Path Compression and Making the Inverse Ackermann Function Appear Natural(ly)

Raimund Seidel

Universität des Saarlandes
Theorem:

Any sequence of $m$ Union, Find operations in a universe of $n$ elements that uses linking by rank and path compression takes time at most

$$O(m \cdot \alpha(m,n) + n)$$
Theorem:

Any sequence of $m$ Union, Find operations in a universe of $n$ elements that uses linking by rank and path compression takes time at most

$$O( m \cdot \alpha(m,n) + n )$$

where $\alpha(m,n)$ is the “Functional Inverse” of the Ackermann Function.
What is this $\alpha(m,n)$ ??
What is this $\alpha(m,n)$? 

Why does this $\alpha(m,n)$ appear in the analysis of path compression?
What is this $\alpha(m,n)$ ??
A two-parameter variation of the inverse Ackermann function can be defined as follows:

\[
\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log_2 n \}.
\]

This function arises in more precise analyses of the algorithms mentioned above, and gives a more refined time bound. In the disjoint-set data structure, \(m\) represents the number of operations while \(n\) represents the number of elements; in the minimum spanning tree algorithm, \(m\) represents the number of edges while \(n\) represents the number of vertices. Several slightly different definitions of \(\alpha(m, n)\) exist; for example, \(\log_2 n\) is sometimes replaced by \(n\), and the floor function is sometimes replaced by a ceiling.
Definition and properties

The Ackermann function is defined recursively for non-negative integers \( m \) and \( n \) as follows:

\[
A(m, n) = \begin{cases} 
  n + 1 & \text{if } m = 0 \\
  A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\
  A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0.
\end{cases}
\]

The Ackermann function can be calculated by a simple function based directly on the definition:
This definition of $\alpha(m,n)$ is not particularly enlightening.
Why does this $\alpha(m,n)$ appear in the analysis of path compression?
Union Find with Path Compressions
Union Find with Path Compressions

Maintain partition of \( S = \{1, 2, \ldots, n\} \)

under operations
Union Find with Path Compressions

Maintain partition of \( S = \{1, 2, \ldots, n\} \)
under operations

\[
\text{Union}(2, 4)
\]
Union Find with Path Compressions

Maintain partition of $S = \{1, 2, \cdots, n\}$ under operations

Union$(2, 4)$

Find$(3) = 6$ (representative element)
Implementation

* forest $\mathcal{F}$ of rooted trees with node set $S$
* one tree for each group in current partition
* root of tree is representative of the group
Implementation

* forest $\mathcal{F}$ of rooted trees with node set $S$
* one tree for each group in current partition
* root of tree is representative of the group

```
1  8  3  
7

2  4  6  5  9
```

Union($2, 4$)

```
2  8  3
1  7
```

"Linking"
Implementation

* forest $\mathcal{F}$ of rooted trees with node set $S$
* one tree for each group in current partition
* root of tree is representative of the group

\[
\begin{align*}
2 & \quad 4 & \quad 6 & \quad 5 & \quad 9 \\
1 & \quad 8 & \quad 3 & \quad 7 \\
\end{align*}
\]

Union(2, 4)

\[
\begin{align*}
 & \quad 4 & \quad 6 & \quad 5 & \quad 9 \\
2 & \quad 8 & \quad 3 & \quad 7 \\
1 & \quad 7 \\
\end{align*}
\]

"Linking"

Find(x) follow path from x to root

"path following"
Heuristic 1: “linking by rank”

- each node \( x \) carries integer \( \text{rk}(x) \)
- initially \( \text{rk}(x) = 0 \)
- as soon as \( x \) is NOT a root, \( \text{rk}(x) \) stays unchanged
- for \( \text{Union}(x, y) \) make node with smaller rank child of the other in case of tie, increment one of the ranks
Heuristic 2: **Path compression**

when performing a $\text{Find}(x)$ operation make all nodes in the “findpath” children of the root
sequence of **Union** and **Find** operation

Explicit cost model:

\[
\text{cost}(\ text{op}) = \# \text{ times some node gets a new parent}
\]

Time for **Union**(x, y) = \( O(1) = O(\text{cost( Union(x,y) )}) \)

Time for **Find**(x) = \( O( \# \text{ of nodes on findpath} ) \)

\[= O( 2 + \text{cost( Find(x) )} ) \]
For analysis assume all **Unions** are performed first, but **Find**-paths are only followed (and compressed) to correct node.
For analysis assume all Unions are performed first, but Find-paths are only followed (and compressed) to correct node.
General path compression in forest $\mathcal{F}$

compress($x, y$)
General path compression in forest $\mathcal{F}$
General path compression in forest $\mathcal{F}$

$\text{cost}(\text{compress}(x,y)) = \# \text{ of nodes that get a new parent}$
Problem formulation

$F$ forest on node set $X$

$C$ sequence of compress operations on $F$

$|C| = \# \text{ of true compress operations in } C$

$\text{cost}(C) = \sum(\text{cost of individual operations})$

How large can $\text{cost}(C)$ be at most, in terms of $|X|$ and $|C|$?
Idea:

For the analysis try "divide and conquer."

Idea:

For the analysis try "divide and conquer."

Question:

How do you "divide"?
**Dissection** of a forest $\mathcal{F}$ with node set $X$:

- partition of $X$ into “top part” $X_t$ and “bottom part” $X_b$
- so that top part $X_t$ is “upwards closed”,
  
  i.e. $x \in X_t \Rightarrow$ every ancestor of $x$ is in $X_t$ also
Dissection of a forest $\mathcal{F}$ with node set $X$:

partition of $X$ into “top part” $X_t$ and “bottom part” $X_b$ so that top part $X_t$ is “upwards closed”, i.e. $x \in X_t \Rightarrow$ every ancestor of $x$ is in $X_t$ also
Dissection of a forest $\mathcal{F}$ with node set $X$:

partition of $X$ into “top part” $X_t$
and “bottom part” $X_b$
so that top part $X_t$ is “upwards closed”,

i.e. $x \in X_t \Rightarrow$ every ancestor of $x$ is in $X_t$ also

Note: $X_t, X_b$ dissection for $\mathcal{F}$
$\mathcal{F}'$ obtained from $\mathcal{F}$ by
sequence of path compressions

$\Rightarrow$ $X_t, X_b$ is dissection for $\mathcal{F}'$
Main Lemma:

\( C \) ... sequence of operations on \( \mathcal{F} \) with node set \( X \), \( X_t, X_b \) dissection for \( \mathcal{F} \) inducing subforests \( \mathcal{F}_t, \mathcal{F}_b \)
Main Lemma:

$C$ ... sequence of operations on $\mathcal{F}$ with node set $X$

$X_t, X_b$ dissection for $\mathcal{F}$ inducing subforests $\mathcal{F}_t, \mathcal{F}_b$

$\Rightarrow \exists$ compression sequences $C_b$ for $\mathcal{F}_b$ and $C_t$ for $\mathcal{F}_t$

with

$$|C_b| + |C_t| \leq |C|$$

and

$$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$
Proof: 1) How to get $C_b$ and $C_t$ from $C$: 
Proof: 1) How to get \( C_b \) and \( C_t \) from \( C \):

compression paths from \( C \)

\[
\text{case 1:} \quad \begin{array}{c}
\downarrow \\
\times \\
\uparrow \\
\times
\end{array} \quad \begin{array}{c}
\downarrow \\
\times
\end{array} \quad \text{into } C_t
\]
Proof: 1) How to get $C_b$ and $C_t$ from $C$:

compression paths from $C$

- case 1: $
  \begin{array}{c}
  \text{Y} \\
  \text{X}
  \end{array}$ into $C_t$

- case 2: $
  \begin{array}{c}
  \text{Y} \\
  \text{X}
  \end{array}$ into $C_b$
Proof: 1) How to get $C_b$ and $C_t$ from $C$:

compression paths from $C$

- **case 1:**
  - $Y$
  - $X$
  - $Y$
  - $X$
  - into $C_t$

- **case 2:**
  - $Y$
  - $X$
  - $Y$
  - $X$
  - into $C_b$

- **case 3:**
  - $Y$
  - $X'$
  - $Y$
  - $X'$
  - $\infty$
  - $Y$
  - $X$
  - into $C_b$
"rootpath compress"
"rootpath compress"

\[
\text{compress}(x, \infty) = 0
\]

\[
\text{cost( compress}(x, \infty)`) = \# \text{ of nodes that get a new parent}
\]

\[
= 0
\]
Proof:

\[ |C_b| + |C_t| \leq |C| \]

compression paths from \( C \)

- **case 1:**
  - \( y \)
  - \( x \)

- **case 2:**
  - \( y \)
  - \( x \)

- **case 3:**
  - \( y' \)
  - \( x' \)
  - \( \infty \)
  - \( x \)
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]

green node gets new green parent: accounted by \( \text{cost}(C_t) \)
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]

- Green node gets new green parent: accounted by \( \text{cost}(C_t) \)
- Brown node gets new brown parent: accounted by \( \text{cost}(C_b) \)
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]

- **Green node gets new green parent:** accounted by \( \text{cost}(C_t) \)
- **Brown node gets new brown parent:** accounted by \( \text{cost}(C_b) \)
- **Brown node gets new green parent: for the first time** accounted by \(|X_b|\)
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]

- **green node gets new green parent:** accounted by \( \text{cost}(C_t) \)
- **brown node gets new brown parent:** accounted by \( \text{cost}(C_b) \)
- **brown node gets new green parent:** accounted by \(|X_b| - \#\text{roots}(F_b)| \)

© Raimund Seidel
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| - \#\text{roots}(F_b) + |C_t| \]
\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| - \#\text{roots}(F_b) + |C_t| \]

- **green node gets new green parent:** accounted by \( \text{cost}(C_t) \)
- **brown node gets new brown parent:** accounted by \( \text{cost}(C_b) \)
- **brown node gets new green parent:** accounted by \( |X_b| - \#\text{roots}(F_b) \)
- **brown node gets new green parent:** accounted by \( |C_t| \)

© Raimund Seidel
Main Lemma':

C ... sequence of operations on $\mathcal{F}$ with node set $X$
$X_t, X_b$ dissection for $\mathcal{F}$ inducing subforests $\mathcal{F}_t, \mathcal{F}_b$

$\Rightarrow \exists$ compression sequences $C_b$ for $\mathcal{F}_b$ and $C_t$ for $\mathcal{F}_t$
with

$|C_b| + |C_t| \leq |C|$

and

$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| - \#\text{roots}(\mathcal{F}_b) + |C_t|$
$f(m,n)$ ... maximum cost of any compression sequence $C$ with $|C|=m$ in an arbitrary forest with $n$ nodes.

Claim: $f(m,n) \leq (m+n) \cdot \log_2 n$
Claim: \[ f(m,n) \leq (m+n) \cdot \log_2 n \]
Claim: \( f(m,n) \leq (m+n) \cdot \log_2 n \)

Proof:

forest \( \mathcal{F} \)

\[ |X| = n \]

\( C \) compression sequence

\[ |C| = m \]
Claim: \( f(m,n) \leq (m+n) \cdot \log_2 n \)

Proof:

forest \( \mathcal{F} \)

\( |X| = n \)

\( \mathcal{F}_b \)

\( |X_b| = n/2 \)

\( \mathcal{F}_t \)

\( |X_t| = |X_b| = n/2 \)

\( C \) compression sequence

\( |C| = m \)
Claim: \( f(m,n) \leq (m+n) \cdot \log_2 n \)

Proof:

forest \( \mathcal{F} \)

\[ |X| = n \]

\( |X_t| = |X_b| = n/2 \)

\( |C| = m \)

Main Lemma \( \Rightarrow \exists C_t, C_b \) \( |C_b| + |C_t| \leq |C| \)

\( m_b + m_t \leq m \)

\( \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \)
Claim: \( f(m,n) \leq (m+n) \cdot \log_2 n \)

Proof:

- forest \( F \)
  - \( |X| = n \)
  - \( |X_t| = |X_b| = n/2 \)

- compression sequence \( C \)
  - \( |C| = m \)

Main Lemma \( \Rightarrow \exists C_t, C_b \) \( |C_b| + |C_t| \leq |C| \)
  - \( m_b + m_t \leq m \)

\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]

Induction: \( \leq (m_b + n/2) \log n/2 + (m_t + n/2) \log n/2 + n/2 + m_t \)
Claim: \( f(m,n) \leq (m+n) \cdot \log_2 n \)

Proof:

- Forest \( \mathcal{F} \)
  - \( |X| = n \)
  - \( |X_t| = |X_b| = n/2 \)

- Compression sequence \( \mathcal{C} \)
  - \( |\mathcal{C}| = m \)

Main Lemma \( \Rightarrow \exists \mathcal{C}_t, \mathcal{C}_b \) \( |\mathcal{C}_b| + |\mathcal{C}_t| \leq |\mathcal{C}| \)

- \( m_b + m_t \leq m \)

\[
\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{C}_b) + \text{cost}(\mathcal{C}_t) + |X_b| + |\mathcal{C}_t|
\]

Induction:

\[
\leq (m_b + n/2) \log n/2 + (m_t + n/2) \log n/2 + n/2 + m_t
\]

\[
\leq (m_b + m_t + n/2 + n/2) \log n/2 + n + m
\]
Claim:   \( f(m,n) \leq (m+n) \cdot \log_2 n \)

Proof:

forest  \( \mathcal{F} \)

\[ |X| = n \]

\( \mathcal{F}_b \)

\[ |X_b| = |X_t| = n/2 \]

\( \mathcal{F}_t \)

\( C \)  compression sequence  \( |C| = m \)

Main Lemma  \( \Rightarrow \exists C_t, C_b \)  \( |C_b| + |C_t| \leq |C| \)

\[ m_b + m_t \leq m \]

\[ \text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t| \]

Induction:

\[ \leq (m_b + n/2) \log n/2 + (m_t + n/2) \log n/2 + n/2 + m_t \]

\[ \leq (m_b + m_t + n/2 + n/2) \log n/2 + n + m \]

\[ \leq (m+n) \cdot \log_2 n/2 + (m+n) = (m+n) \cdot \log_2 n \]
Corollary:

Any sequence of \( m \) Union, Find operations in a universe of \( n \) elements that uses arbitrary linking and path compression takes time at most

\[ O( (m+n) \cdot \log n) \]
Corollary:

Any sequence of $m$ Union, Find operations in a universe of $n$ elements that uses arbitrary linking and path compression takes time at most

$$O((m+n) \cdot \log n)$$

By choosing a dissection that is “unbalanced” in relation to $m/n$ one can prove a better bound of

$$O((m+n) \cdot \log \left\lceil \frac{m}{n} \right\rceil + 1 \cdot n)$$
**Corollary:**

Any sequence of $m$ Union, Find operations in a universe of $n$ elements that uses arbitrary linking and path compression takes time at most

$$O((m+n) \cdot \log n)$$

By choosing a dissection that is “unbalanced” in relation to $m/n$ one can prove a better bound of

$$O((m+n) \cdot \log_{\lceil m/n \rceil + 1} n)$$

**Proof:** exercise
Path compression and union by rank
Brief digression
\[ f : \mathbb{N} \rightarrow \mathbb{R} \]

\[
f^*(n) = \begin{cases} 
0 & \text{if } n \leq 1 \\
1 + f^*(f(n)) & \text{if } n > 1 
\end{cases}
\]

Brief digression
\( f : \mathbb{N} \to \mathbb{R} \)

\[
f^*(n) = \begin{cases} 
0 & \text{if } n \leq 1 \\
1 + f^*(f(n)) & \text{if } n > 1 
\end{cases}
\]

\[
f^*(n) = \min \{ k \mid f(f(\ldots f(n)\ldots)) \leq 1 \}
\]

Brief digression
\( f : \mathbb{N} \rightarrow \mathbb{R} \)

\[
f^*(n) = \begin{cases} 
0 & \text{if } n \leq 1 \\
1 + f^*(f(n)) & \text{if } n > 1 
\end{cases}
\]

\[
f^*(n) = \min \left\{ k \mid f(f(\ldots f(n)\ldots)) \leq 1 \right\}
\]

**Properties:**
- \( f \) a “nice” compaction, i.e. \( f(n) < n - 1 \)
- \( f^* \) a “nice” compaction and
- \( f^* \) “much smaller” than \( f \)
Examples for $f^*$:

<table>
<thead>
<tr>
<th>$f(n)$</th>
<th>$f^*(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n-1$</td>
<td>$n-1$</td>
</tr>
<tr>
<td>$n-2$</td>
<td>$n/2$</td>
</tr>
<tr>
<td>$n-c$</td>
<td>$n/c$</td>
</tr>
<tr>
<td>$n/2$</td>
<td>$\log_2 n$</td>
</tr>
<tr>
<td>$\sqrt{n}$</td>
<td>$\log_c n$</td>
</tr>
<tr>
<td>$\log n$</td>
<td>$\log \log n$</td>
</tr>
<tr>
<td></td>
<td>$\log^* n$</td>
</tr>
</tbody>
</table>
**Def:** \( \mathcal{F} \) forest, \( x \) node in \( \mathcal{F} \)

\[ r(x) = \text{height of subtree rooted at } x \]

\[ ( \quad r(\text{leaf}) = 0 \quad ) \]

\( \mathcal{F} \) is a rank forest, if

for every node \( x \)

for every \( i \) with \( 0 \leq i < r(x) \),

there is a child \( y_i \) of \( x \) with \( r(y_i) = i \).
Path compression and union by rank

**Def:** \( \mathcal{F} \) forest, \( x \) node in \( \mathcal{F} \)

\[ r(x) = \text{height of subtree rooted at } x \]

(\( r(\text{leaf}) = 0 \))

\( \mathcal{F} \) is a rank forest, if

for every node \( x \)

for every \( i \) with \( 0 \leq i < r(x) \),

there is a child \( y_i \) of \( x \) with \( r(y_i) = i \).

**Note:** Union by rank produces rank forests!
Path compression and union by rank

Def: \( \mathcal{F} \) forest, \( x \) node in \( \mathcal{F} \)

\[ r(x) = \text{height of subtree rooted at } x \]

\[ (r(\text{leaf}) = 0) \]

\( \mathcal{F} \) is a \textit{rank forest}, if

for every node \( x \)

for every \( i \) with \( 0 \leq i < r(x) \),

there is a child \( y_i \) of \( x \) with \( r(y_i) = i \).

Note: Union by rank produces rank forests!

Lemma: \( r(x) = r \Rightarrow x \) has at least \( r \) children.
Path compression and union by rank

**Def:** \( \mathcal{F} \) forest, \( x \) node in \( \mathcal{F} \)

\[
\begin{align*}
r(x) &= \text{height of subtree rooted at } x \\
&= \begin{cases} 
0 & \text{if } x \text{ is a leaf} \\
\text{height of largest subtree} & \text{if } x \text{ is not a leaf}
\end{cases}
\end{align*}
\]

\( \mathcal{F} \) is a rank forest, if

for every node \( x \)

for every \( i \) with \( 0 \leq i < r(x) \),

there is a child \( y_i \) of \( x \) with \( r(y_i) = i \).

**Note:** Union by rank produces rank forests!

**Lemma:** \( r(x) = r \Rightarrow x \) has at least \( r \) children

and at least \( 2^r \) descendants.
Inheritance Lemma:

\[ \mathcal{F} \text{ rank forest with maximum rank } r \text{ and node set } X \]

\[ s \in \mathbb{N} : \quad X_{>s} = \{ x \in X \mid r(x) > s \} \quad \mathcal{F}_{>s} \quad \text{induced forests} \]

\[ X_{\leq s} = \{ x \in X \mid r(x) \leq s \} \quad \mathcal{F}_{\leq s} \]
Inheritance Lemma:

$\mathcal{F}$ rank forest with maximum rank $r$ and node set $X$

$s \in \mathbb{N}$:

$X_{>s} = \{ x \in X \mid r(x) > s \}$ \hspace{1cm} $\mathcal{F}_{>s}$ induced forests

$X_{\leq s} = \{ x \in X \mid r(x) \leq s \}$ \hspace{1cm} $\mathcal{F}_{\leq s}$

i) $X_{\leq s}, X_{>s}$ is a dissection for $\mathcal{F}$

ii) $\mathcal{F}_{\leq s}$ is a rank forest with maximum rank $\leq s$

iii) $\mathcal{F}_{>s}$ is a rank forest with maximum rank $\leq r-s-1$
Inheritance Lemma:

\[ \mathcal{F} \text{ rank forest with maximum rank } r \text{ and node set } X \]

\[ s \in \mathbb{N}: \quad X_{>s} = \{ x \in X \mid r(x) > s \} \quad \mathcal{F}_{>s} \quad \text{induced forests} \]

\[ X_{\leq s} = \{ x \in X \mid r(x) \leq s \} \quad \mathcal{F}_{\leq s} \]

i) \( X_{\leq s}, X_{>s} \) is a dissection for \( \mathcal{F} \)

ii) \( \mathcal{F}_{\leq s} \) is a rank forest with maximum rank \( \leq s \)

iii) \( \mathcal{F}_{>s} \) is a rank forest with maximum rank \( \leq r-s-1 \)
Inheritance Lemma:

\( F \) rank forest with maximum rank \( r \) and node set \( X \)

\[ s \in \mathbb{N}: \quad X_{>s} = \{ x \in X \mid r(x) > s \} \quad \quad F_{>s} \quad \text{induced forests} \]

\[ X_{\leq s} = \{ x \in X \mid r(x) \leq s \} \quad \quad F_{\leq s} \]

i) \( X_{\leq s}, X_{>s} \) is a dissection for \( F \)

ii) \( F_{\leq s} \) is a rank forest with maximum rank \( \leq s \)

iii) \( F_{>s} \) is a rank forest with maximum rank \( \leq r-s-1 \)

Proofs: exercise
\[ f(m,n,r) = \text{maximum cost of any compression sequence } C, \text{ with } |C|=m, \text{ in rank forest } \mathcal{F} \text{ with } n \text{ nodes and maximum rank } r. \]
\[ f(m,n,r) = \text{maximum cost of any compression sequence } C, \text{ with } |C|=m, \text{ in rank forest } F \text{ with } n \text{ nodes and maximum rank } r. \]

**Trivial bounds:**

\[ f(m,n,r) \leq (r-1) \cdot n \]

\[ f(m,n,r) \leq (r-1) \cdot m \]
\[ f(m,n,r) = \text{maximum cost of any compression sequence } C, \text{ with } |C|=m, \text{ in rank forest } \mathcal{F} \text{ with } n \text{ nodes and maximum rank } r. \]

**Trivial bounds:**

\[ f(m,n,r) \leq (r-1) \cdot n \]

\[ f(m,n,r) \leq (r-1) \cdot m \]

\[ f(m,n,r) \leq m + (r-2) \cdot n \]
\[
\text{cost}(C) \leq \text{cost}(C_t) + \text{cost}(C_b) + |X_b| - \#\text{rts}(F_b) + |C_t|
\]
\[
\begin{align*}
\text{cost}(C) & \leq \text{cost}(C_t) + \text{cost}(C_b) + |X_b| - \#rts(F_b) + |C_t| \\
& \leq f(m_t,n_t,r-s-1) +
\end{align*}
\]
\[
\text{cost}(C) \leq \text{cost}(C_t) + \text{cost}(C_b) + |X_b| - \#rts(F_b) + |C_t| \\
\leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + 
\]
\[
\begin{align*}
\text{cost}( C ) & \leq \text{cost}( C_t ) + \text{cost}( C_b ) + |X_b| - \#\text{rts}( \mathcal{F}_b ) + |C_t| \\
& \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n-n_t -
\end{align*}
\]
\[ \text{cost}(C) \leq \text{cost}(C_t) + \text{cost}(C_b) + |X_b| - \#\text{rts}(\mathcal{F}_b) + |C_t| \]
\[ \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n-n_t - (s+1)n_t + \]
Each node in $\mathcal{F}_t$ has at least $s+1$ children in $\mathcal{F}_b$, and they must all be different roots of $\mathcal{F}_b$. 

\[
\text{cost}(C) \leq \text{cost}(C_t) + \text{cost}(C_b) + |X_b| - \#rts(\mathcal{F}_b) + |C_t| \\
\leq f(m_t, n_t, r-s-1) + f(m_b, n_b, s) + n - n_t - (s+1) \cdot n_t +
\]
Each node in $\mathcal{F}_t$ has at least $s+1$ children in $\mathcal{F}_b$, and they must all be different roots of $\mathcal{F}_b$. 

$$\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{C}_t) + \text{cost}(\mathcal{C}_b) + |X_b| - \#\text{rts}(\mathcal{F}_b) + |\mathcal{C}_t|$$

$$\leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n-n_t - (s+1)n_t + m_t$$
Each node in $\mathcal{F}_t$ has at least $s+1$ children in $\mathcal{F}_b$, and they must all be different roots of $\mathcal{F}_b$.

$$f(m,n,r) \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)n_t + m_t$$
f(m, n, r) \leq f(m_t, n_t, r-s-1) + f(m_b, n_b, s) + n - (s+2) \cdot n_t + m_t

n_t + n_b = n
m_t + m_b \leq m
0 \leq s < r
\[ f(m, n, r) \leq f(m_t, n_t, r-s-1) + f(m_b, n_b, s) + n - (s+2) \cdot n_t + m_t \]

\[
n_t + n_b = n \\
m_t + m_b \leq m
\]

Assume: \[ f(M, N, R) \leq k \cdot M + N \cdot g(R) \]
\[ f(m,n,r) \leq f(m_+,n_+,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_+ + m_+ \]

\[
\begin{align*}
  n_+ + n_b &= n \\
  m_+ + m_b &\leq m \\
  0 \leq s < r
\end{align*}
\]

Assume: \( f(M,N,R) \leq k\cdot M + N\cdot g(R) \)

\[
\begin{align*}
  f(m,n,r) &\leq k\cdot m_+ + n_+\cdot g(r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_+ + m_+ \\
  &\leq k\cdot m_+ + n_+\cdot g(r) + f(m_b,n_b,s) + n - s\cdot n_+ + m_+
\end{align*}
\]
\[ f(m,n,r) \leq f(m_+,n_+,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_+ + m_+ \]

\[ n_+ + n_b = n \quad 0 \leq s < r \]

\[ m_+ + m_b \leq m \]

Assume: \[ f(M,N,R) \leq k\cdot M + N\cdot g(R) \]

\[ f(m,n,r) \leq k\cdot m_+ + n_+\cdot g(r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_+ + m_+ \]

\[ \leq k\cdot m_+ + n_+\cdot g(r) + f(m_b,n_b,s) + n - s\cdot n_+ + m_+ \]

choose \[ s = g(r) \]
\[ f(m, n, r) \leq f(m_t, n_t, r-s-1) + f(m_b, n_b, s) + n - (s+2)n_t + m_t \]

\[ n_t + n_b = n \quad 0 \leq s < r \]

\[ m_t + m_b \leq m \]

Assume: \[ f(M, N, R) \leq k \cdot M + N \cdot g(R) \]

\[ f(m, n, r) \leq k \cdot m_t + n_t \cdot g(r-s-1) + f(m_b, n_b, s) + n - (s+2)n_t + m_t \]

\[ \leq k \cdot m_t + n_t \cdot g(r) + f(m_b, n_b, s) + n - s \cdot n_t + m_t \]

Choose: \[ s = g(r) \]

\[ f(m, n, r) \leq (k+1) \cdot m_t + f(m_b, n_b, s) + n \]

\[ \leq (k+1) \cdot m_t + f(m_b, n, s) + n \]
\[ s = g(r) \]

\[ f(m, n, r) \leq (k+1) \cdot m + f(m_b, n, s) + n \]
\[ s = g(r) \]

\[ f(m,n,r) \leq (k+1) \cdot m_t + f(m_b,n,s) + n \]

\[ -(k+1) \cdot (m_b+m_t) \]
\[ s = g(r) \]

\[ f(m,n,r) \leq (k+1) \cdot m_t + f(m_b,n,s) + n \]

\[ -(k+1) \cdot (m_b + m_t) \]
\[ \begin{align*}
\text{s} &= g(r) \\
\text{f}(m, n, r) &\leq (k+1) \cdot m_t + \text{f}(m_b, n, s) + n - (k+1) \cdot (m_b + m_t) \\
\text{f}(m, n, r) - (k+1) \cdot m &\leq \text{f}(m_b, n, s) - (k+1) \cdot m_b + n
\end{align*} \]
\[ s = g(r) \]

\[ f(m,n,r) \leq (k+1) \cdot m_t + f(m_b,n,s) + n \]

\[ -(k+1) \cdot (m_b + m_t) \]

\[ f(m,n,r) - (k+1) \cdot m \leq f(m_b,n,s) - (k+1) \cdot m_b + n \]

\[ \phi(m,n,r) \leq \phi(m_b,n,g(r)) + n \]
\[ s = g(r) \]

\[ f(m, n, r) \leq (k+1) \cdot m + f(m_b, n, s) + n \]

\[ f(m, n, r) -(k+1) \cdot m \leq f(m_b, n, s) - (k+1) \cdot m_b + n \]

\[ \phi(m, n, r) \leq \phi(m_b, n, g(r)) + n \]

\[ \leq (\phi(m_{bb}, n, g(g(r))) + n) + n \]
\[ s = g(r) \]

\[ f(m,n,r) \leq (k+1) \cdot m + f(m_b,n,s) + n - (k+1) \cdot (m_b + m) \]

\[ f(m,n,r) - (k+1) \cdot m \leq f(m_b,n,s) - (k+1) \cdot m_b + n \]

\[ \phi(m,n,r) \leq \phi(m_b,n,g(r)) + n \]

\[ \leq (\phi(m_{bb},n,g(g(r))) + n) + n \]

\[ \leq ((\phi(m_{bbb},n,g(g(g(r)))) + n) + n) + n \]
\[ s = g(r) \]

\[
f(m,n,r) \leq (k+1) \cdot m_t + f(m_b,n,s) + n \quad \text{subject to} \quad -(k+1) \cdot (m_b + m_t) \]

\[
f(m,n,r) - (k+1) \cdot m \leq f(m_b,n,s) - (k+1) \cdot m_b + n
\]

\[
\phi(m,n,r) \leq \phi(m_b,n,g(r)) + n
\]

\[
\leq (\phi(m_{bb},n,g(g(r))) + n) + n
\]

\[
\leq ((\phi(m_{bbb},n,g(g(g(r)))) + n) + n) + n
\]

\[
\phi(m,n,r) \leq n \cdot g^*(r)
\]
\[ s = g(r) \]

\[ f(m,n,r) \leq (k+1) \cdot m_{+} + f(m_{b},n,s) + n \]

\[ (k+1) \cdot (m_{b} + m_{+}) \]

\[ f(m,n,r) - (k+1) \cdot m \leq f(m_{b},n,s) - (k+1) \cdot m_{b} + n \]

\[ \phi(m,n,r) \leq \phi(m_{b},n,g(r)) + n \]

\[ \leq (\phi(m_{bb},n,g(g(r))) + n) + n \]

\[ \leq ((\phi(m_{bbb},n,g(g(g(r)))) + n) + n) + n \]

\[ \phi(m,n,r) \leq n \cdot g^{*}(r) \]

\[ f(m,n,r) \leq (k+1) \cdot m + n \cdot g^{*}(r) \]
Shifting Lemma:

If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)
then also \( f(m,n,r) \leq (k+1) \cdot m + n \cdot g^*(r) \)
Shifting Lemma:

If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+1) \cdot m + n \cdot g^*(r) \)

Shifting Corollary:

If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{** \ldots *}(r) \)

for any \( i \geq 0 \)
If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{** \cdots *}(r)$

for any $i \geq 0$
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)
then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{** \ldots *}(r) \)
for any \( i \geq 0 \)

Trivial bound: \( f(m,n,r) \leq n \cdot (r-1) \)
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{**\cdots *}(r) \)

for any \( i \geq 0 \)

**Trivial bound:** \( f(m,n,r) \leq n \cdot (r-1) \)

\[ = 0 \cdot m + n \cdot (r-1) \]
If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$, then also
$f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{{* \ldots *}}(r)$
for any $i \geq 0$

Trivial bound:
$f(m,n,r) \leq n \cdot (r-1)$

\[= 0 \cdot m + n \cdot (r-1)\]

\[g(r) = r-1\]
\[g^*(r) = r-1\]
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)
then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{** \ldots *}(r) \)
for any \( i \geq 0 \)

**Trivial bound:** \( f(m,n,r) \leq m + n \cdot (r-2) \)
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{** \ldots \ast}(r) \)

for any \( i \geq 0 \)

Trivial bound: \( f(m,n,r) \leq m + n \cdot (r-2) \)

\[
= 1 \cdot m + n \cdot (r-2)
\]
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{**\ldots\ast}(r) \)

for any \( i \geq 0 \)

Trivial bound: \( f(m,n,r) \leq m + n \cdot (r-2) \)

\[ = 1 \cdot m + n \cdot (r-2) \]

\( g(r) = r-2 \)
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{**\ldots\ast}(r) \)

for any \( i \geq 0 \)

Trivial bound: \( f(m,n,r) \leq m + n \cdot (r-2) \)

\[ g(r) = r-2 \]
\[ g^*(r) = r/2 \]
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \), then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{* \cdot \star \cdot \star \cdot \star}(r) \) for any \( i \geq 0 \).

Trivial bound: \( f(m,n,r) \leq m + n \cdot (r-2) \)

\[ = 1 \cdot m + n \cdot (r-2) \]

\( g(r) = r-2 \)

\( g^{\star}(r) = r/2 \)

\( f(m,n,r) \leq 2 \cdot m + n \cdot (r/2) \)
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{** \ldots *}(r) \)

for any \( i \geq 0 \)

Trivial bound: \( f(m,n,r) \leq m + n \cdot (r-2) \)

\[ = 1 \cdot m + n \cdot (r-2) \]

\( g(r) = r-2 \)

\( g^*(r) = r/2 \)

\( g^{**}(r) = \log r \)
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)
then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{**...**}(r) \)
for any \( i \geq 0 \)

Trivial bound: \( f(m,n,r) \leq m + n \cdot (r-2) \)
\[
= 1 \cdot m + n \cdot (r-2)
\]
\( g(r) = r-2 \)
\( g^{*}(r) = r/2 \)
\( g^{**}(r) = \log r \)
If \( f(m, n, r) \leq k \cdot m + n \cdot g(r) \), then also \( f(m, n, r) \leq (k+i) \cdot m + n \cdot g^{**...**}(r) \) for any \( i \geq 0 \).
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{** \ldots *}(r) \)

for any \( i \geq 0 \)

We know bound: \( f(m,n,r) \leq 3 \cdot m + n \cdot \log r \)
If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)
then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{**...*}(r) \)
for any \( i \geq 0 \)

We know bound: \( f(m,n,r) \leq 3 \cdot m + n \cdot \log r \)

Therefore for any \( i \geq 0 \):
\[
f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**...*}(r)
\]
For any $i \geq 0$:  $f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{** \ldots *}(r)$
For any \( i \geq 0 \) : \( f(m, n, r) \leq (3+i) \cdot m + n \cdot \log^{** \ldots \ast}(r) \)

Choice of \( i \) :
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots\ast}(r)$$

Choice of $i$:

Define $\alpha(r) = \min\{ i \mid \log^{**\ldots\ast}(r) \leq i \}$
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{\ldots \ldots}(r)$$

Choice of $i$:

Define $\alpha(r) = \min\{ i \mid \log^{\ldots \ldots}(r) \leq i \}$

Here is your definition of the Inverse Ackermann Function!!
For any \( i \geq 0 \):

\[
f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots*}(r)
\]

**Choice of \( i \):**

Define \( \alpha(r) = \min\{ i \mid \log^{**\ldots*}(r) \leq i \} \)

\[
f(m,n,r) \leq (m+n)(3+\alpha(r))
\]
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots\ast}(r)$$

Choice of $i$:

Define $\alpha(r) = \min\{ i \mid \log^{**\ldots\ast}(r) \leq i \}$

$$f(m,n,r) \leq (m+n)(3+\alpha(r))$$

$$\leq (m+n)(3+\alpha(\log n))$$
For any $i \geq 0$: $f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots\cdot}(r)$

Choice of $i$:
For any $i \geq 0$:
$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots\ast}(r)$$

**Choice of $i$:**

For $t \geq 1$ define $\alpha_t(r) = \min\{ i \mid \log^{**\ldots\ast}(r) \leq t \}$
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{** \cdots} (r)$$

Choice of $i$:

For $t \geq 1$ define:

$$\alpha_t(r) = \min \{ i \mid \log^{** \cdots} (r) \leq t \}$$

Here is a parametrized definition of the Inverse Ackermann Function!!
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i)m + n \cdot \log^{**\ldots*}(r)$$

Choice of $i$:

For $t \geq 1$ define $\alpha_t(r) = \min\{ i \mid \log^{**\ldots*}(r) \leq t \}$

$$f(m,n,r) \leq (3+\alpha_t(r))m + n \cdot t$$
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots\ast}(r)$$

Choice of $i$:

For $t \geq 1$ define $\alpha_t(r) = \min\{i \mid \log^{**\ldots\ast}(r) \leq t\}$

$$f(m,n,r) \leq (3+\alpha_t(r)) \cdot m + n \cdot t$$

choose $t = 1 + m/n$
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots*}(r)$$

**Choice of $i$:**

For $t \geq 1$ define $\alpha_t(r) = \min\{ i \mid \log^{**\ldots*}(r) \leq t \}$

$$f(m,n,r) \leq (3+\alpha_t(r)) \cdot m + n \cdot t$$

Choose $t = 1 + \frac{m}{n}$

$$f(m,n,r) \leq (4+\alpha_{1+m/n}(r)) \cdot m + n$$
For any $i \geq 0$:

$$f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{** \cdots *}(r)$$

Choice of $i$:

For $t \geq 1$ define

$$\alpha_t(r) = \min\{ i \mid \log^{** \cdots *}(r) \leq t \}$$

Then

$$f(m,n,r) \leq (3+\alpha_t(r)) \cdot m + n \cdot t$$

choose $t = 1+m/n$

$$f(m,n,r) \leq (4+\alpha_{1+m/n}(r)) \cdot m + n$$

$$\leq (4+\alpha_{1+m/n}(\log n)) \cdot m + n$$
Theorem:

Any sequence of \( m \) Union, Find operations in a universe of \( n \) elements that uses linking by rank and path compression takes time at most

\[
O( m \cdot \alpha(m,n) + n )
\]
Bob Tarjan 1975

Theorem:

Any sequence of $m$ Union, Find operations in a universe of $n$ elements that uses linking by rank and path compression takes time at most

$$O(m \cdot \alpha(m,n) + n)$$

$$f(m,n,r) \leq (4 + \alpha^{1+m/n}(\log n)) \cdot m + n$$
**Theorem:**

Any sequence of $m$ Union, Find operations in a universe of $n$ elements that uses linking by rank and path compression takes time at most

$$O(m \cdot \alpha(m,n) + n)$$

where

$$f(m,n,r) \leq (4 + \alpha_{1+m/n}(\log n)) \cdot m + n$$

$$\alpha(m,n) = \alpha_{1+m/n}(\log n)$$
**Shifting Lemma:**

If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+1) \cdot m + n \cdot g^*(r) \)

**Shifting Corollary:**

If \( f(m,n,r) \leq k \cdot m + n \cdot g(r) \)

then also \( f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{**...**(r)} \)

for any \( i \geq 0 \)

**Definition of \( \alpha \):**

\[
\alpha(r) = \min\{ i \mid \log^{**...**}(r) \leq i \}
\]
Odds and Ends
Odds and Ends

We used $f(m,n,r) \leq 1 \cdot m + n \cdot (r-2)$
We used \( f(m,n,r) \leq 1 \cdot m + n \cdot (r-2) \) to get

\[
\text{for any } i \geq 0 : \quad f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{**\ldots\ast}(r)
\]
Odds and Ends

We used \( f(m,n,r) \leq 1 \cdot m + n \cdot (r-2) \) to get

\[
\text{for any } i \geq 0 : \quad f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{\ldots}(r)
\]

Actually \( f(m,n,r) \leq 1 \cdot m + n \cdot \log r \)
Odds and Ends

We used $f(m,n,r) \leq 1 \cdot m + n \cdot (r-2)$ to get

$$\begin{aligned} &\text{for any } i \geq 0: \quad f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^{*** \cdots \ast}(r) \\
\end{aligned}$$

Actually $f(m,n,r) \leq 1 \cdot m + n \cdot \log r$ (Exercise)
We used \( f(m,n,r) \leq 1 \cdot m + n \cdot (r-2) \) to get

\[
\text{for any } i \geq 0 : \quad f(m,n,r) \leq (3+i) \cdot m + n \cdot \log^\ast \ast \ast \ast \ast (r)
\]

Actually \( f(m,n,r) \leq 1 \cdot m + n \cdot \log r \) (Exercise)

and therefore

\[
\text{for any } i \geq 0 : \quad f(m,n,r) \leq (1+i) \cdot m + n \cdot \log^\ast \ast \ast \ast (r)
\]
Odds and Ends

Actually \( f(m,n,r) \leq 1 \cdot m + n \cdot \log^* r \) (difficult Exercise)

and therefore

\[
\text{For any } i \geq 0 : \quad f(m,n,r) \leq i \cdot m + n \cdot \log^{** \cdots ^*}(r)
\]
Odds and Ends

$f(m, n, r)$ for small values of $r$
f(m,n,r) for small values of r

f(m,n,0) = 0  f(m,n,1) = 0  f(m,n,2) ≤ m
Odds and Ends

\[ f(m, n, r) \] for small values of \( r \)

\[ f(m, n, 0) = 0 \quad f(m, n, 1) = 0 \quad f(m, n, 2) \leq m \]

\[ f(m, n, r) \leq m + n \quad \text{for } r \leq 8, \text{ i.e. for } n < 512 \]
Odds and Ends

\[ f(m,n,r) \] for small values of \( r \)

\[ f(m,n,0) = 0 \quad f(m,n,1) = 0 \quad f(m,n,2) \leq m \]

\[ f(m,n,r) \leq m + n \quad \text{for} \ r \leq 8, \text{i.e. for} \ n < 512 \]

\[ f(m,n,r) \leq m + 2n \quad \text{for} \ r \leq 202, \text{i.e. for} \ n < 2^{203} \]
f(m,n,r) for small values of r

f(m,n,0) = 0   f(m,n,1) = 0   f(m,n,2) ≤ m

f(m,n,r) ≤ m + n   for r≤ 8, i.e. for n<512

f(m,n,r) ≤ m + 2n   for r≤ 202, i.e. for n<2^{203}

(difficult exercises)
Odds and Ends

Similar proof for $O( m \cdot \alpha(m,n) + n )$ bound also works for

* linking by weight and path compression

* linking by rank and *generalized path compaction*
Odds and Ends

Similar proof for $O(m \cdot \alpha(m,n) + n)$ bound also works for

* linking by weight and path compression
* linking by rank and generalized path compaction

Open problem:

simple top-down approach for proving lower bounds