Implicit Surfaces & Solid Representations

COS 426
3D Object Representations

- Desirable properties of an object representation
  - Easy to acquire
  - Accurate
  - Concise
  - Intuitive editing
  - Efficient editing
  - Efficient display
  - Efficient intersections
  - Guaranteed validity
  - Guaranteed smoothness
  - etc.

Large Geometric Model Repository
Georgia Tech
3D Object Representations

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Large Geometric Model Repository
Georgia Tech
3D Object Representations

• Points
  ◦ Range image
  ◦ Point cloud

• Surfaces
  ◦ Polygonal mesh
  ◦ Subdivision
  ◦ Parametric
    ➢ Implicit

• Solids
  ◦ Voxels
  ◦ BSP tree
  ◦ CSG
  ◦ Sweep

• High-level structures
  ◦ Scene graph
  ◦ Application specific
Implicit Surfaces

- Represent surface with function over all space
Implicit Surfaces

- Surface defined implicitly by function
Implicit Surfaces

- Surface defined implicitly by function:
  - \( f(x, y, z) = 0 \) (on surface)
  - \( f(x, y, z) < 0 \) (inside)
  - \( f(x, y, z) > 0 \) (outside)
Implicit Surfaces

- Normals defined by partial derivatives
  - $\text{normal}(x, y, z) = \text{normalize}(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$
Implicit Surface Properties

(1) Efficient check for whether point is inside

- Evaluate $f(x,y,z)$ to see if point is inside/outside/on
- Example: ellipsoid

$$f(x, y, z) = \left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 - 1$$

H&B Figure 10.10
(2) Efficient surface intersections

- Substitute to find intersections
  
  Ray: \( P = P_0 + tV \)
  Sphere: \(|P - O|^2 - r^2 = 0\)

Substituting for \( P \), we get:
\[ |P_0 + tV - O|^2 - r^2 = 0 \]

Solve quadratic equation:
\[ at^2 + bt + c = 0 \]

where:
\[ a = 1 \]
\[ b = 2 \ V \cdot (P_0 - O) \]
\[ c = |P_0 - C|^2 - r^2 = 0 \]
Implicit Surface Properties

(3) Efficient boolean operations (CSG)

- How would you implement: Union? Intersection? Difference?
Implicit Surface Properties

(4) Efficient topology changes

- Surface is not represented explicitly!
Implicit Surface Properties

(4) Efficient topology changes

- Surface is not represented explicitly!
Comparison to Parametric Surfaces

- Implicit
  - Efficient intersections & topology changes

- Parametric
  - Efficient “marching” along surface & rendering

\[ \begin{align*}
  p &= (\cos(\alpha), \sin(\alpha)), \alpha \in [0, 2\pi] \\
  p &= (\pm(1-t^2)/(1+t^2), 2t/(1+t^2)), t \in [-1, 1] \\
  p_x^2 + p_y^2 - 1 &= 0
\end{align*} \]
Implicit Surface Representations

• How do we define implicit function?
  ○ $f(x,y,z) =$ ?
Implicit Surface Representations

- How do we define implicit function?
  - Algebraics
  - Voxels
  - Basis functions
  - Others
Implicit Surface Representations

• How do we define implicit function?
  ➢ Algebraics
    ◦ Voxels
    ◦ Basis functions
    ◦ Others
Algebraic Surfaces

- Implicit function is polynomial
  \[ f(x,y,z) = ax^d + by^d + cz^d + dx^{d-1}y + dx^{d-1}z + dy^{d-1}x + \ldots \]

\[ f(x, y, z) = \left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 - 1 \]

H&B Figure 10.10
Algebraic Surfaces

• Most common form: quadrics
  - $f(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz + 2gx + 2hy + 2jz + k$

• Examples
  - Sphere
  - Ellipsoid
  - Torus
  - Paraboloid
  - Hyperboloid
Algebraic Surfaces

- Higher degree algebraics

Cubic

Quartic

Degree six
Algebraic Surfaces

- Equivalent parametric surface
  - Tensor product patch of degree m and n curves yields algebraic function with degree 2mn

Bicubic patch has degree 18!
Algebraic Surfaces

- Intersection
  - Intersection of degree m and n algebraic surfaces yields curve with degree mn

Intersection of bicubic patches has degree 324!
Algebraic Surfaces

• Function extends to infinity
  ◦ Must trim to get desired patch (this is difficult!)
Implicit Surface Representations

• How do we define implicit function?
  ◦ Algebraics
  ➢ Voxels
  ◦ Basis functions
Voxels

• Regular array of 3D samples (like image)
  ◦ Samples are called *voxels* ("volume pixels")

www.volumegraphics.com
Voxels

- Example isosurfaces

SUNY Stoney Brook

Princeton University
**Voxels**

- Regular array of 3D samples (like image)
  - Apply reconstruction filter to determine $f(x,y,z)$
  - Isosurface at $f(x,y,z) = 0$ defines surface
Voxels

- Iso-surface extraction algorithm
  - e.g., Marching cubes
Voxels

• Iso-surface extraction algorithm
  ◦ e.g., Marching cubes (15 cases)
Voxel Storage

- $O(n^3)$ storage for $n \times n \times n$ grid
  - 1 billion voxels for 1000 x 1000 x 1000
Implicit Surface Representations

- How do we define implicit function?
  - Algebraics
  - Voxels
  - Basis functions
Basis functions

• Implicit function is sum of basis functions
  ◦ Example:

\[
f(P) = a_0 e^{-b_0 d(P, P_0)^2} + a_1 e^{-b_1 d(P, P_1)^2} + \cdots - \tau
\]
Radial Basis Functions

- Blobby molecules
  
  \[ D(r) = ae^{-br^2} \]

- Meta balls
  
  \[ D(r) = \begin{cases} 
  a(1 - \frac{3r^2}{b^2}) & 0 \leq r \leq b/3 \\
  \frac{3a}{2}(1 - \frac{r}{b})^2 & b/3 \leq r \leq b \\
  0 & b \leq r 
  \end{cases} \]

- Soft objects
  
  \[ D(r) = \begin{cases} 
  a(1 - \frac{4r^6}{9b^6}) + \frac{17r^4}{9b^4} - \frac{22r^2}{9b^2} & r \leq b \\
  0 & r \geq b 
  \end{cases} \]
Blobby Models

- Implicit function is sum of Gaussians

\[ f(P) = a_0 e^{-b_0 d(P, P_0)^2} + a_1 e^{-b_1 d(P, P_1)^2} + \cdots - \tau \]
Blobby Models

• Sum of two blobs
Blobby Models

- Sum of four blobs
Blobby Model of Face

(a) $N = 1$

(b) $N = 2$
Blobby Model of Face

(c) $N = 10$

(d) $N = 35$
Blobby Model of Face

(e) $N = 70$

(f) $N = 243$
Blobby Model of Head

(a) $N = 1$

(b) $N = 2$
Blobby Model of Head

(c) $N = 20$

(d) $N = 60$
Blobby Model of Head

(e) $N = 120$

(f) $N = 451$
Blobby Models

Objects resulting from CSG of implicit soft objects and other primitives

Menon
Variational Implicit Surfaces
Variational Implicit Surfaces
Implicit Surface Summary

• Advantages:
  ○ Easy to test if point is on surface
  ○ Easy to compute intersections/unions/differences
  ○ Easy to handle topological changes

• Disadvantages:
  ○ Indirect specification of surface
  ○ Hard to describe sharp features
  ○ Hard to enumerate points on surface
    » Slow rendering
# Summary

<table>
<thead>
<tr>
<th>Feature</th>
<th>Polygonal Mesh</th>
<th>Implicit Surface</th>
<th>Parametric Surface</th>
<th>Subdivision Surface</th>
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3D Object Representations

• Points
  ◦ Range image
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• Surfaces
  ◦ Polygonal mesh
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  ◦ Parametric
  ◦ Implicit

• Solids
  ◦ Voxels
  ◦ BSP tree
  ◦ CSG
  ◦ Sweep

• High-level structures
  ◦ Scene graph
  ◦ Application specific
Solid Modeling

- Represent solid interiors of objects

www.volumeographics.com
Motivation 1

• Some acquisition methods generate solids

Airflow Inside a Thunderstorm
(Bob Wilhelmson, University of Illinois at Urbana-Champaign)

Visible Human
(National Library of Medicine)
Motivation 2

- Some applications require solids
  - Examples: medicine, CAD/CAM

SUNY Stoney Brook

Intergraph Corporation
Motivation 3

• Some operations are easier with solids
  ◦ Example: union, difference, intersection

Union

Difference
3D Object Representations

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- **High-level structures**
  - Scene graph
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Voxels

- Regular array of 3D samples (like image)
  - Samples are called voxels ("volume pixels")

www.volumegraphics.com
Voxels

• Store properties of solid object with each voxel
  ◦ Occupancy
  ◦ Color
  ◦ Density
  ◦ Temperature
  ◦ etc.

Engine Block
Stanford University

Visible Human
(National Library of Medicine)
Voxel Processing

• Signal processing (just like images)
  ◦ Reconstruction
  ◦ Resampling

• Typical operations
  ◦ Blur
  ◦ Edge detect
  ◦ Warp
  ◦ etc.

• Often fully analogous to image processing
Voxel Boolean Operations

- Compare objects voxel by voxel
  - Trivial

\[ \cup \quad = \quad \cap \]

\[ \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4}
\end{array} \]
Voxel Display

- Slicing
  - Draw 2D image resulting from intersecting voxels with a plane
Voxel Display

• Isosurface rendering
  ○ Interpolate samples stored on regular grid
  ○ Isosurface at \( f(x,y,z) = 0 \) defines surface
Voxel Display

- Ray casting
  - Integrate density along rays: compositing!

Engine Block
Stanford University
Voxel Display

- Extended ray-casting
  - **Transfer functions:**
    - Map voxel values to opacity and material
  - Normals (for lighting) from density gradient

Bruckner et al. 2007
Voxels

- **Advantages**
  - Simple, intuitive, unambiguous
  - Same complexity for all objects
  - Natural acquisition for some applications
  - Trivial boolean operations

- **Disadvantages**
  - Approximate
  - Not affine invariant
  - Expensive display
  - Large storage requirements
Voxels

- What resolution should be used?
Quadtrees & Octrees

• Refine resolution of voxels hierarchically
  ◦ More concise and efficient for non-uniform objects

Uniform Voxels

Quadtree

FvDFH Figure 12.21
Quadtree Processing

- Hierarchical versions of voxel methods
  - Finding neighbor cell requires traversal of hierarchy: expected/amortized $O(1)$
Quadtree Boolean Operations

\[ A \cup B \]

\[ A \cap B \]
3D Object Representations

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BSP Trees

Object

Binary Spatial Partition

Binary Tree
BSP Trees

- Key properties
  - visibility ordering (later)
  - hierarchy of convex regions
3D Object Representations

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Constructive Solid Geometry (CSG)

- Represent solid object as hierarchy of boolean operations
  - Union
  - Intersection
  - Difference
CSG Acquisition

• Interactive modeling programs
  ◦ Intuitive way to design objects
CSG Acquisition

- Interactive modeling programs
  - Intuitive way to design objects

H&B Figure 9.9
CSG Boolean Operations

- Create a new CSG node joining subtrees
  - Union
  - Intersection
  - Difference
• Ray casting
3D Object Representations

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Sweeps

- Swept volume
  - Sweep one curve along path of another curve

Demetri Terzopoulos
Sweeps

- Surface of revolution
  - Take a curve and rotate it about an axis
Sweeps

- Surface of revolution
  - Take a curve and rotate it about an axis
## Summary

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<tr>
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<th>Octree</th>
<th>BSP</th>
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<tr>
<td>Accurate</td>
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<td>Efficient boolean operations</td>
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