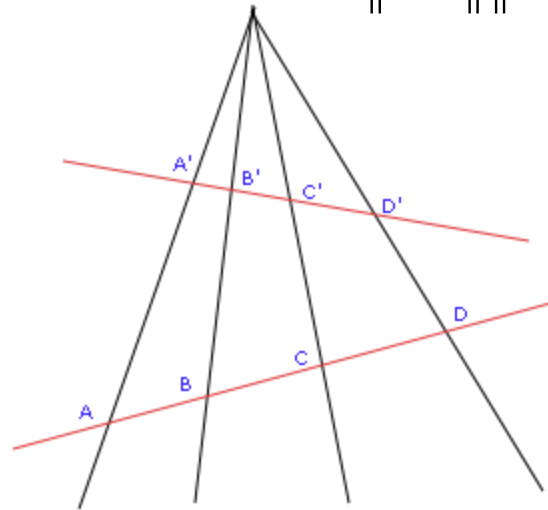
- Will be useful to model (pinhole) cameras: can represent camera projection in same framework as modeling transformations



Cross-Ratio

- Definition: for 4 collinear points A, B, C, D

$$(A, B; C, D) = \frac{\|AC\| \|BD\|}{\|AD\| \|BC\|}$$



- Projective Invariant: $(A, B; C, D) = (A', B'; C', D')$

Overview



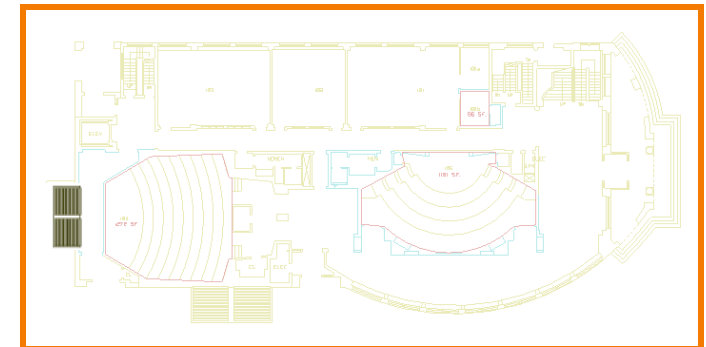
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 - Matrix representation
 - **Matrix composition**
 - 3D transformations

Matrix Composition

- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{pmatrix} \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{pmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\mathbf{p}' = \mathbf{T}(tx,ty) \quad \mathbf{R}(\Theta) \quad \mathbf{S}(sx,sy) \quad \mathbf{p}$$

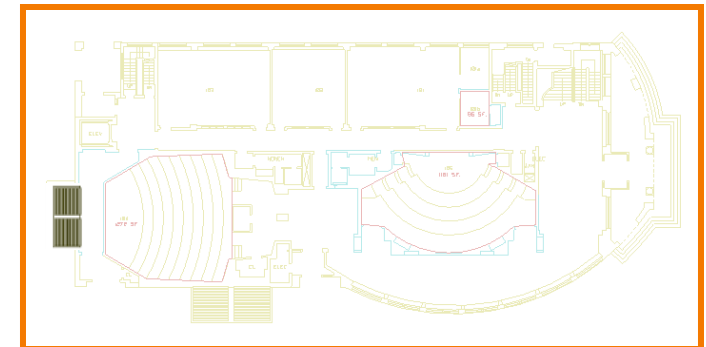




Matrix Composition

- Matrices are a convenient and efficient way to represent a sequence of transformations
 - General purpose representation
 - Hardware matrix multiply
 - Efficiency with premultiplication
 - » Matrix multiplication is associative

$$\mathbf{p}' = (T * (R * (S * \mathbf{p})))$$
$$\mathbf{p}' = (T * R * S) * \mathbf{p}$$



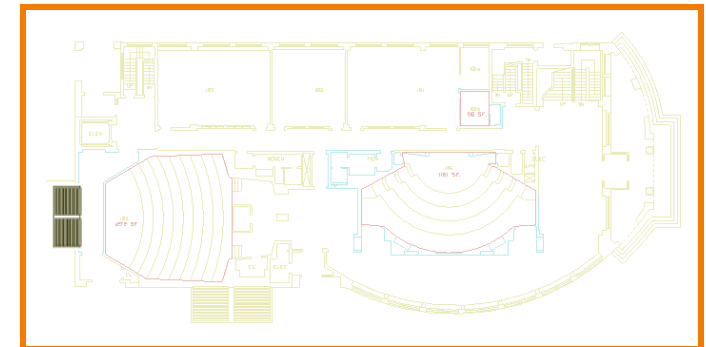
Matrix Composition

- Be aware: order of transformations matters
 - » Matrix multiplication is **not** commutative

$$p' = T * R * S * p$$

←—————→

“Global” “Local”

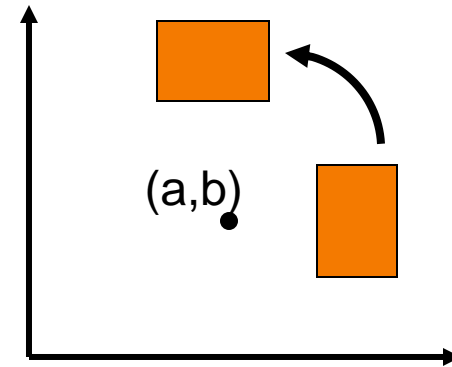


Matrix Composition

- Rotate by Θ around arbitrary point (a,b)
 - $M = T(a,b) * R(\Theta) * T(-a,-b)$

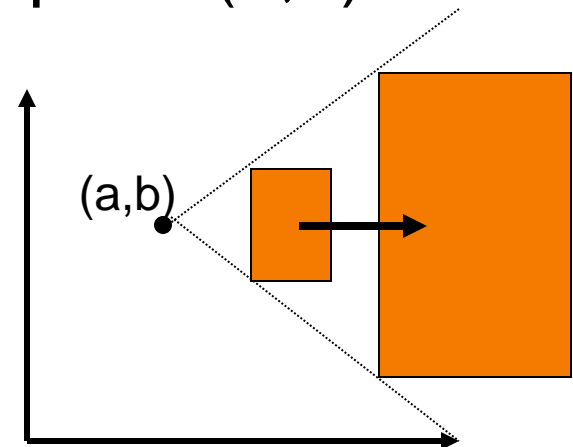
The trick:

First, translate (a,b) to the origin.
Next, do the rotation about origin.
Finally, translate back.



- Scale by s_x, s_y around arbitrary point (a,b)
 - $M = T(a,b) * S(s_x, s_y) * T(-a,-b)$

(Use the same trick.)



Overview



- Scene graphs
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3D Transformations

- Same idea as 2D transformations
 - Homogeneous coordinates: (x,y,z,w)
 - 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations



$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror over X axis



Basic 3D Transformations

Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 & 0 \\ \sin \Theta & \cos \Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

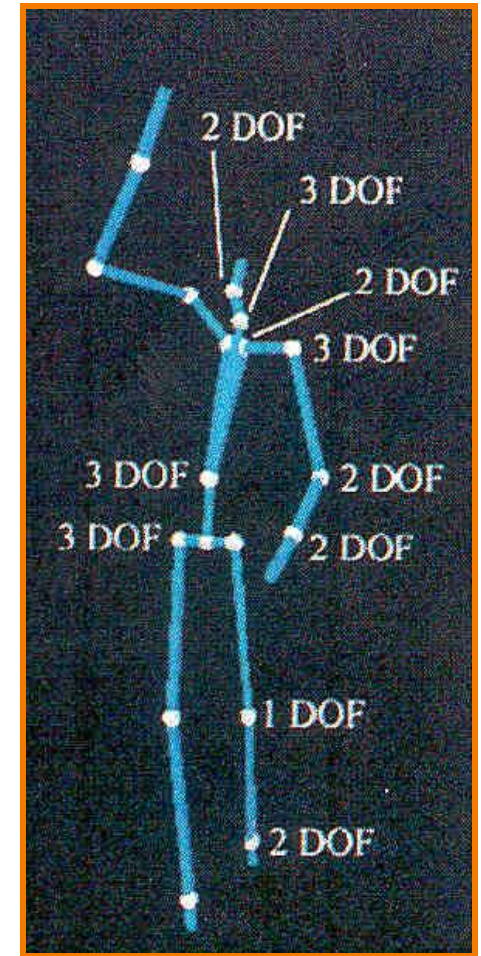
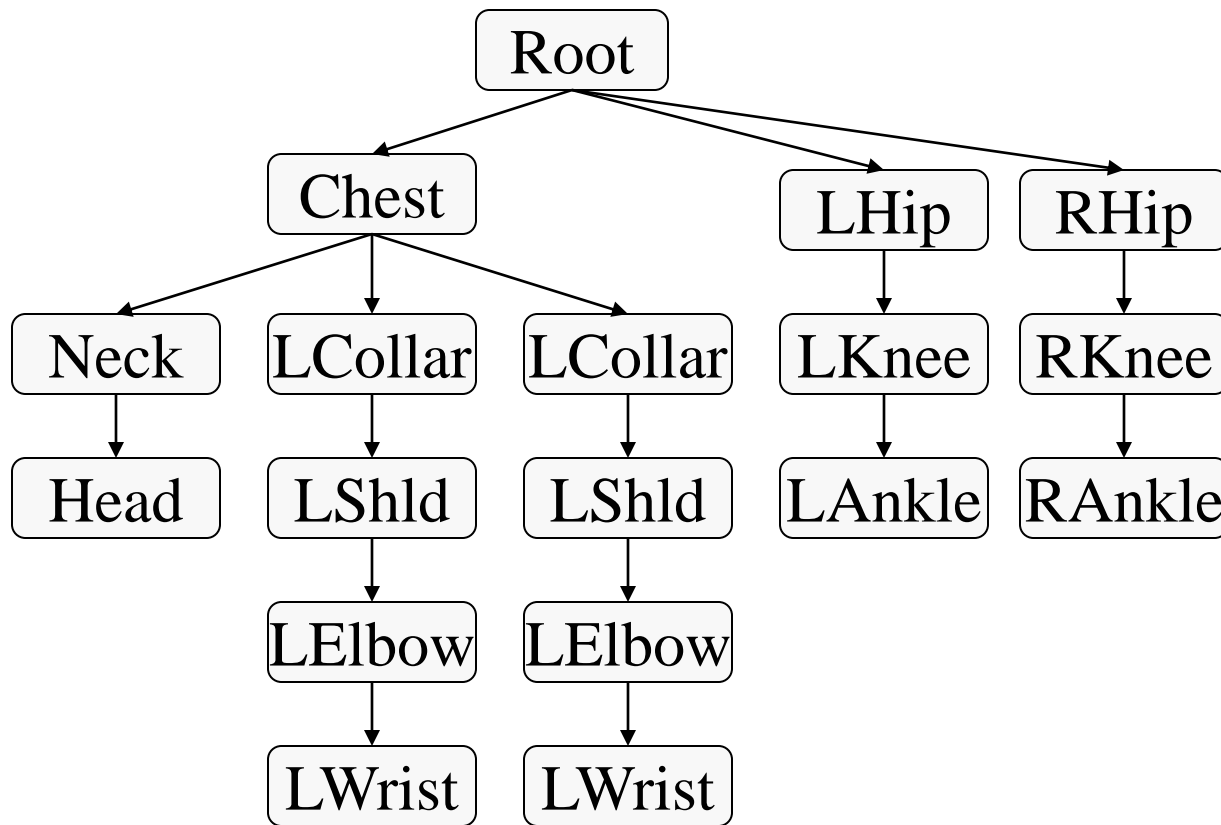
Rotate around Y axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos \Theta & 0 & -\sin \Theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \Theta & 0 & \cos \Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around X axis:

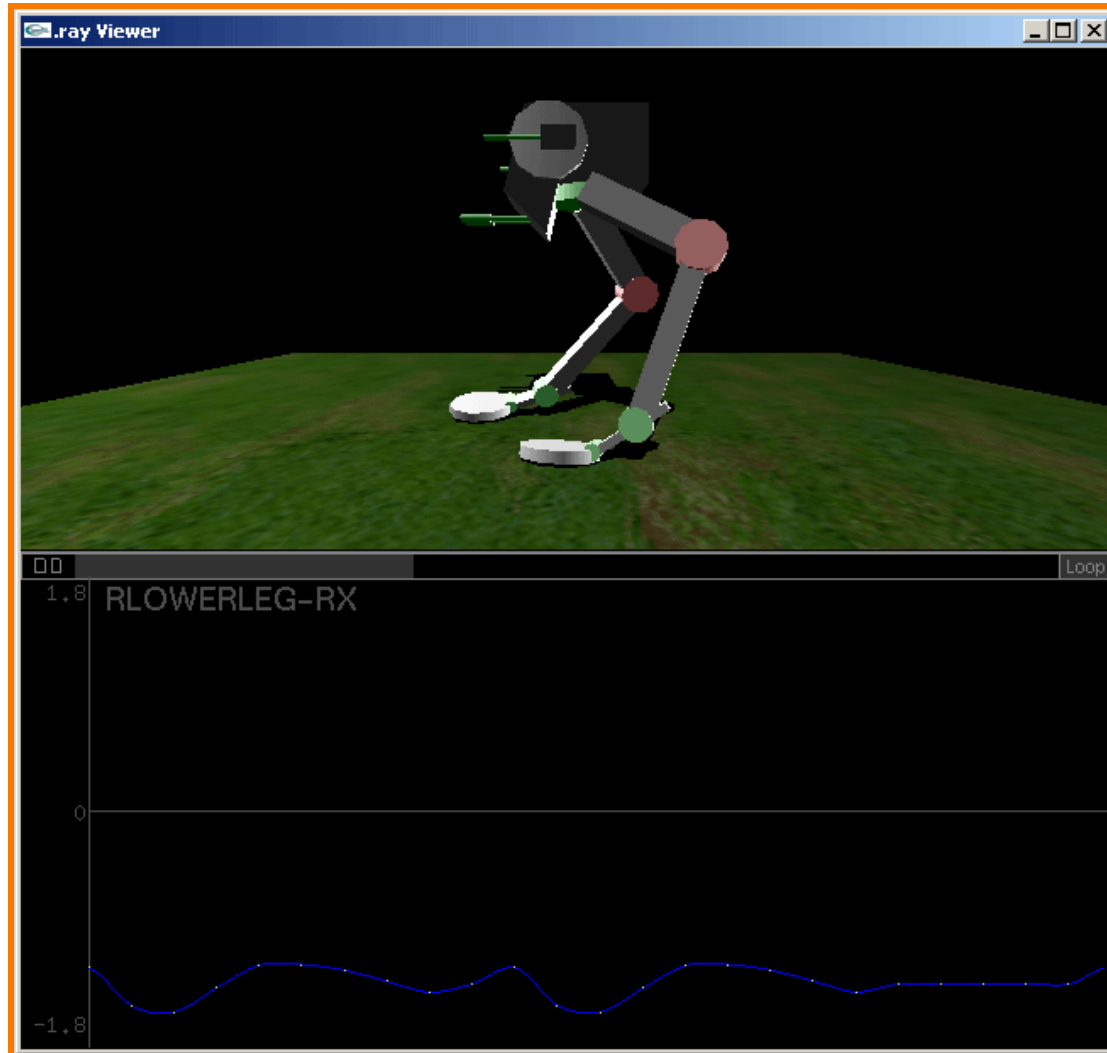
$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \Theta & -\sin \Theta & 0 \\ 0 & \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Transformations in Scene Graphs



Rose et al. '96

Transformations in Scene Graphs



Summary



- Scene graphs
 - Hierarchical
 - Modeling transformations
 - Bounding volumes
- Coordinate systems
 - World coordinates
 - Modeling coordinates
- 3D modeling transformations
 - Represent most transformations by 4x4 matrices
 - Composite with matrix multiplication (order matters)