# Parametric Surfaces

COS 426

# 3D Object Representations

- Raw data
  - Voxels
  - Point cloud
  - Range image
  - Polygons
- Surfaces
  - Mesh
  - Subdivision
  - Parametric
  - Implicit
- Solids
  - Octree
  - BSP tree
  - CSG
  - Sweep
- High-level structures
  - Scene graph
  - Application specific

# 3D Object Representations

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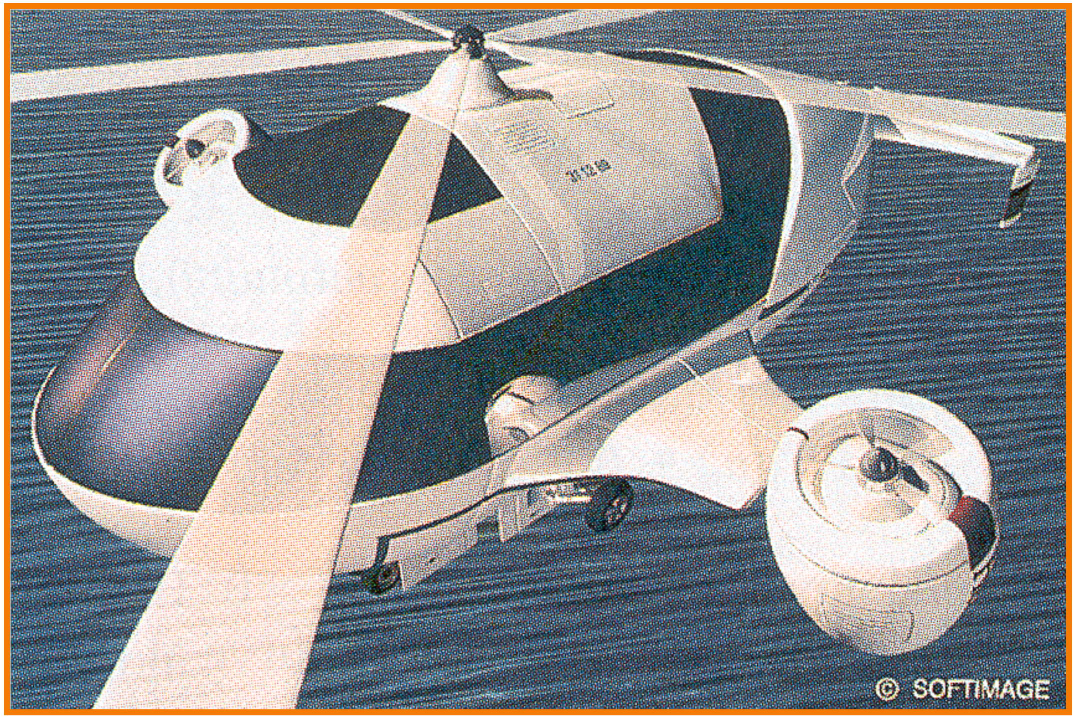
- High-level structures
  - Scene graph
  - Application specific

# Parametric Surfaces

- Applications
  - Design of smooth surfaces in cars, ships, etc.




H&B Figure 10.46

# Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

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- ➢ Parametric curves
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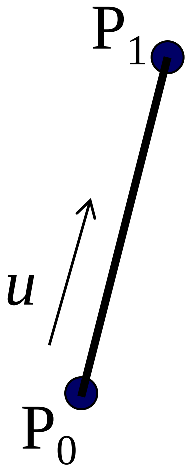
# Parametric Curves

- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$




- Example: line segment

$$f_x(u) = (1-u)x_0 + ux_1$$

$$f_y(u) = (1-u)y_0 + uy_1$$

$$u \in [0..1]$$

H&B Figure 10.10

# Parametric Curves

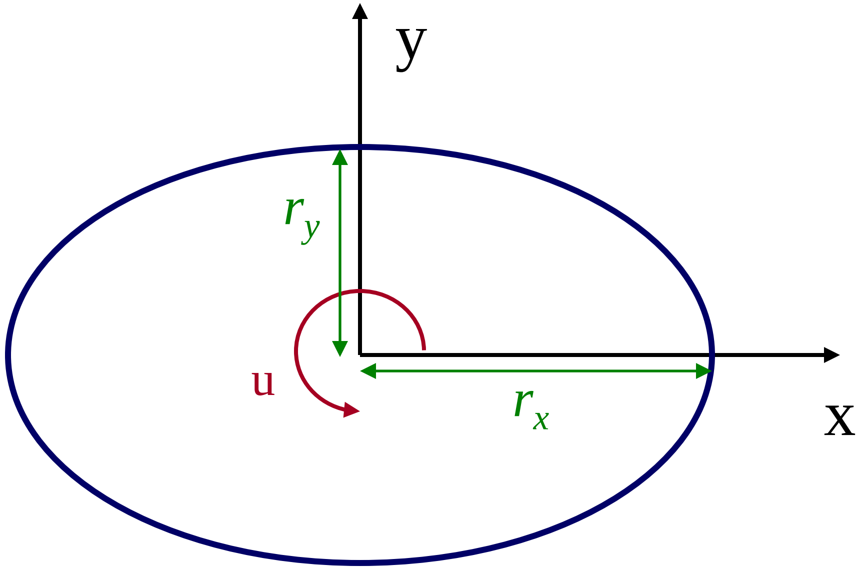
- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$

- Example: ellipse

$$f_x(u) = r_x \cos \frac{u}{2\pi}$$

$$f_y(u) = r_y \sin \frac{u}{2\pi}$$

$$u \in [0..1]$$

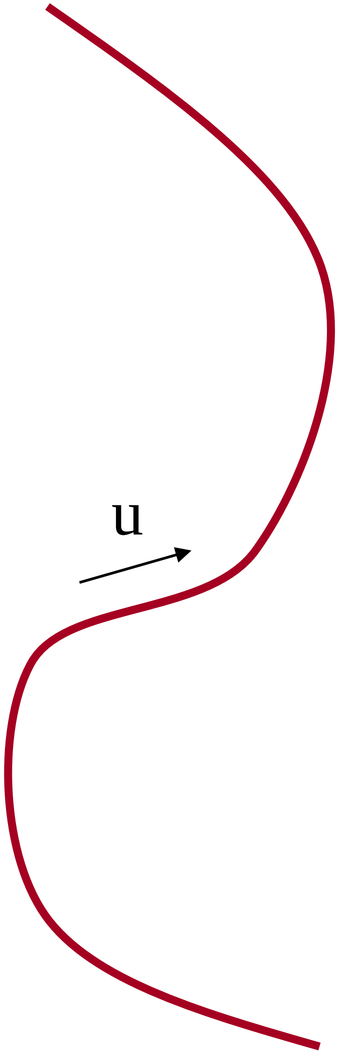



H&B Figure 10.10

# Parametric curves

How to easily define arbitrary curves?

$$x = f_x(u)$$
$$y = f_y(u)$$



# Parametric curves

How to easily define arbitrary curves?

$$x = f_x(u)$$
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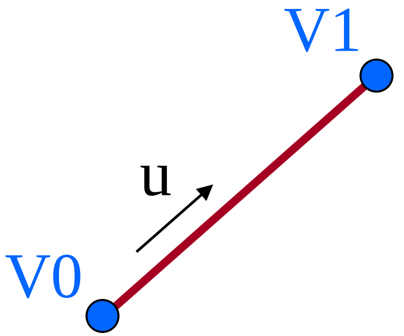



Use functions that “blend” control points

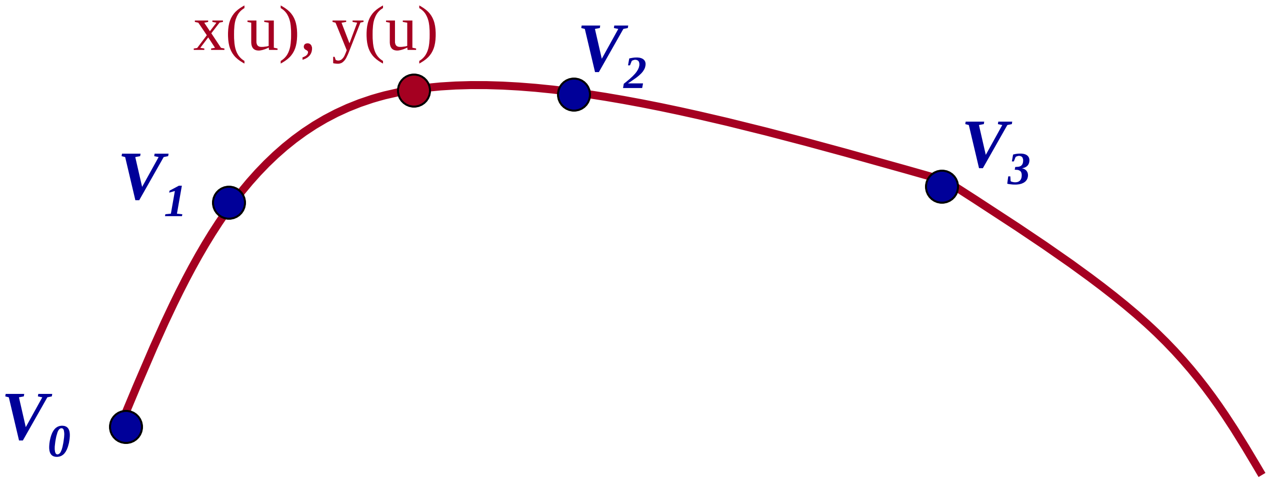
$$x = f_x(u) = V0_x*(1 - u) + V1_x*u$$
$$y = f_y(u) = V0_y*(1 - u) + V1_y*u$$

# Parametric curves

More generally:

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$



# Parametric curves

What B(u) functions should we use?

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$

# Parametric curves

What B(u) functions should we use?

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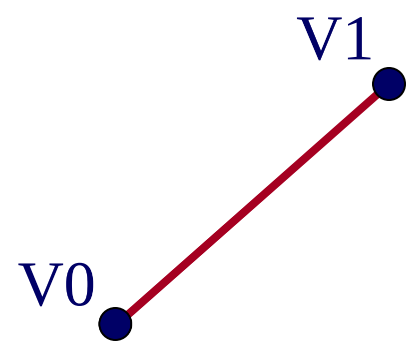


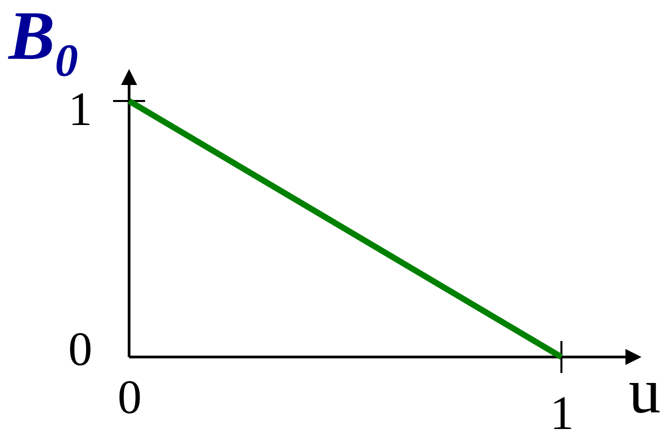


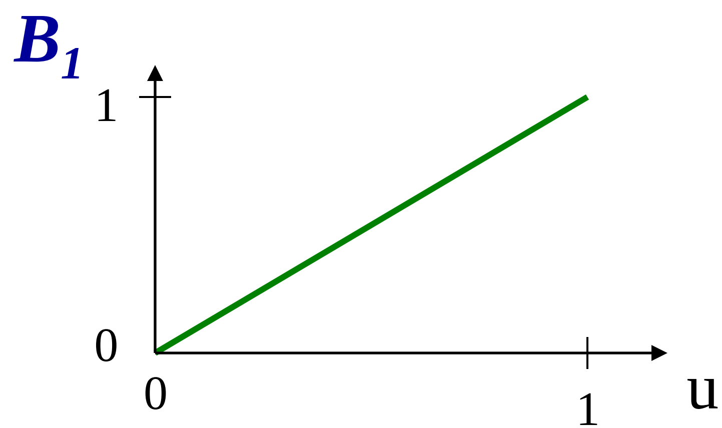

# Parametric curves

What B(u) functions should we use?

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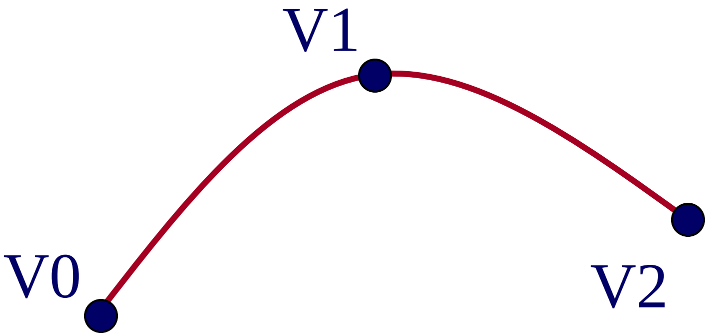
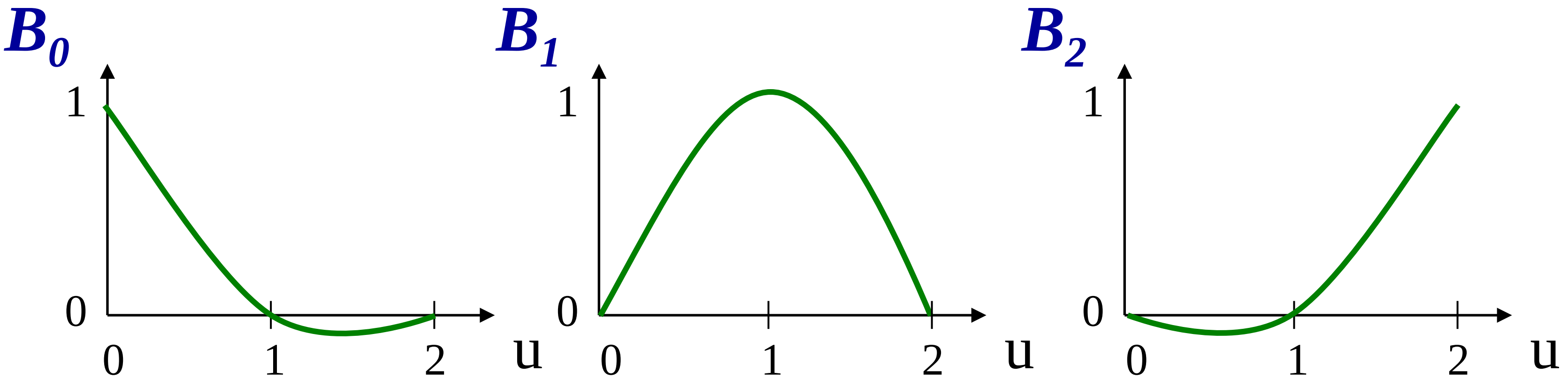
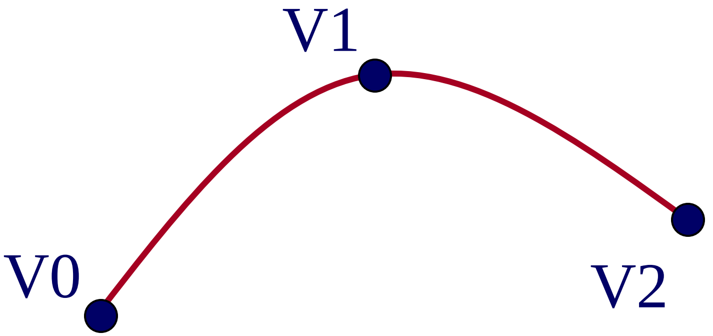


# Parametric Polynomial Curves

- Polynomial blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

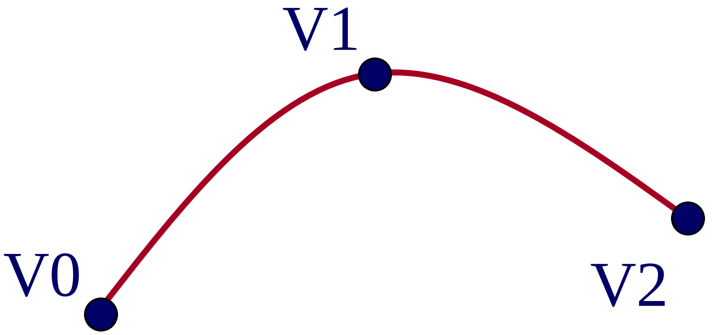


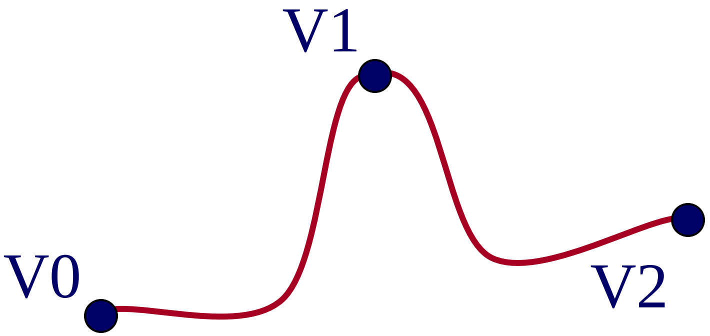

- Advantages of polynomials
  - Easy to compute
  - Infinitely continuous
  - Easy to derive curve properties

# Parametric Polynomial Curves

- Polynomial blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

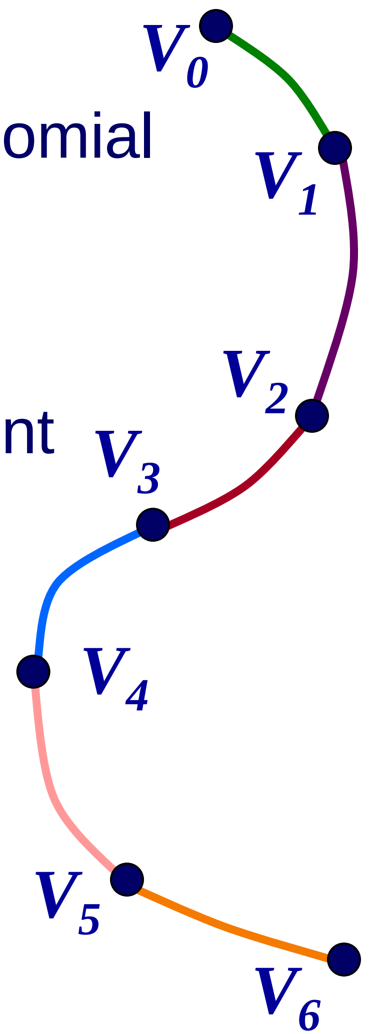



- What degree polynomial?
  - Easy to compute
  - Easy to control
  - Expressive

# Piecewise Parametric Polynomial Curves

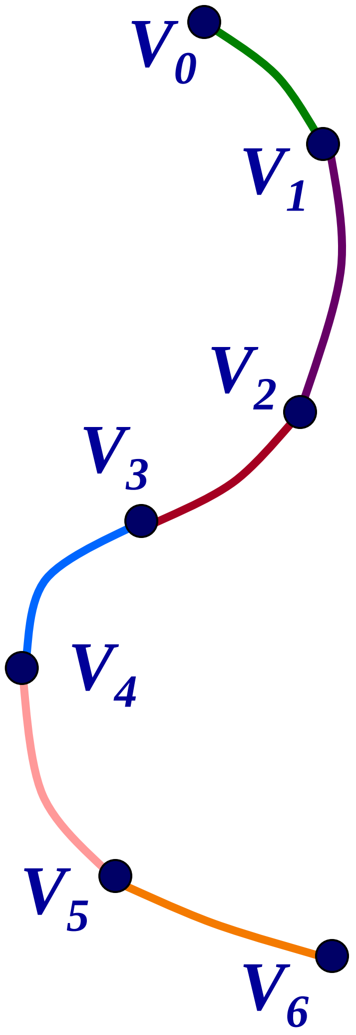
- Splines:
  - Split curve into segments
  - Each segment defined by low-order polynomial blending subset of control vertices
- Motivation:
  - Same blending functions for every segment
  - Prove properties from blending functions
  - Provides local control & efficiency
- Challenges
  - How choose blending functions?
  - How determine properties?



# Cubic Splines

- Some properties we might like to have:
  - Local control
  - Continuity
  - Interpolation?
  - Convex hull?

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$




Blending functions determine properties

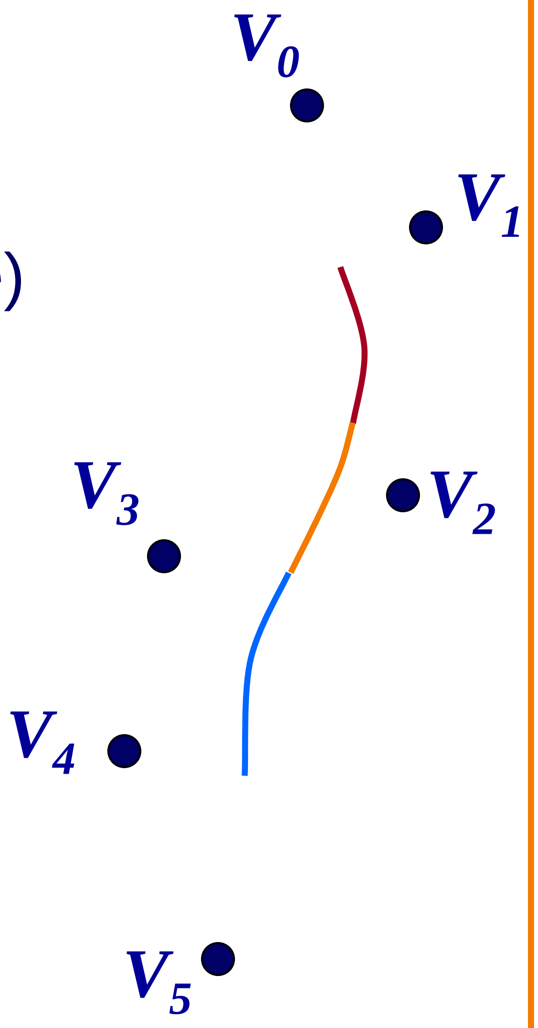
Properties determine blending functions

# Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

# Cubic B-Splines

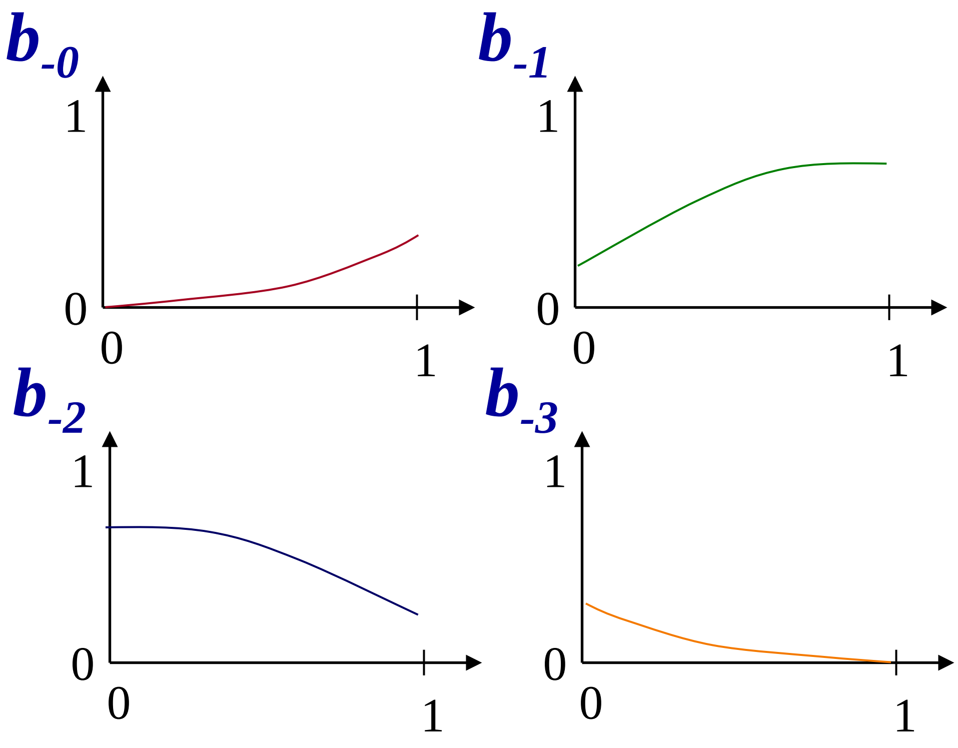
- Properties:
  - Local control
  - $C^2$ continuity at joints (infinitely continuous within each piece)
  - Approximating
  - Convex hull

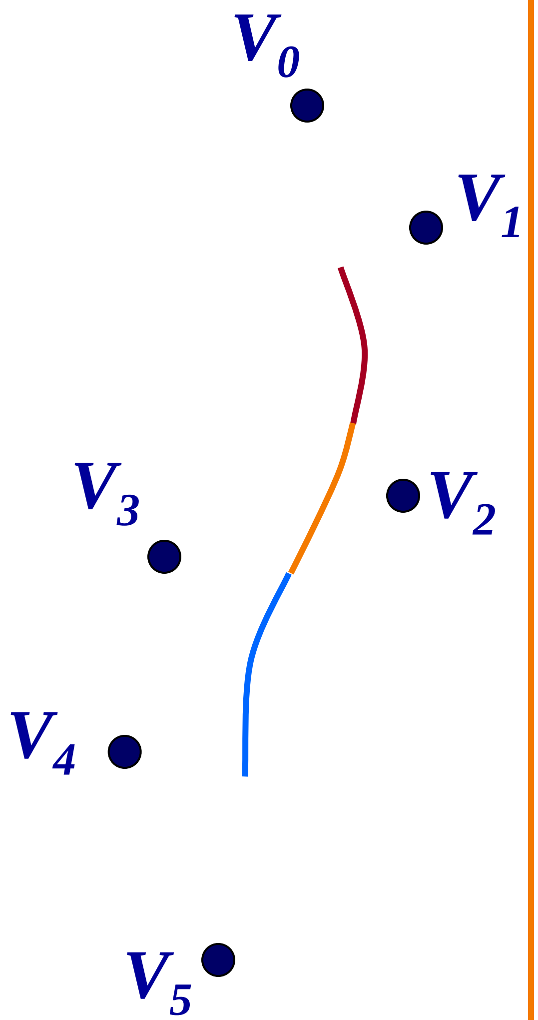


# Cubic B-Spline Blending Functions

Blending functions:

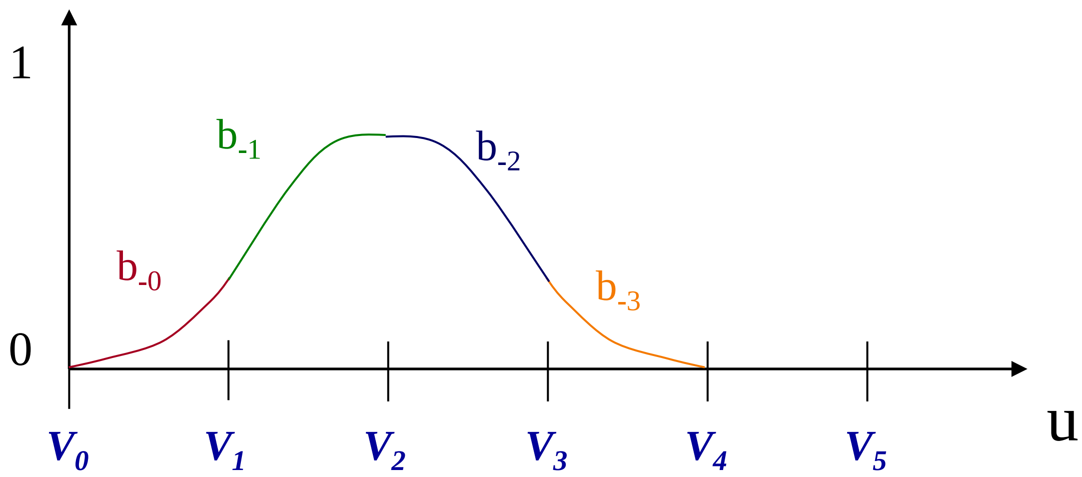
$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

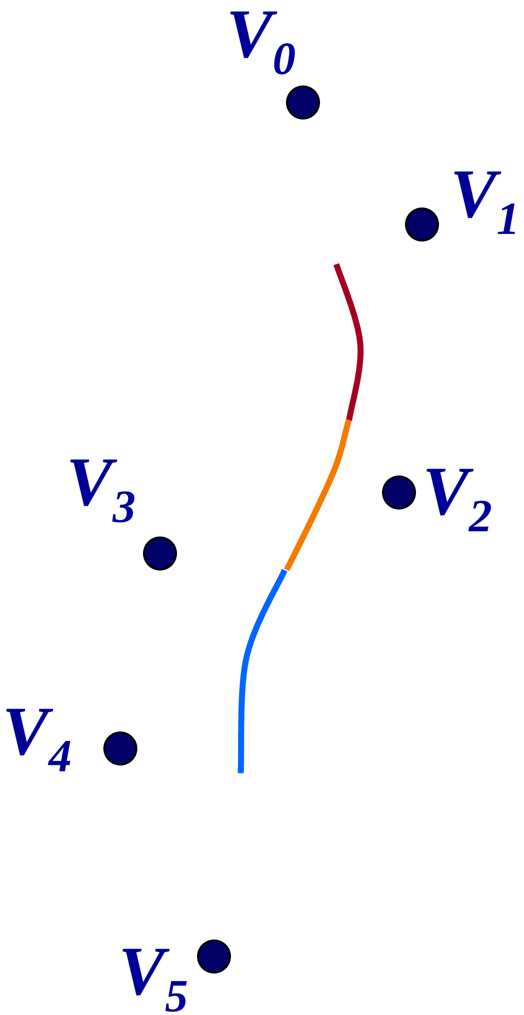


# Cubic B-Spline Blending Functions

- How derive blending functions?
  - Cubic polynomials
  - Local control
  - $C^2$ continuity
  - Convex hull

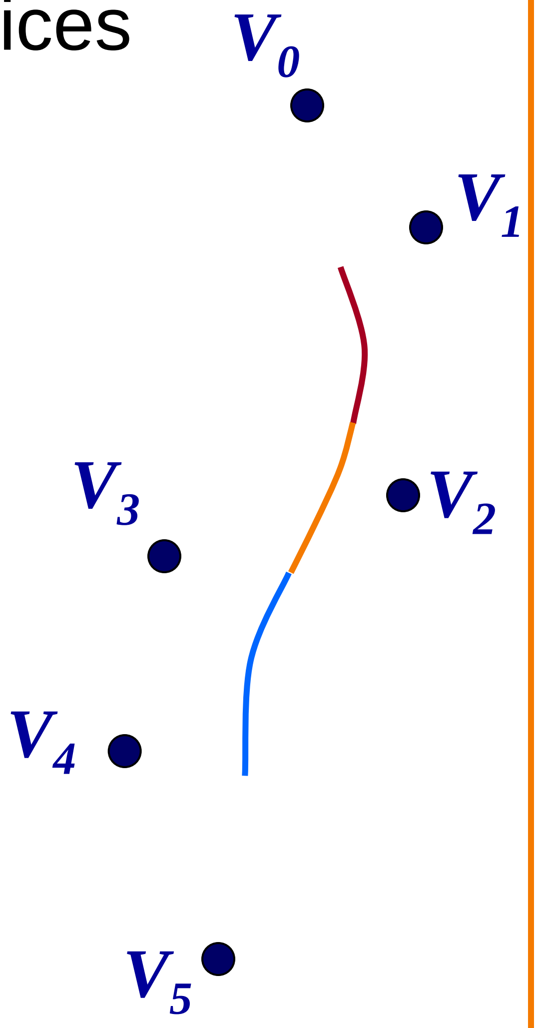


# Cubic B-Spline Blending Functions

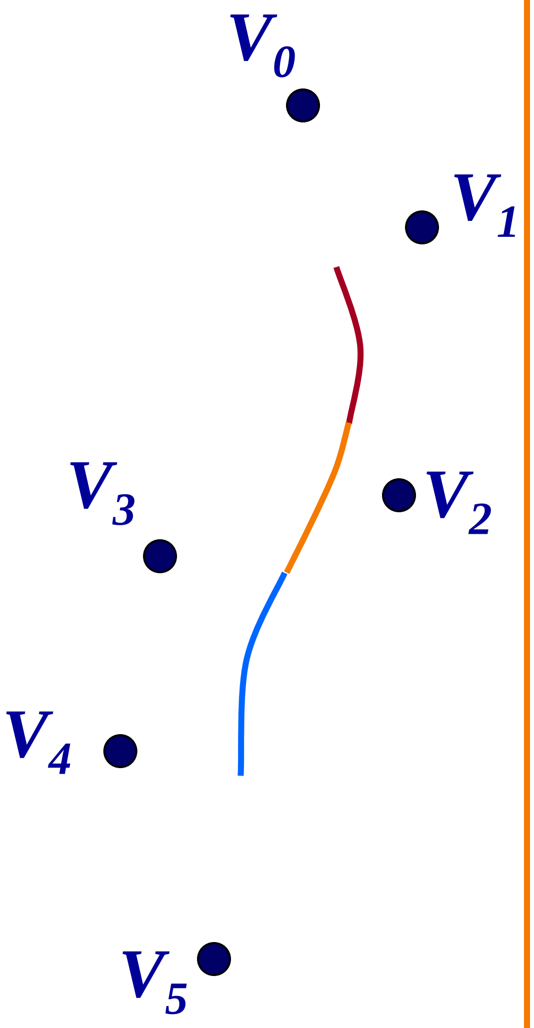
- Four cubic polynomials for four vertices
  - 16 variables (degrees of freedom)
  - Variables are $a_i$, $b_i$, $c_i$, $d_i$ for four blending functions

$$b_{-0}(u) = a_0 u^3 + b_0 u^2 + c_0 u^1 + d_0$$
$$b_{-1}(u) = a_1 u^3 + b_1 u^2 + c_1 u^1 + d_1$$
$$b_{-2}(u) = a_2 u^3 + b_2 u^2 + c_2 u^1 + d_2$$
$$b_{-3}(u) = a_3 u^3 + b_3 u^2 + c_3 u^1 + d_3$$



# Cubic B-Spline Blending Functions

- $C^2$ continuity implies 15 constraints
  - Position of two curves same
  - Derivative of two curves same
  - Second derivatives same



# Cubic B-Spline Blending Functions

Fifteen continuity constraints:

$$
\begin{aligned}
0 &= b_{-0}(0) \\
b_{-0}(1) &= b_{-1}(0) \\
b_{-1}(1) &= b_{-2}(0) \\
b_{-2}(1) &= b_{-3}(0) \\
b_{-3}(1) &= 0
\end{aligned}
\qquad
\begin{aligned}
0 &= b_{-0}'(0) \\
b_{-0}'(1) &= b_{-1}'(0) \\
b_{-1}'(1) &= b_{-2}'(0) \\
b_{-2}'(1) &= b_{-3}'(0) \\
b_{-3}'(1) &= 0
\end{aligned}
\qquad
\begin{aligned}
0 &= b_{-0}''(0) \\
b_{-0}''(1) &= b_{-1}''(0) \\
b_{-1}''(1) &= b_{-2}''(0) \\
b_{-2}''(1) &= b_{-3}''(0) \\
b_{-3}''(1) &= 0
\end{aligned}
$$

One more convenient constraint:

$$b_{-0}(0) + b_{-1}(0) + b_{-2}(0) + b_{-3}(0) = 1$$

# Cubic B-Spline Blending Functions

- Solving the system of equations yields:

$$b_{-3}(u) = -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6}$$
$$b_{-2}(u) = \tfrac{1}{2}u^3 - u^2 + \tfrac{2}{3}$$
$$b_{-1}(u) = -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6}$$
$$b_{-0}(u) = \tfrac{1}{6}u^3$$

# Cubic B-Spline Blending Functions

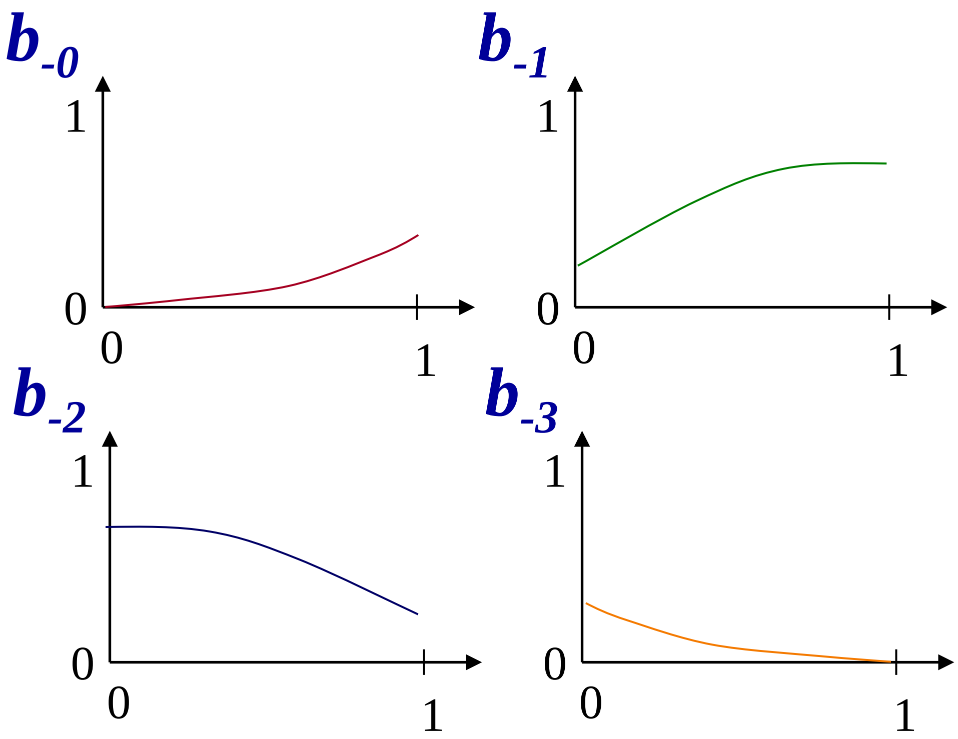
- In matrix form:

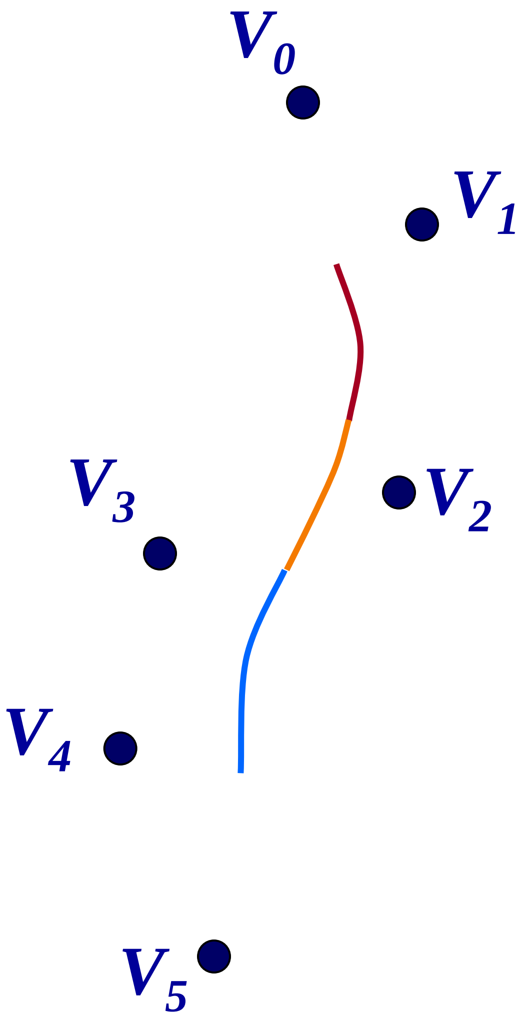
$$Q(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

# Cubic B-Spline Blending Functions

In plot form:

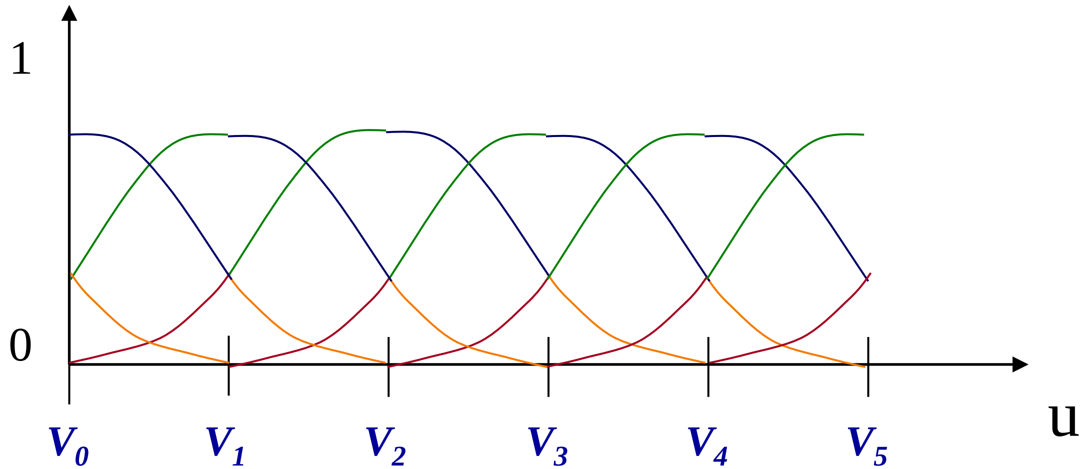
$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

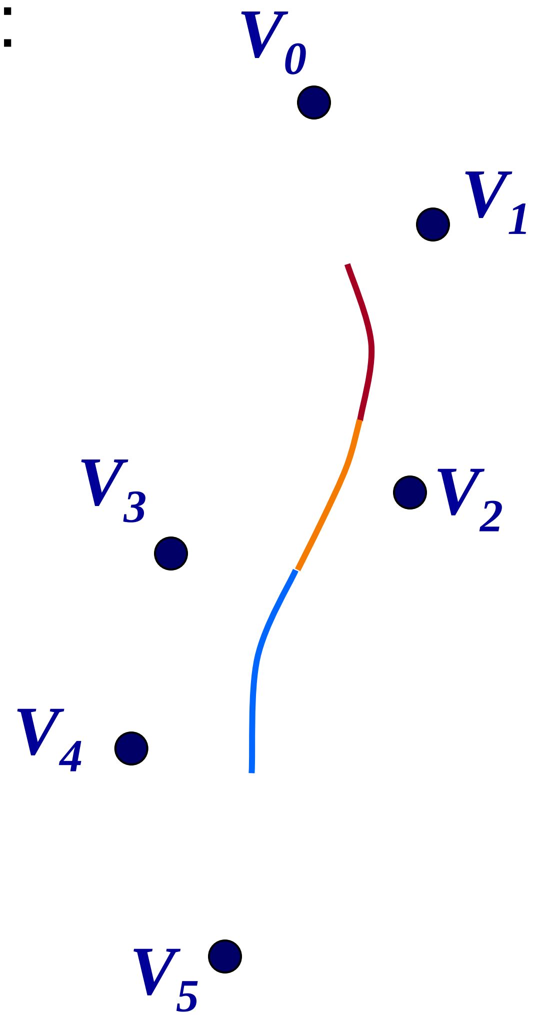


# Cubic B-Spline Blending Functions

- Blending functions imply properties:
  - Local control
  - Approximating
  - $C^2$ continuity
  - Convex hull



# Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier

- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

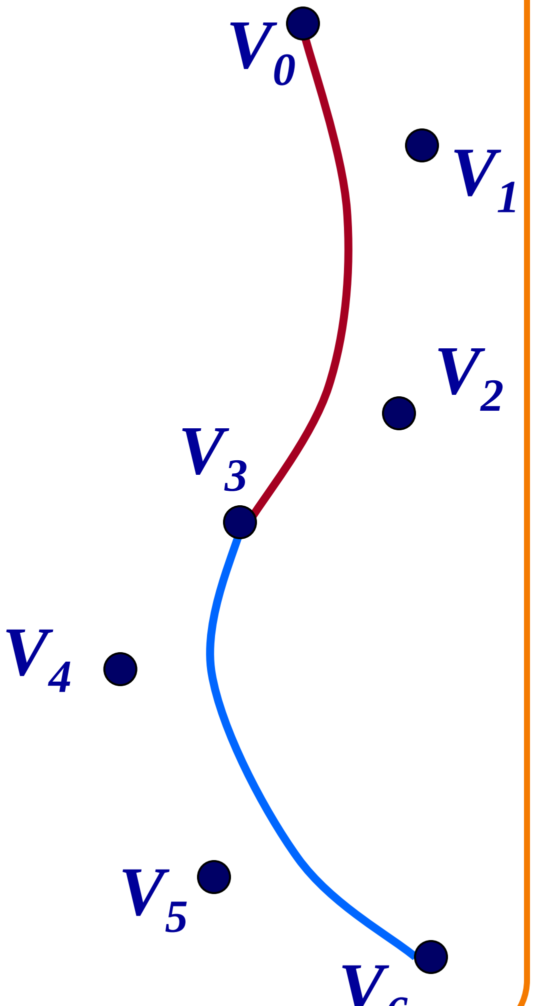
# Cubic Bézier

- Developed around 1960 by both
  - Pierre Bézier (Renault)
  - Paul de Casteljau (Citroen)

- Properties:
  - Local control
  - Continuity depends on control points
  - Interpolating (every third)

Properties determine blending functions

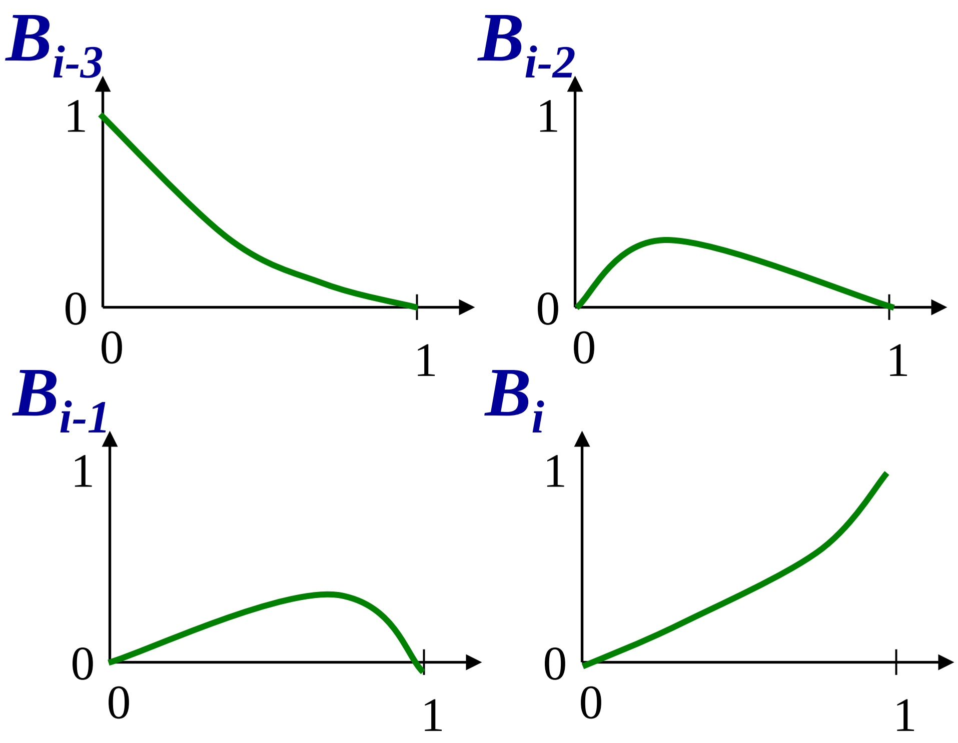
Blending functions determine properties

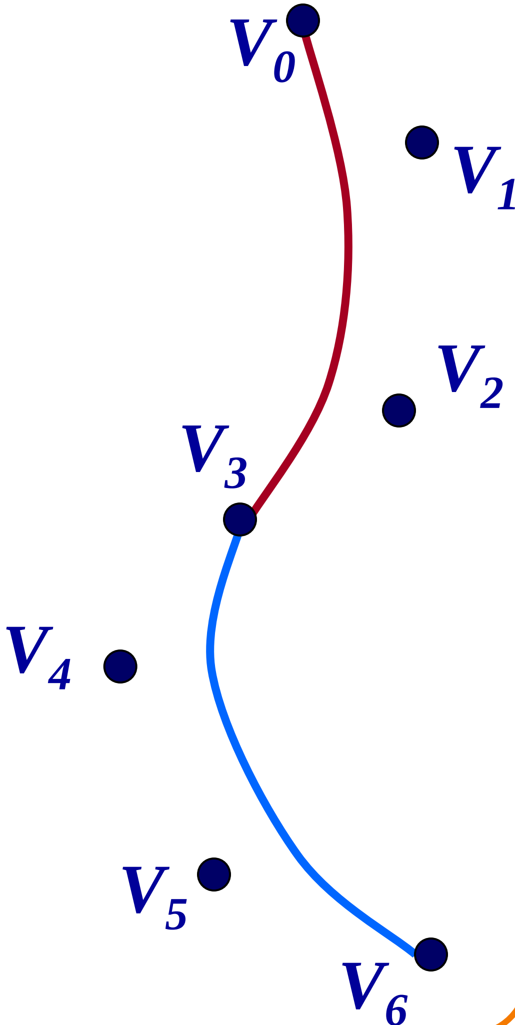


# Cubic Bézier Curves

Blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$



# Cubic Bézier Curves

Bézier curves in matrix form:

$$Q(u) = \sum_{i=0}^{n} V_i \binom{n}{i} u^i (1-u)^{n-i}$$

$$= (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u) V_2 + u^3 V_3$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{M}_{\textbf{Bézier}}} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

# Basic properties of Bézier Curves

- Endpoint interpolation:

$$Q(0) = V_0$$

$$Q(1) = V_n$$

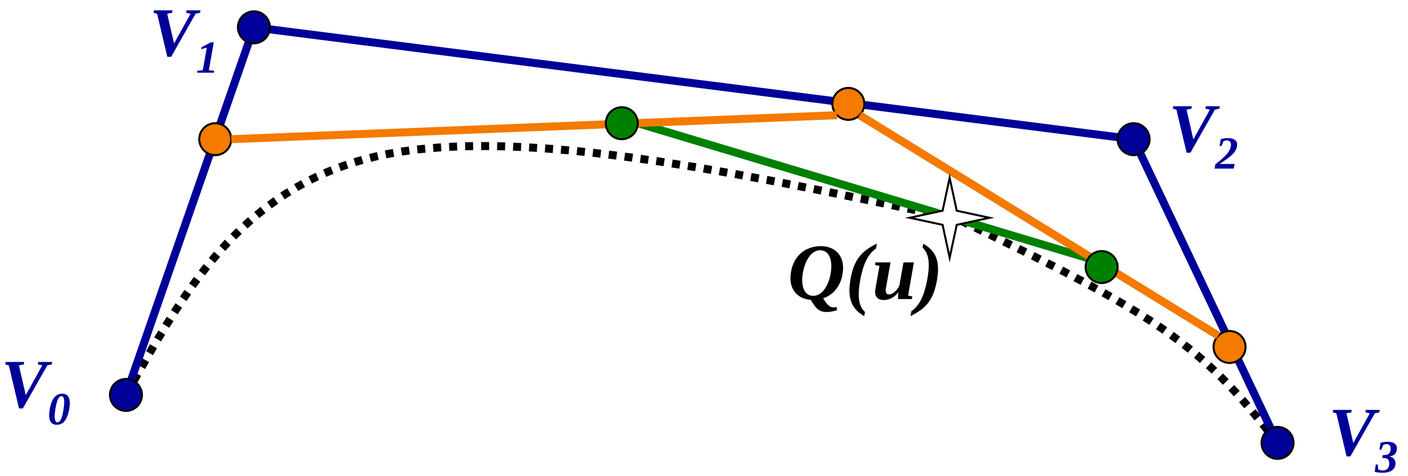
- Convex hull:
  - Curve is contained within convex hull of control polygon

- Symmetry

$$Q(u) \text{ defined by } \{V_0, \ldots, V_n\} \;\equiv\; Q(1-u) \text{ defined by } \{V_n, \ldots, V_0\}$$

# Bézier Curves

- Curve $Q(u)$ can also be defined by nested interpolation:

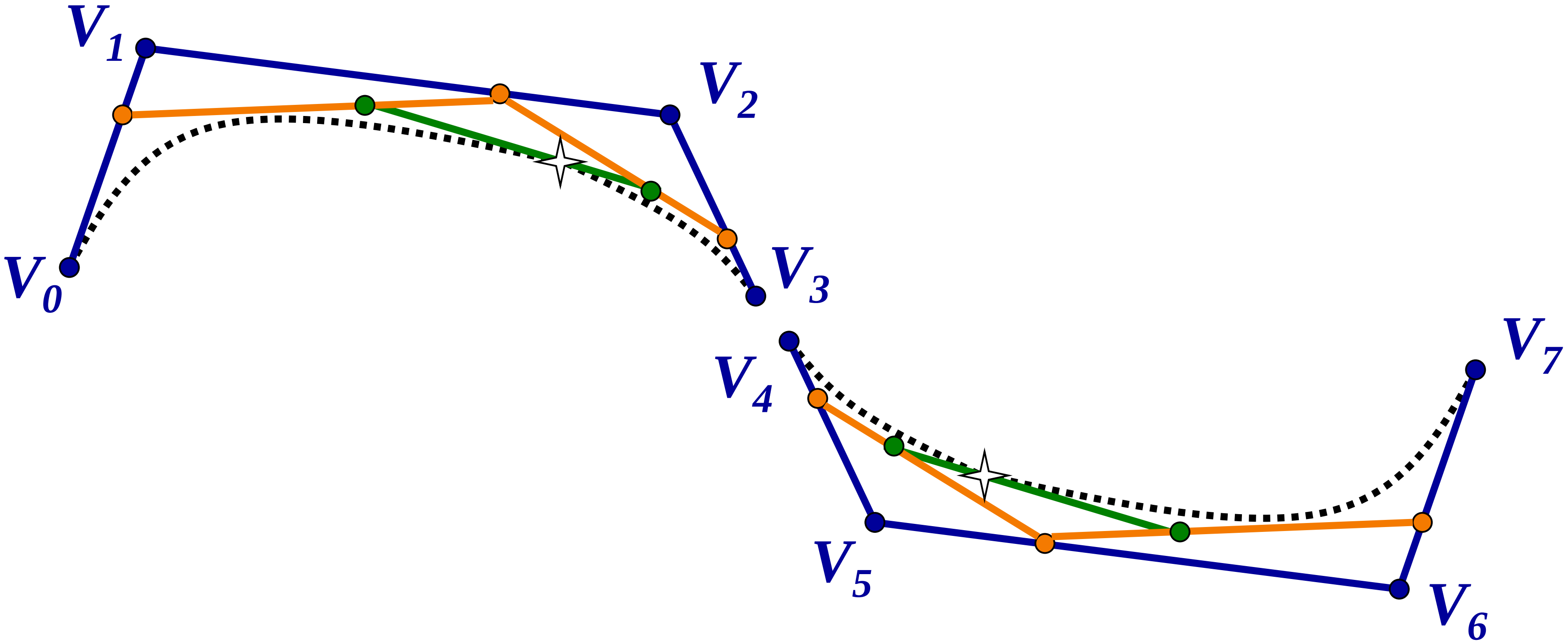



$V_i$ are *control points*

$\{V_0, V_1, \ldots, V_n\}$ is *control polygon*

# Enforcing Bézier Curve Continuity

- $C^0$: $V_3 = V_4$
- $C^1$: $V_5 - V_4 = V_3 - V_2$
- $C^2$: $V_6 - 2V_5 + V_4 = V_3 - 2V_2 + V_1$



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  - Cubic Bézier

➢ Parametric surfaces
  - Bi-cubic B-Spline
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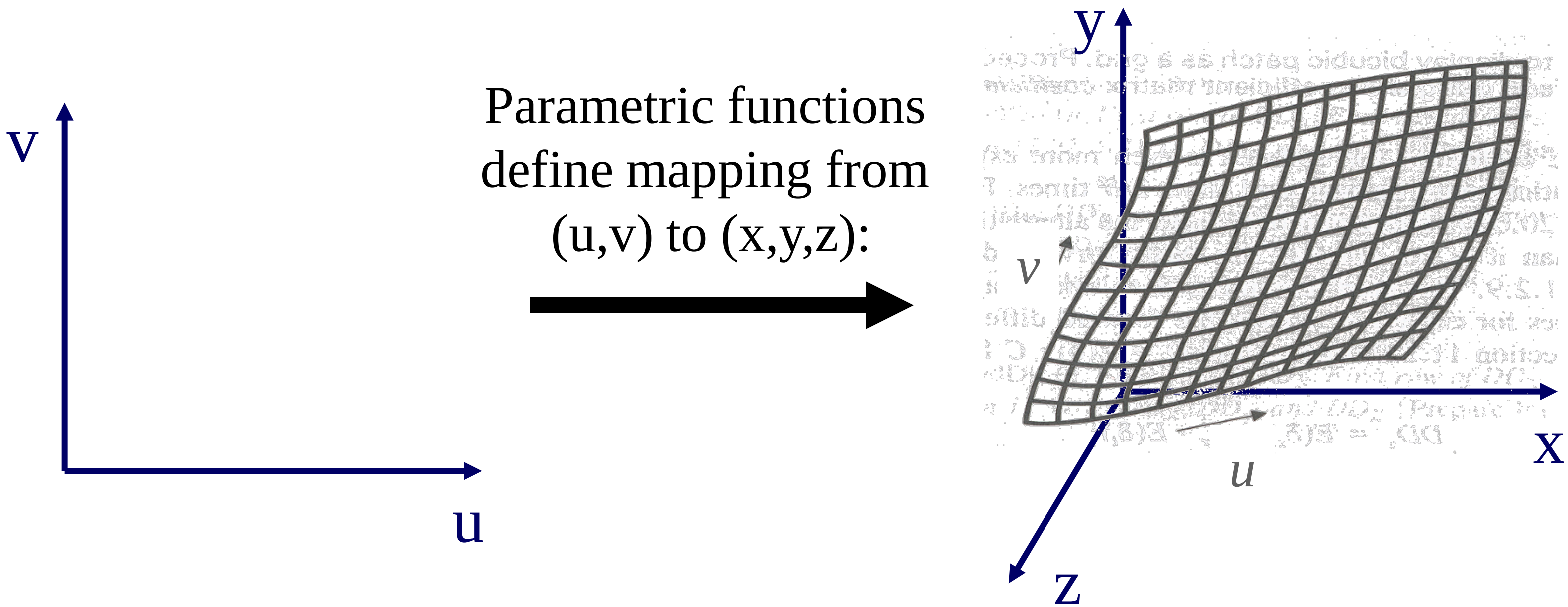
# Parametric Surfaces

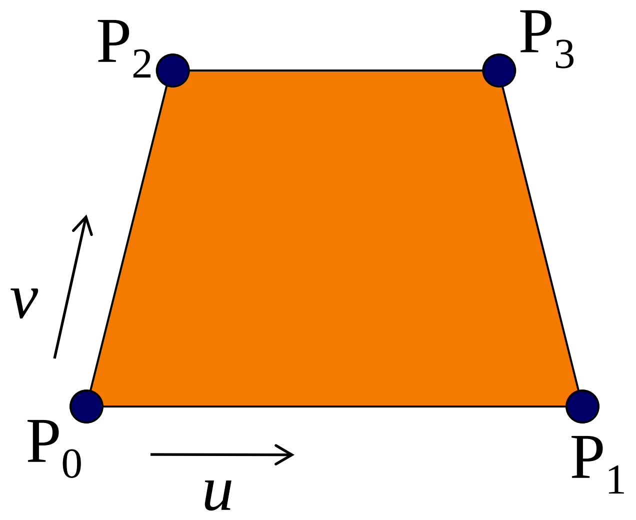
- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$




FvDFH Figure 11.42

# Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$




- Example: quadrilateral

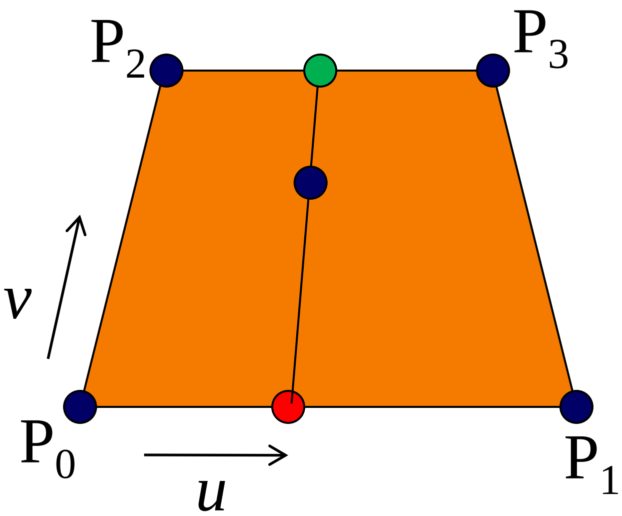
$$f_x(u,v) = (1-v)\big((1-u)x_0 + ux_1\big) + v\big((1-u)x_2 + ux_3\big)$$

$$f_y(u,v) = (1-v)\big((1-u)y_0 + uy_1\big) + v\big((1-u)y_2 + uy_3\big)$$

$$f_z(u,v) = (1-v)\big((1-u)z_0 + uz_1\big) + v\big((1-u)z_2 + uz_3\big)$$

# Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$




- Example: quadrilateral

$$f_x(u,v) = (1-v)\left((1-u)x_0 + ux_1\right) + v\left((1-u)x_2 + ux_3\right)$$

$$f_y(u,v) = (1-v)\left((1-u)y_0 + uy_1\right) + v\left((1-u)y_2 + uy_3\right)$$

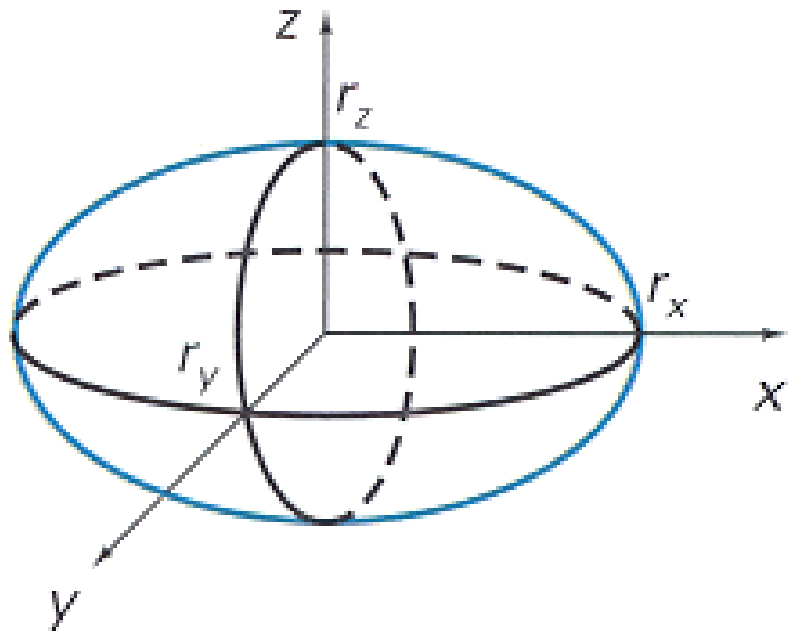
$$f_z(u,v) = (1-v)\left((1-u)z_0 + uz_1\right) + v\left((1-u)z_2 + uz_3\right)$$

# Parametric Surfaces

- Defined by parametric functions:
    - $x = f_x(u,v)$
    - $y = f_y(u,v)$
    - $z = f_z(u,v)$

- Example: ellipsoid

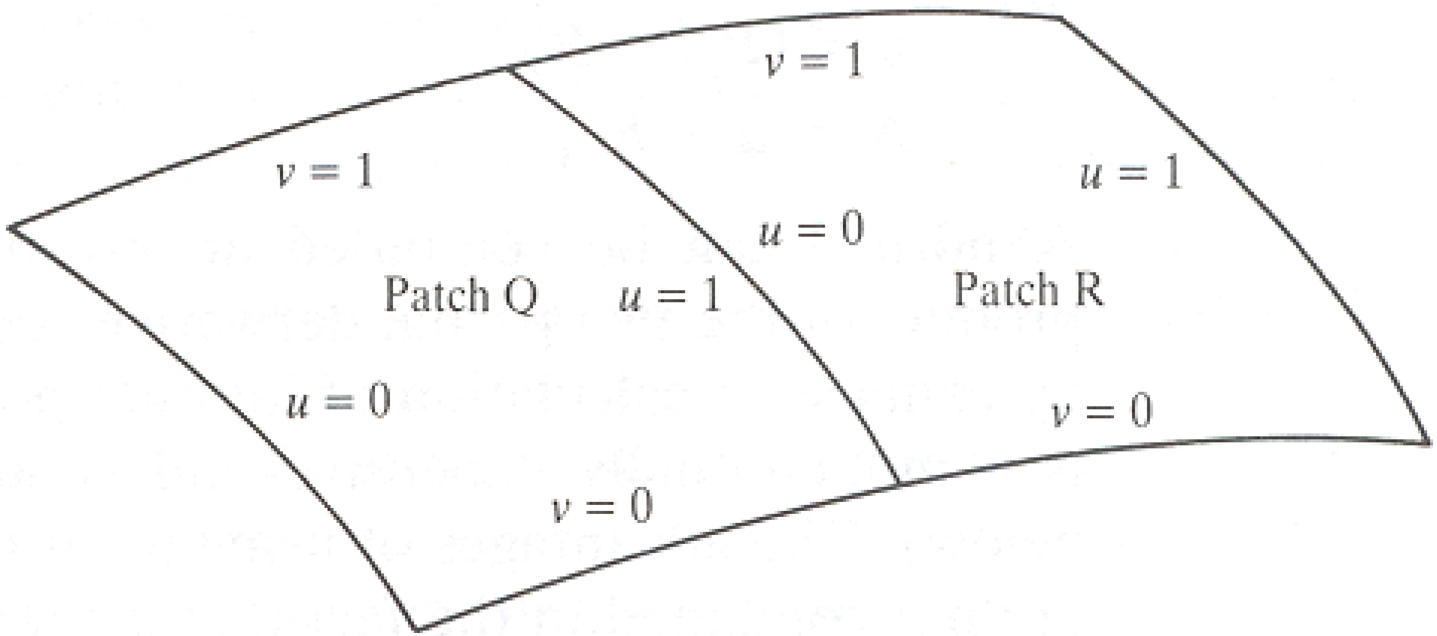
$$f_x(u,v) = r_x \cos v \cos u$$
$$f_y(u,v) = r_y \cos v \sin u$$
$$f_z(u,v) = r_z \sin v$$




H&B Figure 10.10

# Piecewise Polynomial Parametric Surfaces

- Surface is partitioned into parametric patches:

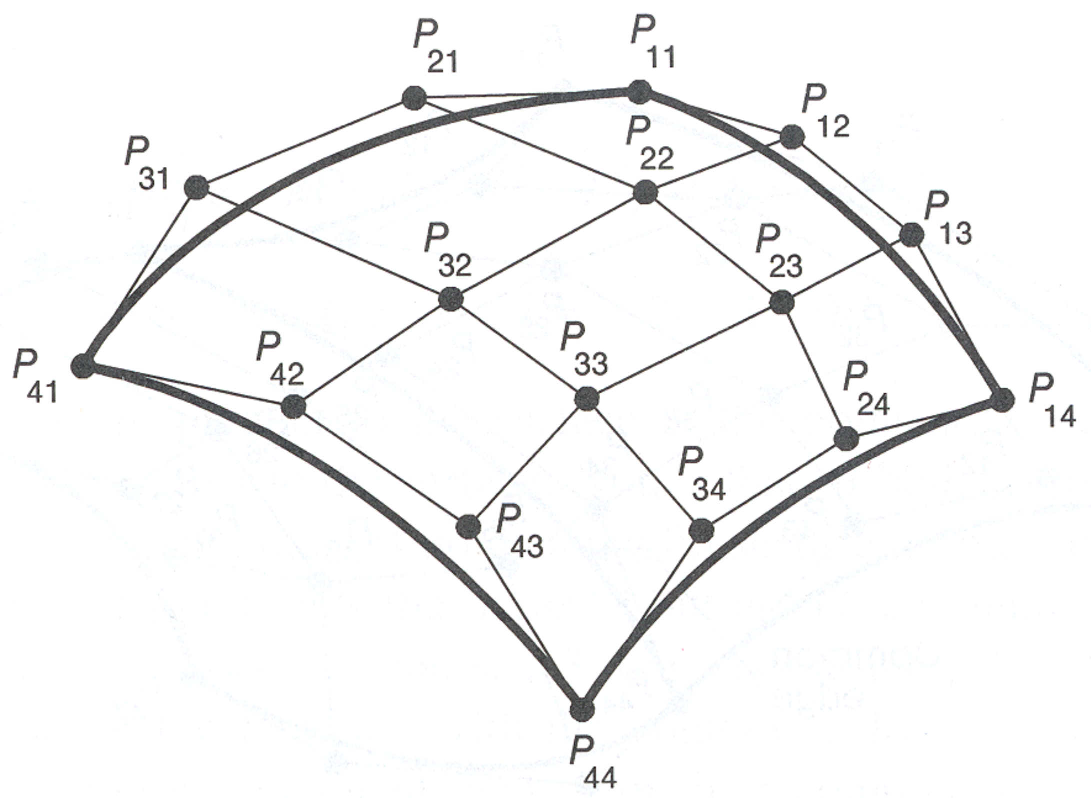



Same ideas as parametric splines!

Watt Figure 6.25

# Parametric Patches

- Each patch is defined by blending control points

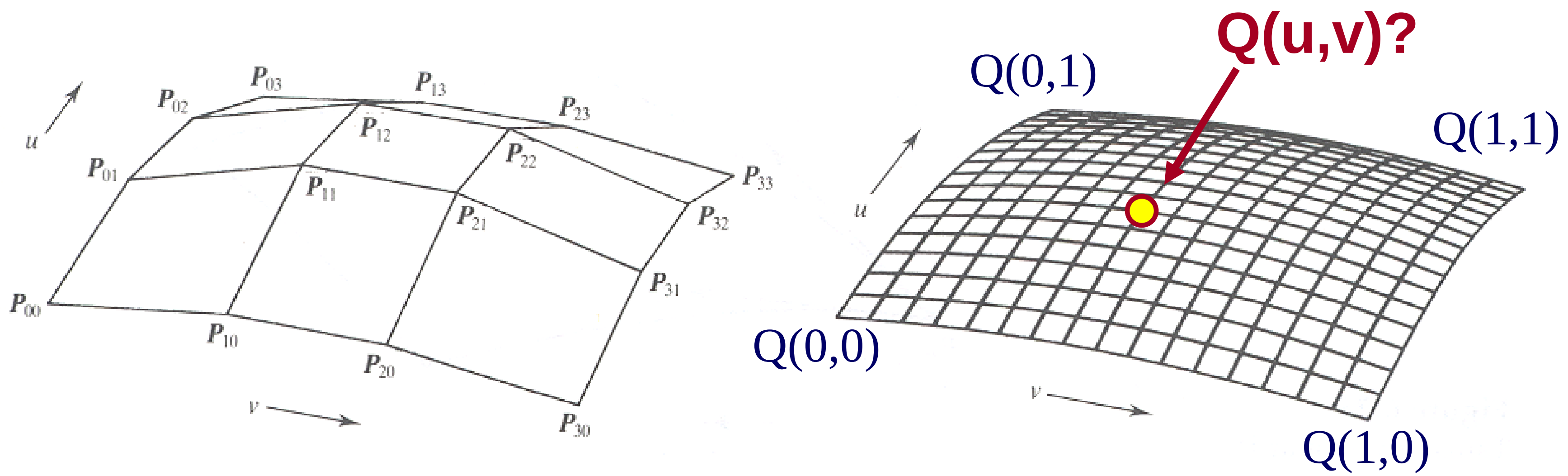



Same ideas as parametric curves!

FvDFH Figure 11.44

# Parametric Patches

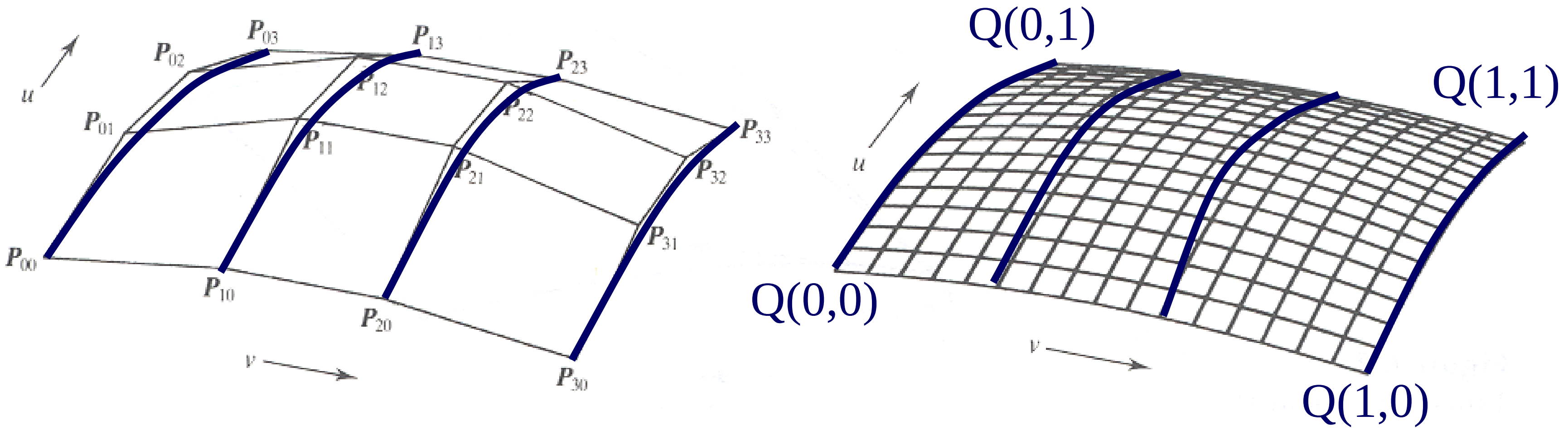
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points




Watt Figure 6.21

# Parametric Patches

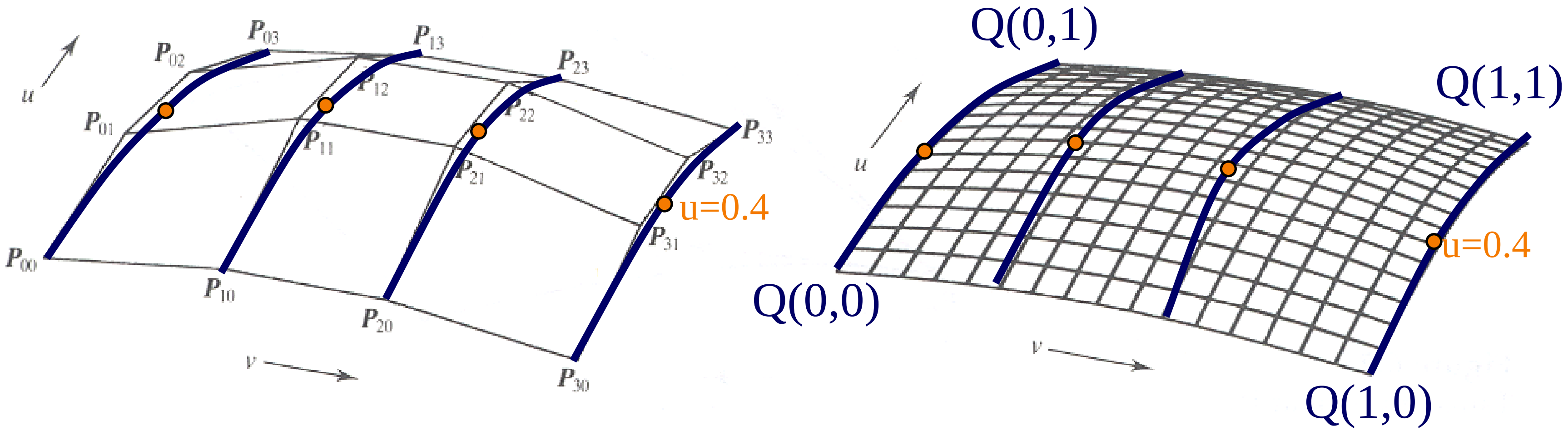
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points




Watt Figure 6.21

# Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points




Watt Figure 6.21

# Parametric Patches

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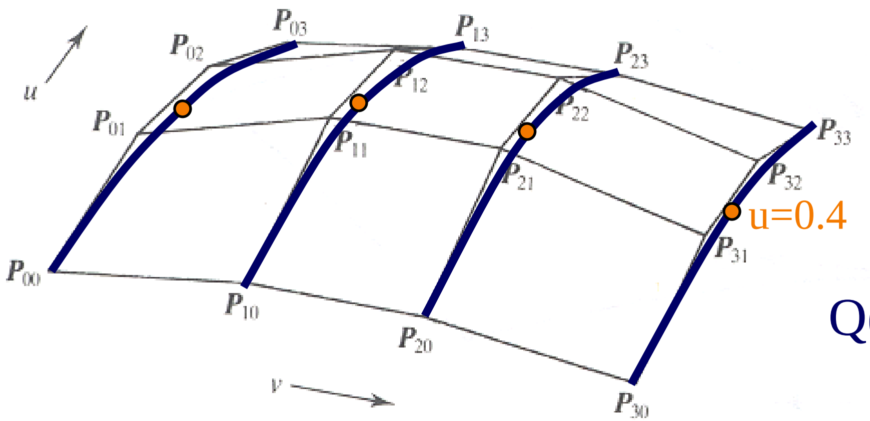
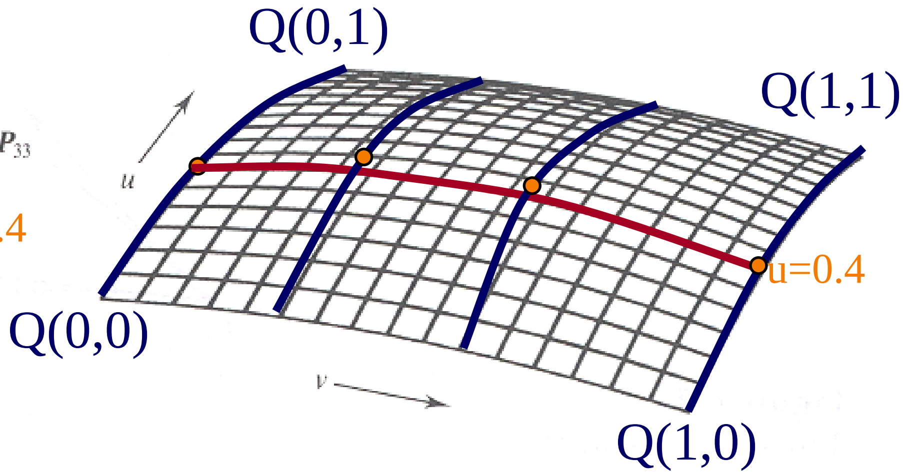
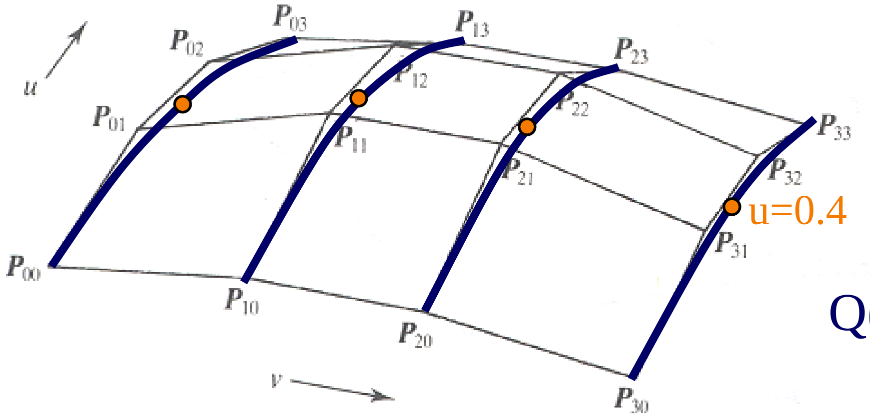


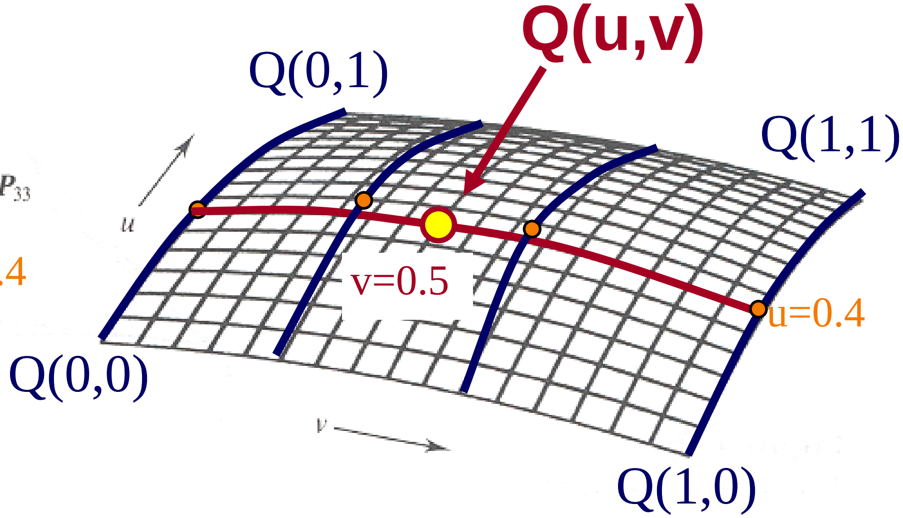




Watt Figure 6.21

# Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points




Watt Figure 6.21

# Parametric Bicubic Patches

Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:

$$Q(u,v) = \mathbf{UM}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}$$

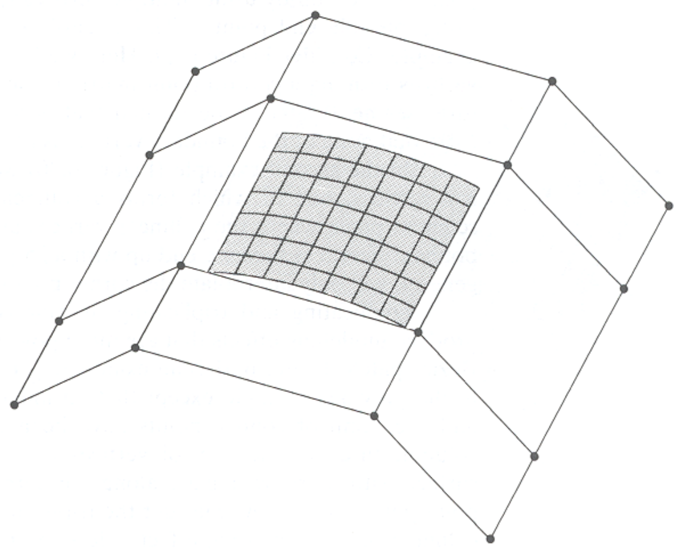
$$\mathbf{U} = [u^3 \quad u^2 \quad u \quad 1] \qquad \mathbf{V} = [v^3 \quad v^2 \quad v \quad 1]$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bézier, B-spline, etc.)

# B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathbf{B-Spline}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}^{\mathbf{T}}_{\mathbf{B-Spline}} \mathbf{V}$$

$$\mathbf{M}_{\mathbf{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$

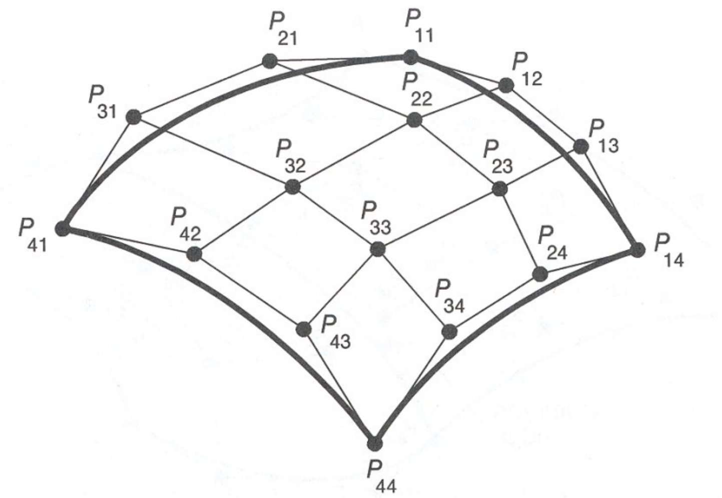


Watt Figure 6.28

# Bézier Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathbf{Bezier}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}_{\mathbf{Bezier}}^{\mathbf{T}} \mathbf{V}$$

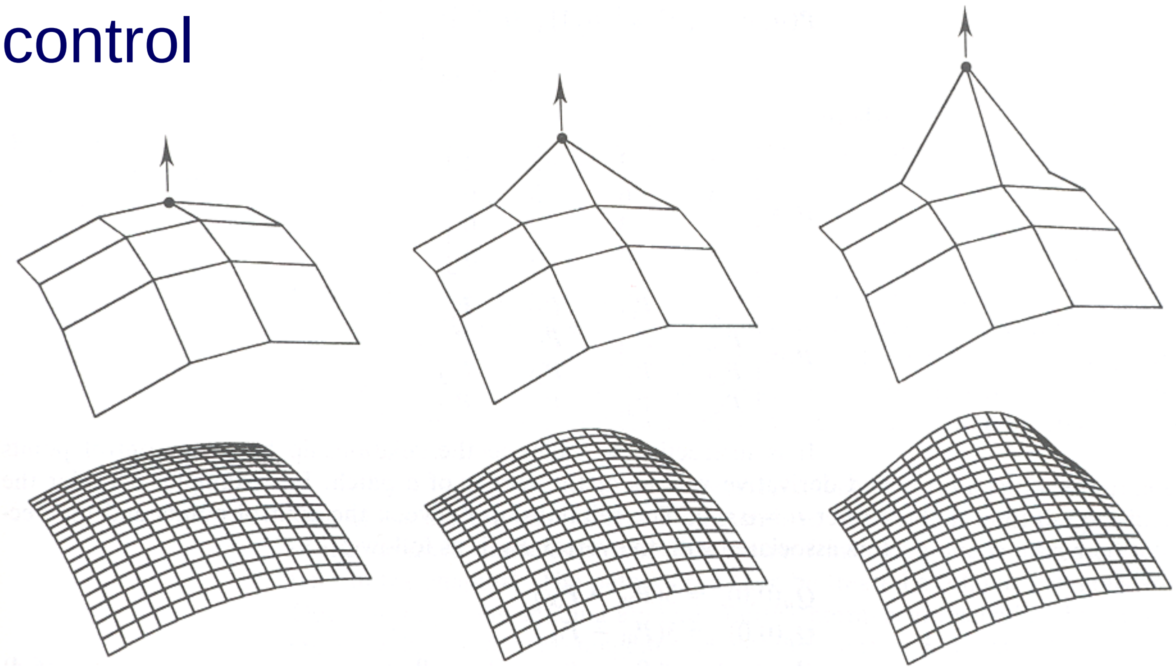
$$\mathbf{M}_{\mathbf{Bezier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$




FvDFH Figure 11.42

# Bézier Patches

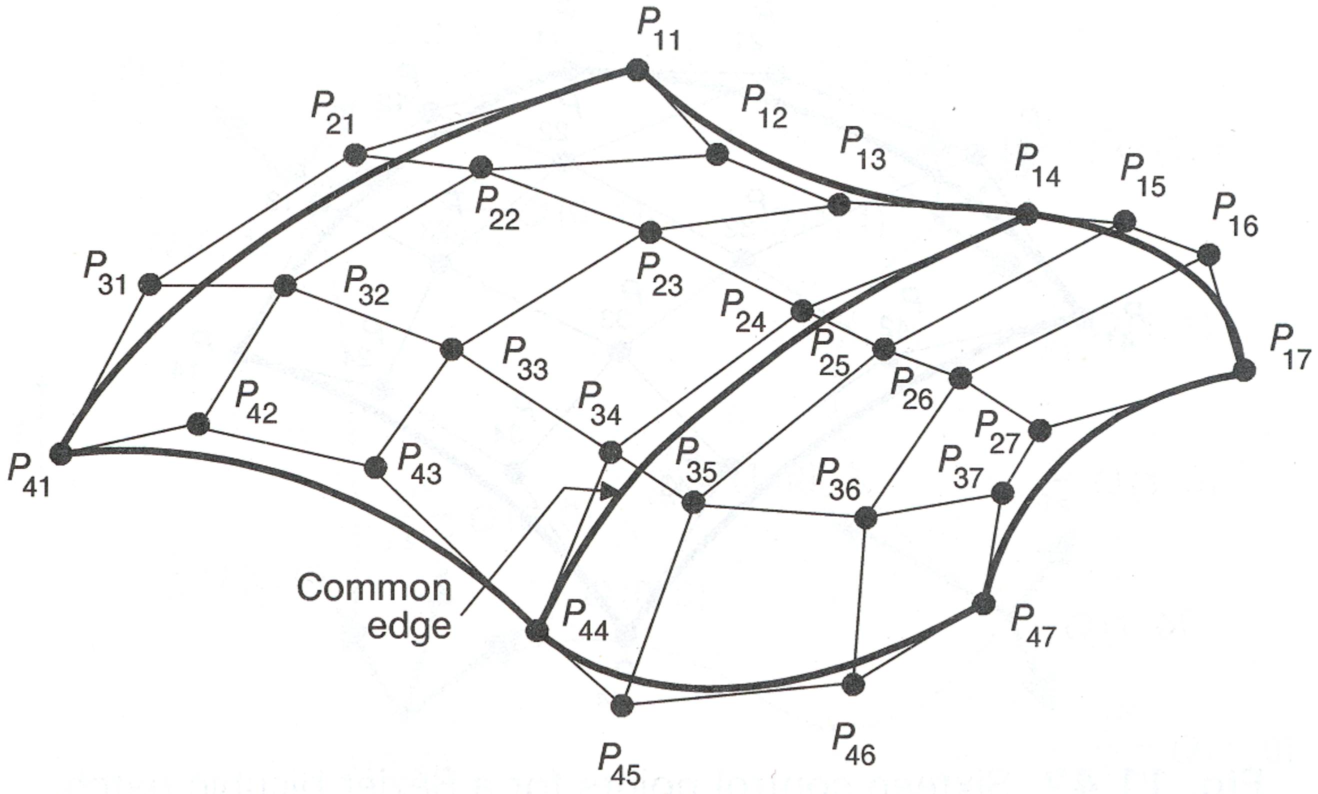
- Properties:
  - Interpolates four corner points
  - Convex hull
  - Local control



Watt Figure 6.22

# Bézier Surfaces

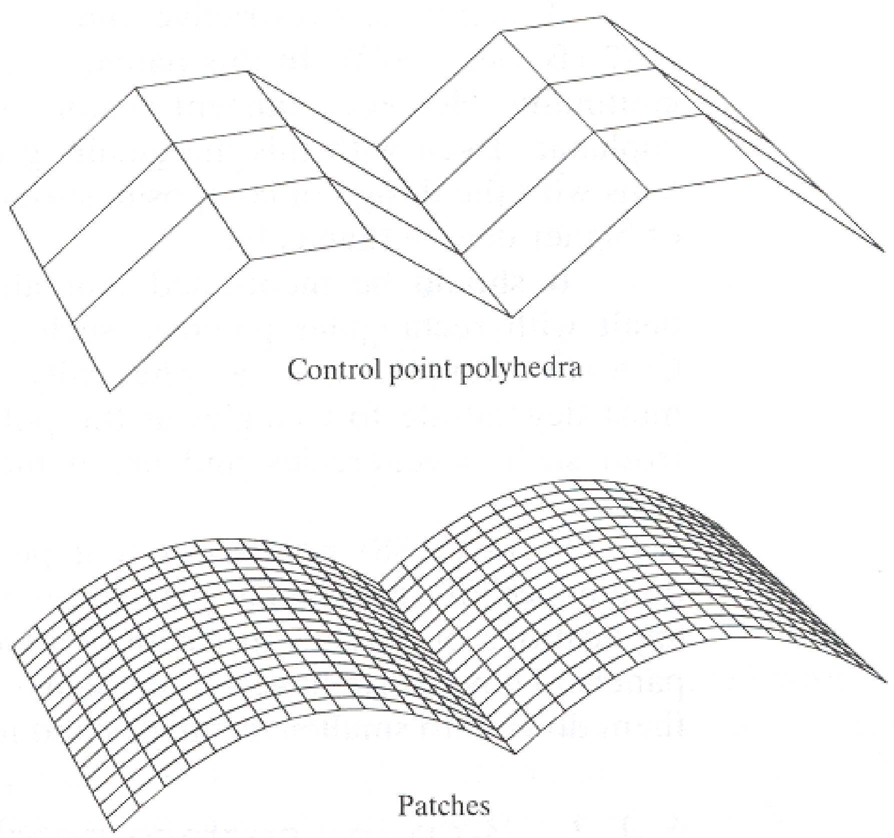
- Continuity constraints are similar to the ones for Bézier splines




FvDFH Figure 11.43

# Bézier Surfaces

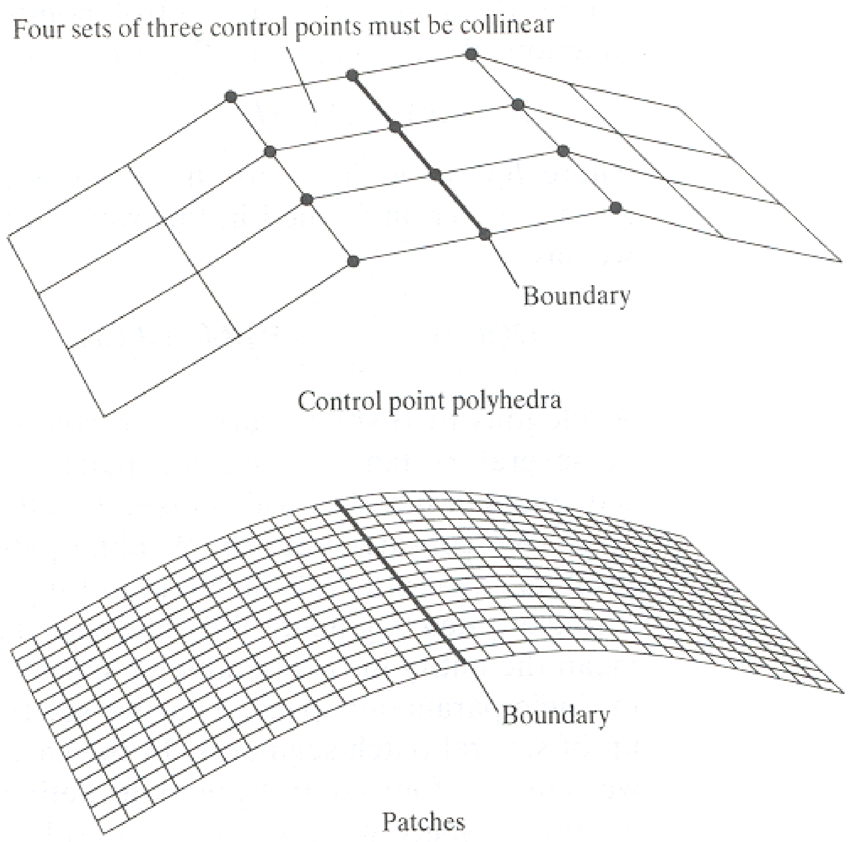
- $C^0$ continuity requires aligning boundary curves




Watt Figure 6.26a

# Bézier Surfaces

- $C^1$ continuity requires aligning boundary curves and derivatives




Watt Figure 6.26b

# Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface
  - Possible to describe complex shapes
- Disadvantages:
  - Control mesh must be quadrilaterals
  - Continuity constraints difficult to maintain:
    $C^0$ easy, $C^1$ possible, $C^2$ hard at extraordinary vertices
  - Hard to find intersections

# Comparison

| Feature | Polygonal Mesh | Parametric Surface | Subdivision Surface |
|---|---|---|---|
| Accurate | No | Yes | Yes |
| Concise | No | Yes | Yes |
| Intuitive specification | No | Yes | No |
| Local support | Yes | Yes | Yes |
| Affine invariant | Yes | Yes | Yes |
| Arbitrary topology | Yes | No | Yes |
| Guaranteed continuity | No | Yes | Yes |
| Natural parameterization | No | Yes | No |
| Efficient display | Yes | Yes | Yes |
| Efficient intersections | No | No | No |