COS 423 Lecture 11
Finding Minimum Spanning Trees Faster

Concurrent greedy with cleanup

For certain families of graphs, concurrent greedy with cleanup runs in O(*n*) time

Requirements:

(i) All graphs in the family are sparse: $m = O(n)$

(ii) The family is closed under *edge contraction*: combine both ends of the edge into a single vertex, with an edge to any vertex that was adjacent to either end

If $m \leq cn$, concurrent greedy with cleanup takes $O(c(n + n/2 + n/4 + \ldots)) = O(n)$ time

Trees: $m < n$, closed under contraction, no cleanups needed

Planar graphs: $m < 3n$, closed under contraction

For concurrent greedy (with or without cleanups) to run in $O(m)$ time on a arbitrary graph, we need a way to “thin” a graph: using red rule, color red all but $O(n)$ edges, in $O(m)$ time.

Faster algorithms for general graphs

O($m$lglg$n$) Yao 1975, packets

Run concurrent greedy algorithm, but with only m/lg$n$ edges. To do this, group edges incident to each vertex into packets, each of size lg$n$ (with at most one small packet per vertex). Give the main algorithm only the minimum-weight edge in each packet.

Time to find packet minima is O($m$lglg$n$).

O($m$lg*n) Fredman & Tarjan 1984, F-heaps

Store the set of edges incident to each blue tree in an F-heap (or rank-pairing heap). Run single-source greedy until blue tree is big enough; then choose an unconnected source and run single-source from it. Repeat until all blue trees are big enough. Clean up. Repeat.

Algorithm is a hybrid of single-source and concurrent greedy with cleanup. Each round takes O(m) time. If blue trees have size at least k before a round, they have size at least 2^k after $\rightarrow$ lg*n rounds.

O($m$lglg*n) Gabow et al. 1986, F-heaps + packets

O(m) Karger et al. 1995, thinning + random sampling

O($m\alpha(n)$) Chazelle 1998, soft heaps + complicated hybrid algorithm

O(minimum) Pettie & Ramachandran 2002, Chazelle's algorithm with fixed-depth recursion + brute force for small subproblems

The power of random sampling

Concurrent greedy + thinning

How to thin?

A related question: MST verification

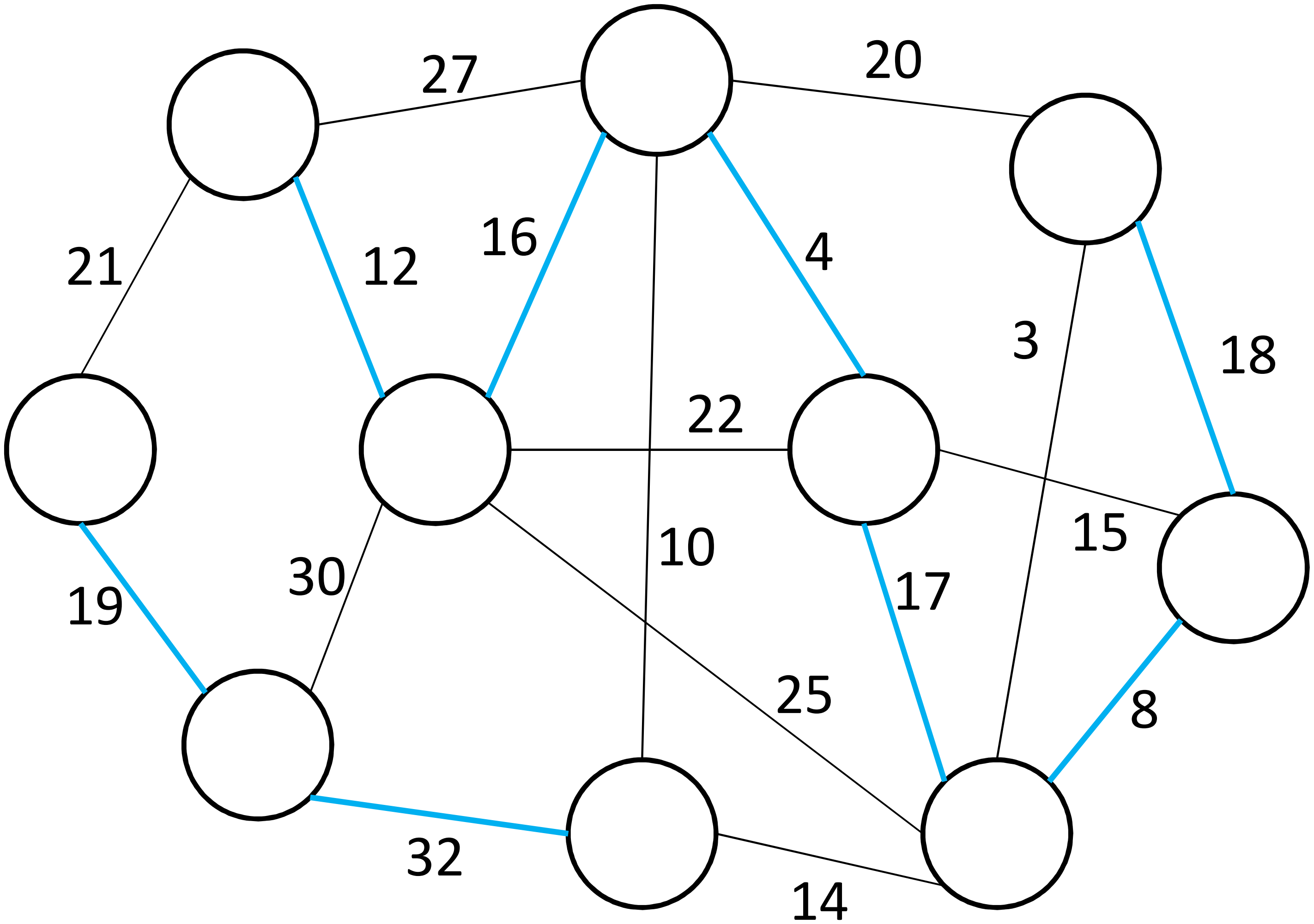
Given a spanning tree T*, is it an MST?*

Yes, if and only if every non-tree edge (*v*, *w*) has maximum weight on the cycle formed with the path in *T* joining *v* and *w*

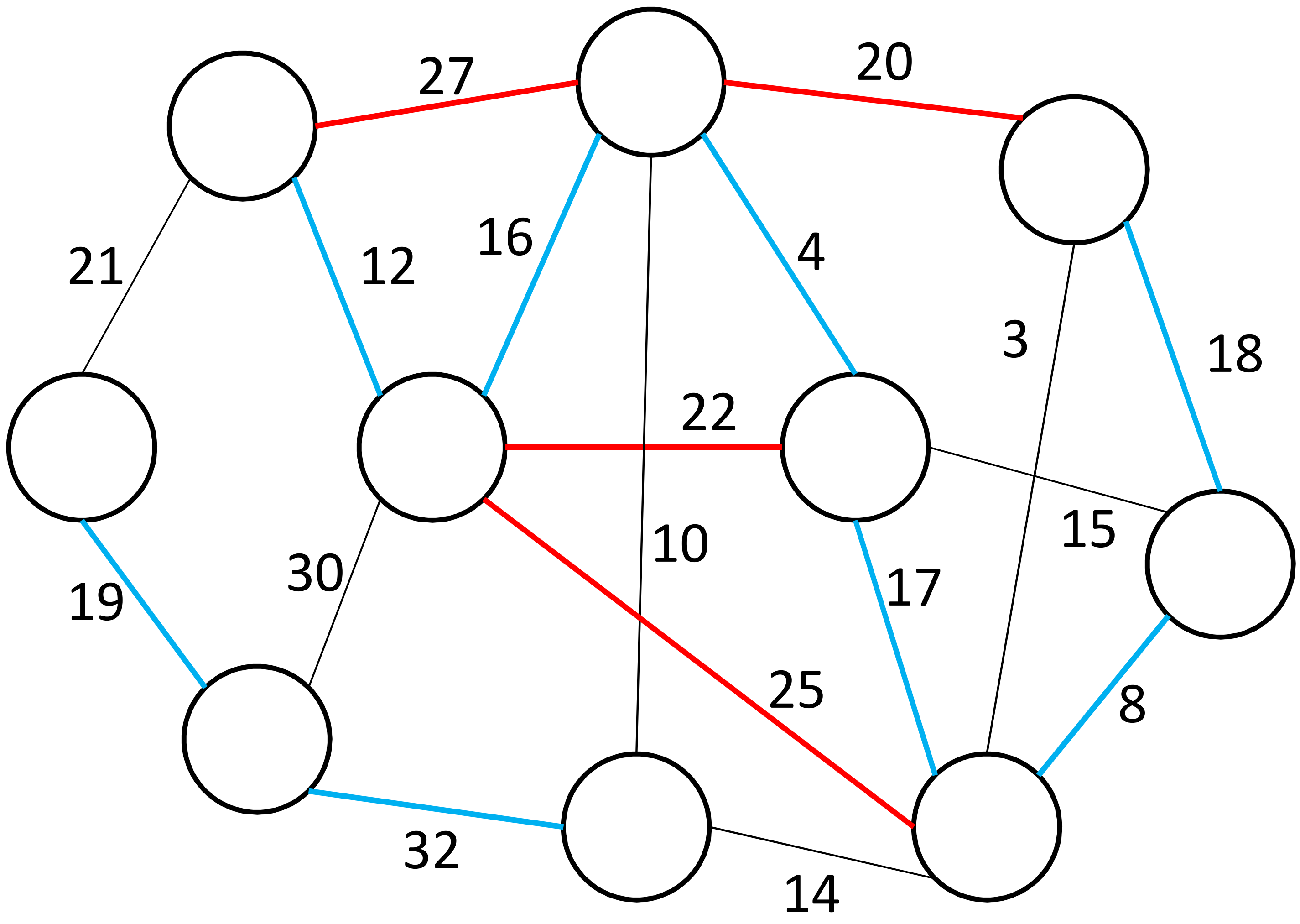
Proof: Red rule

Use the same idea to thin: given any forest (set of vertex-disjoint trees), can color red any non-tree edge whose ends are in the same tree and whose weight is maximum on the cycle formed with tree edges

Thinning using a forest



Thinning using a forest



How to find maxima on tree paths?

For now, assume O(*m*)

How to find a good forest? Best is an MST, but too expensive to compute

Good enough: an MSF (minimum spanning forest) of a random sample of the edges. (The sample subgraph may not be connected)

Randomized minimum spanning tree algorithm

Concurrent greedy with occasional thinning

Let b = #blue trees, initially n,

e = #uncolored edges, initially m

c = a constant to be chosen later

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while $b > 1$ do
    if $e < cb$ then one pass of concurrent greedy
                 else thinning step
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Thinning step

Sample the uncolored edges by adding each edge with probability ½, independently

Find an MSF of the sample by applying the MST algorithm recursively to each connected component of the sample

Color red all sampled edges not in the MSF and all non-sampled edges maximum on a cycle with MSF edges

After thinning, expected #uncolored edges $\leq 2b$

Expected running time

$R(e) \leq \text{O}(e) + R(e - b/2)$ if sparse

$\leq \text{O}(e) + R(e/2) + T(2b)$ if dense

Sparse: $e < cb \rightarrow b/2 > e/(2c)$

$\rightarrow e - b/2 < e(1 - 1/(2c))$

Dense: $e \geq cb \rightarrow 2b \leq 2e/c$

$\rightarrow e/2 + 2b \leq e(1/2 + 4e/c)$

$c = 5 \rightarrow R(e) \leq \text{O}(e) + R(9e/10) = \text{O}(e)$

After thinning, expected #uncolored edges ≤ $2b$

Proof: Think of building the MSF *F* of the sample in the following way: Process the edges in increasing order by weight. To process (*v*, *w*), flip coin. If heads, put (*v*, *w*) in sample: if ends in same tree, color red; otherwise, add to *F*. If tails, not in sample: if ends in same tree, color red; otherwise, not in sample, not colored.

Proof (cont.): Do coin flip after testing whether ends are in same tree: if ends in same tree, color red; otherwise, flip coin, add to *F* if heads. This change has no effect on the outcome: *F* is the same, as is the set of red edges. (The outcomes of the coin flips on the red edges have no effect.)

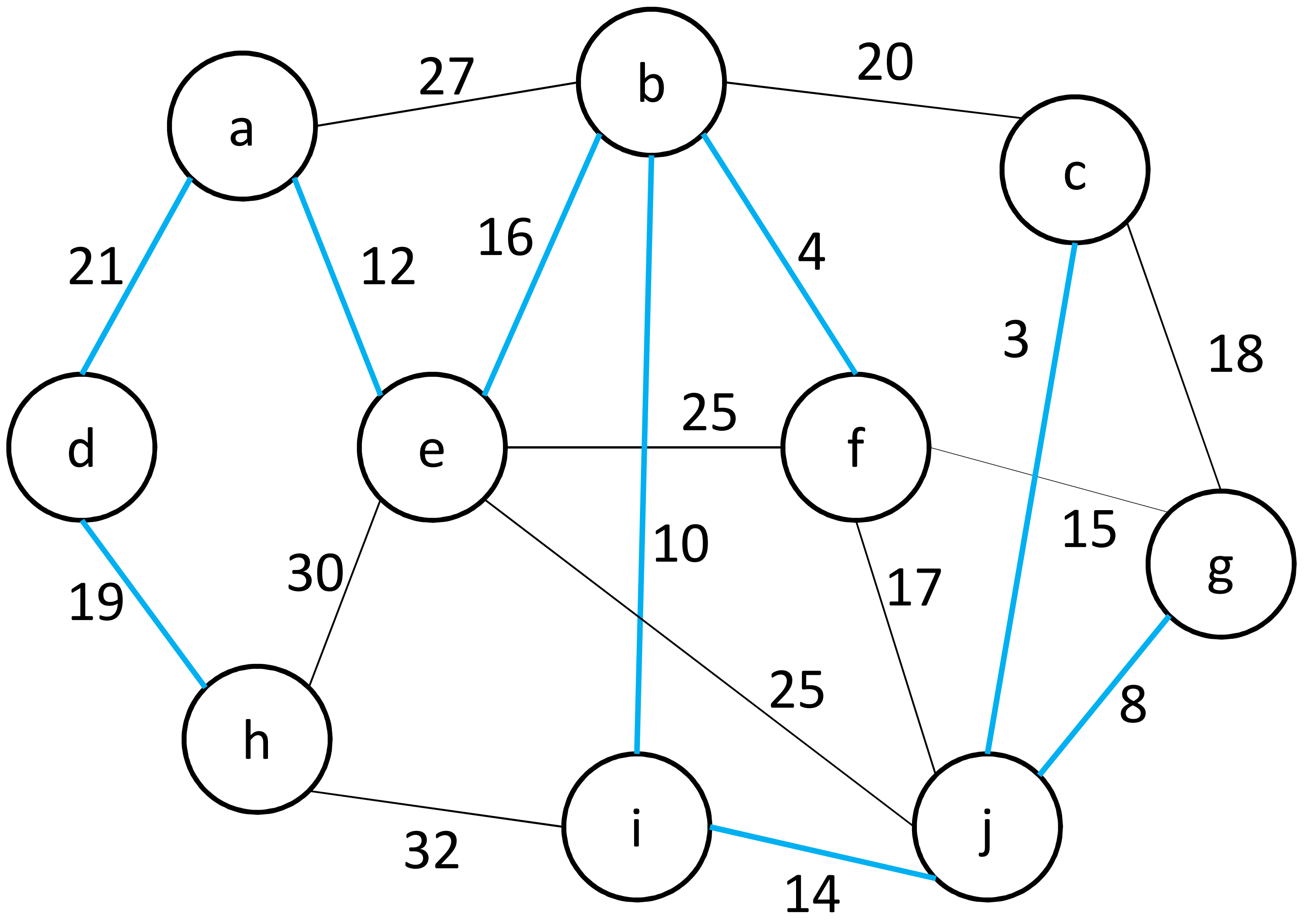
Expected #uncolored edges = expected #coin flips = expected #flips until $b - 1$ heads. Each flip increases expected #heads by ½ → expected #flips = $2(b - 1)$.

Finding maxima on tree paths

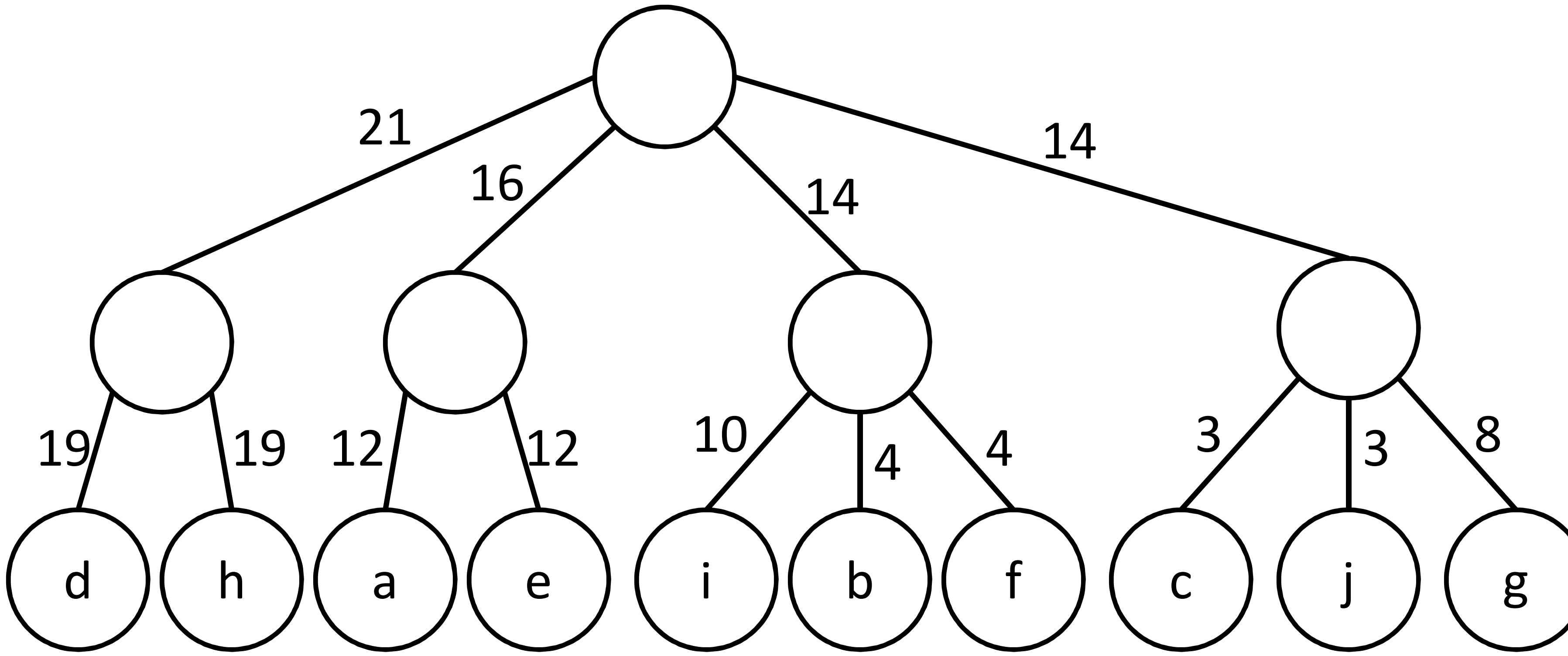
Convert to a problem on a shallow tree

Given tree T with edge weights, the *Borůvka tree* $B(T)$ is formed from T by running the concurrent greedy algorithm on T. Tree B contains one node for each blue tree formed. Each leaf of B is a vertex of T; each non-leaf is a blue tree containing >1 vertex; the root is the final blue tree. Node x is the parent of node y in B if y is a blue tree before some pass k and x is the blue tree containing the vertices of y after pass k. The weight of edge (x, y) is the weight of the edge incident to y selected during pass k.

Minimum spanning tree



Borůvka tree



If T has n vertices, B has $<2n$ nodes, $\leq n/2^k$ of depth k

The concurrent greedy algorithm can build the Borůvka tree of the MST as it builds the MST

$T(v, w)$ = path joining vertices v and w in T

$B(v, w)$ = path joining nodes v and w in B

$p(v)$ = parent of v in B

For any v, w in T, $\max\{c(x, y) \mid (x, y) \text{ on } T(v, w)\} = \max\{c(x, y) \mid (x, y) \text{ on } B(v, w)\}$

Proof:

(≤): Let (x, y) have maximum weight on $T(v, w)$. Let U be a blue tree that selects (x, y). Deleting (x, y) from T forms a cut X, Y with one of v and w in X and the other in Y. Let x and v be in X, so y and w are in Y. Since (x, y) has maximum weight on $T(x, y)$, v but not w is in U. The edge $(U, p(U))$ has weight $c(x, y)$, and this edge is on $B(v, w)$.

(≥): Let $(U, p(U))$ be any edge on B(v, w). Let v be in U, so w is not in U. Let (x, y) be the edge on $T(v, w)$ with exactly one end in U. Then $c(x, y) \geq c(U, p(U))$.