7.5 Reductions

- designing algorithms
- establishing lower bounds
- intractability

Algorithms in Java, 4th Edition · Robert Sedgewick and Kevin Wayne · Copyright © 2009 · January 26, 2010 8:57:56 AM

Bird's-eye view

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, …
linearithmic	N log N	sorting, convex hull, closest pair, farthest pair, …
quadratic	N^2	???
	…	
exponential	c^N	???

Frustrating news. Huge number of problems have defied classification.

Bird's-eye view

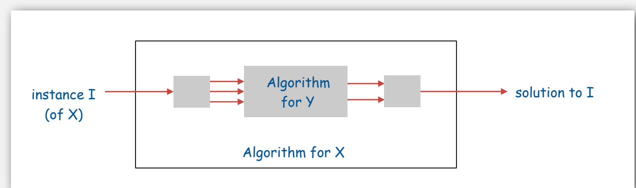
Desiderata. Classify problems according to computational requirements.

Desiderata'.
Suppose we could (couldn't) solve problem X efficiently.
What else could (couldn't) we solve efficiently?

" Give me a lever long enough and a fulcrum on which to place it, and I shall move the world. " — Archimedes

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



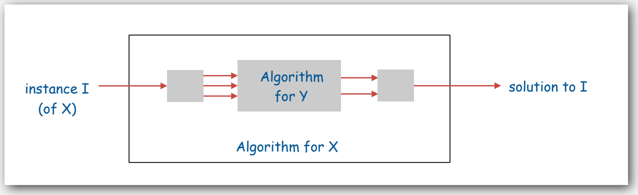
Cost of solving X = total cost of solving Y + cost of reduction.

perhaps many calls to Y on problems of different sizes

preprocessing and postprocessing

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 1. [element distinctness reduces to sorting]
To solve element distinctness on N integers:

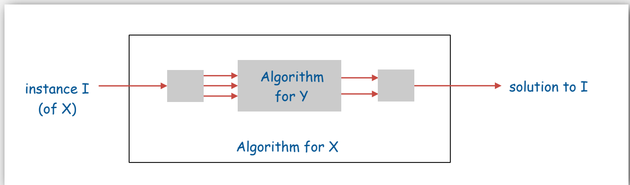
- Sort N integers.
- Check adjacent pairs for equality.

cost of sorting
cost of reduction

Cost of solving element distinctness. N log N + N

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 2. [3-collinear reduces to sorting]
To solve 3-collinear instance on N points in the plane:

- For each point, sort other points by polar angle.
  - check adjacent triples for collinearity

cost of sorting
cost of reduction

Cost of solving 3-collinear. N^2 log N + N^2.

- designing algorithms
- establishing lower bounds
- intractability

Reduction: design algorithms

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.

Design algorithm. Given algorithm for Y, can also solve X.

Ex.

- Element distinctness reduces to sorting.
- 3-collinear reduces to sorting.
- PERT reduces to topological sort. [see digraph lecture]
- h-v line intersection reduces to 1D range searching. [see geometry lecture]
- Burrows-Wheeler transform reduces to suffix sort. [see assignment 8]

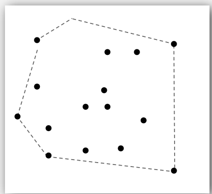
Mentality. Since I know how to solve Y, can I use that algorithm to solve X?

programmer's version: I have code for Y. Can I use it for X?

Convex hull reduces to sorting

Sorting. Given N distinct integers, rearrange them in ascending order.

Convex hull. Given N points in the plane, identify the extreme points of the convex hull (in counter-clockwise order).



convex hull

1251432
2861534
3988818
4190745
13546464
89885444
43434213
34435312

sorting

Proposition. Convex hull reduces to sorting.

Pf. Graham scan algorithm.

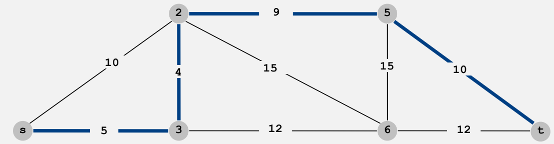
cost of sorting

cost of reduction

Cost of convex hull. N log N + N.

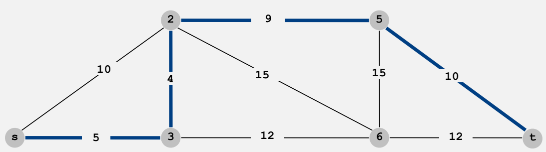
Shortest path on graphs and digraphs

Proposition. Undirected shortest path (with nonnegative weights) reduces to directed shortest path.

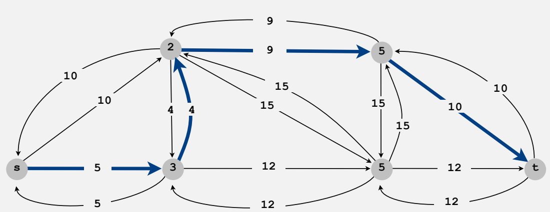


Shortest path on graphs and digraphs

Proposition. Undirected shortest path (with nonnegative weights) reduces to directed shortest path.

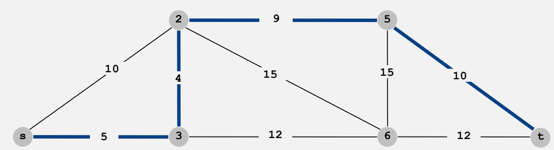


Pf. Replace each undirected edge by two directed edges.



Shortest path on graphs and digraphs

Proposition. Undirected shortest path (with nonnegative weights) reduces to directed shortest path.



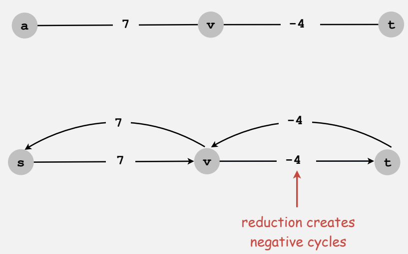
Cost of undirected shortest path. E log E + E.

cost of shortest path in digraph

cost of reduction

Shortest path with negative weights

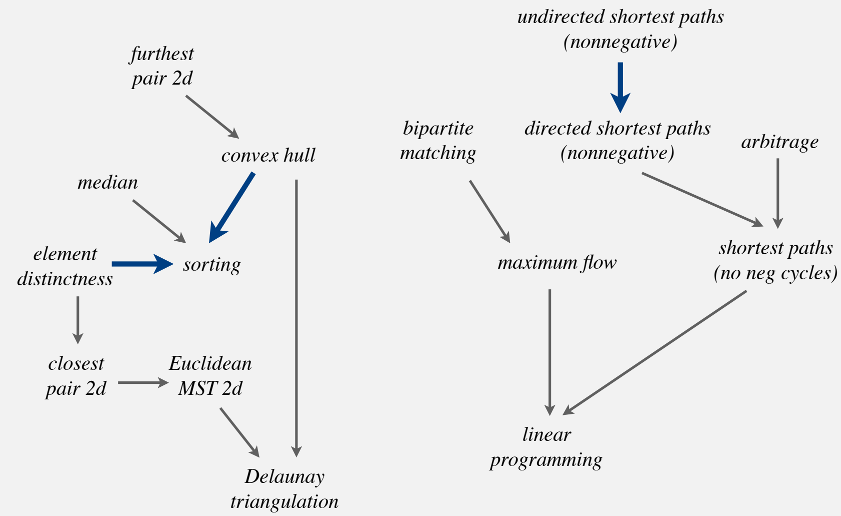
Caveat. Reduction is invalid in networks with negative weights (even if no negative cycles).



Remark. Can still solve shortest path problem in undirected graphs (if no negative cycles), but need more sophisticated techniques.

reduces to weighted non-bipartite matching (!)

Some reductions involving familiar problems



- designing algorithms
- **linear programming**
- establishing lower bounds
- establishing intractability
- classifying problems

Linear Programming

What is it? [see ORF 307]

- Quintessential tool for optimal allocation of scarce resources
- Powerful and general problem-solving method

Why is it significant?

- Widely applicable.
- Dominates world of industry.
- Fast commercial solvers available: CPLEX, OSL.
- Powerful modeling languages available: AMPL, GAMS.
- Ranked among most important scientific advances of 20th century.

Ex: Delta claims that LP saves $100 million per year.

Present context. Many important problems reduce to LP.

Applications

Agriculture. Diet problem.
Computer science. Compiler register allocation, data mining.
Electrical engineering. VLSI design, optimal clocking.
Energy. Blending petroleum products.
Economics. Equilibrium theory, two-person zero-sum games.
Environment. Water quality management.
Finance. Portfolio optimization.
Logistics. Supply-chain management.
Management. Hotel yield management.
Marketing. Direct mail advertising.
Manufacturing. Production line balancing, cutting stock.
Medicine. Radioactive seed placement in cancer treatment.
Operations research. Airline crew assignment, vehicle routing.
Physics. Ground states of 3-D Ising spin glasses.
Plasma physics. Optimal stellarator design.
Telecommunication. Network design, Internet routing.
Sports. Scheduling ACC basketball, handicapping horse races.

Linear programming

Model problem as maximizing an objective function subject to constraints.

Input: real numbers a_{ij}, c_j, and b_i.

Output: real numbers x_j.

n variables

$$\begin{array}{ll} \text{maximize} & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\ \text{subject to the constraints} & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \le b_1 \\ & a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \le b_2 \\ & \ldots \\ & a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \le b_m \\ & x_1, x_2, \ldots, x_n \ge 0 \end{array}$$

m equations

matrix version

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to the constraints} & A\, x \le b \\ & x \ge 0 \end{array}$$

Solutions. [see ORF 307]

- Simplex algorithm has been used for decades to solve practical LP instances.
- Newer algorithms guarantee fast solution.

Linear programming

"Linear programming"

- Process of formulating an LP model for a problem.
- Solution to LP for a specific problem gives solution to the problem.
- Equivalent to "reducing the problem to LP."

1. Identify variables.
2. Define constraints (inequalities and equations).
3. Define objective function.

Examples:

- Shortest paths
- Maximum flow.
- Bipartite matching.

  . . .
- [a very long list]

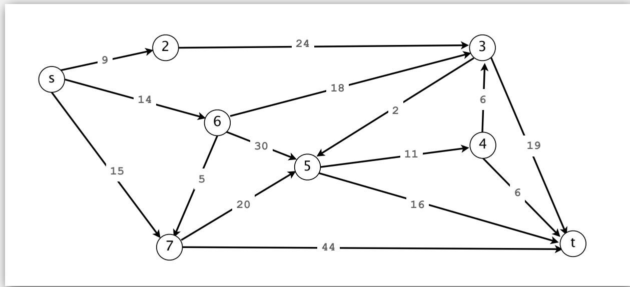
stay tuned (next)

Single-source shortest-paths problem (revisited)

Given. Weighted digraph, single source s.

Distance from s to v. Length of the shortest path from s to v .

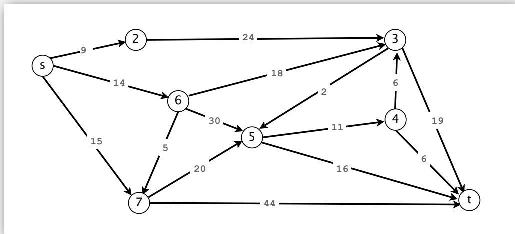
Goal. Find distance (and shortest path) from s to every other vertex.



Single-source shortest-paths problem reduces to LP

LP formulation.

- One variable per vertex, one inequality per edge.
- Interpretation: x_i = length of shortest path from s to i.

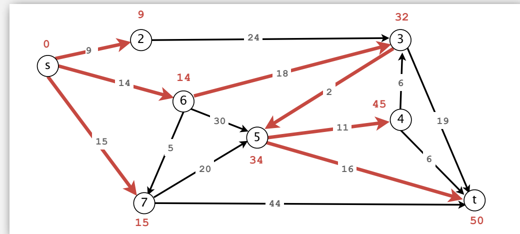


maximize	x_t
subject to the constraints	$x_s + 9 \ge x_2$
	$x_s + 14 \ge x_6$
	$x_s + 15 \ge x_7$
	$x_2 + 24 \ge x_3$
	$x_3 + 2 \ge x_5$
	$x_3 + 19 \ge x_t$
	$x_4 + 6 \ge x_3$
	$x_4 + 6 \ge x_t$
	$x_5 + 11 \ge x_4$
	$x_5 + 16 \ge x_t$
	$x_6 + 18 \ge x_3$
	$x_6 + 30 \ge x_5$
	$x_6 + 5 \ge x_7$
	$x_7 + 20 \ge x_5$
	$x_7 + 44 \ge x_t$
	$x_s = 0$

Single-source shortest-paths problem reduces to LP

LP formulation.

- One variable per vertex, one inequality per edge.
- Interpretation: x_i = length of shortest path from s to i.



maximize	x_t
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	$x_s + 14 \ge x_6$
	$x_s + 15 \ge x_7$
	$x_2 + 24 \ge x_3$
	$x_3 + 2 \ge x_5$
	$x_3 + 19 \ge x_t$
	$x_4 + 6 \ge x_3$
	$x_4 + 6 \ge x_t$
	$x_5 + 11 \ge x_4$
	$x_5 + 16 \ge x_t$
	$x_6 + 18 \ge x_3$
	$x_6 + 30 \ge x_5$
	$x_6 + 5 \ge x_7$
	$x_7 + 20 \ge x_5$
	$x_7 + 44 \ge x_t$
	$x_s = 0$

$x_s = 0$	$x_5 = 34$
$x_2 = 9$	$x_6 = 14$
$x_3 = 32$	$x_7 = 15$
$x_4 = 45$	$x_t = 50$

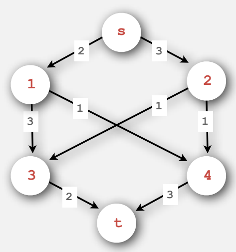
solution

Maxflow problem

Given: Weighted digraph, source s, destination t.

Interpret edge weights as capacities

- Models material flowing through network
- Ex: oil flowing through pipes
- Ex: goods in trucks on roads
- [many other examples]



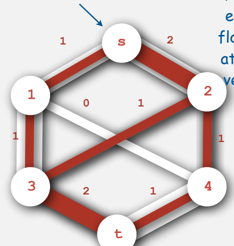
Flow: A different set of edge weights

- flow does not exceed capacity in any edge
- flow at every vertex satisfies equilibrium [flow in equals flow out]

flow out of s is 3

flow in equals flow out at each vertex

flow ≤ capacity in every edge



flow in to t is 3

Goal: Find maximum flow from s to t.

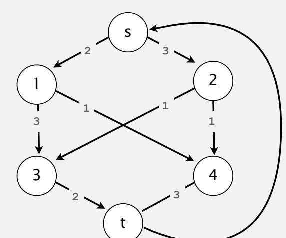
Maximum flow reduces to LP

One variable per edge.
One inequality per edge, one equality per vertex.

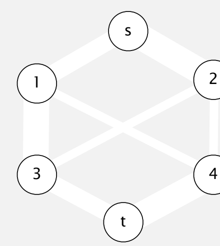
maximize	$x_{3t} + x_{4t}$
subject to the constraints	$x_{s1} \le 2$
	$x_{s2} \le 3$
	$x_{13} \le 3$
	$x_{14} \le 1$
	$x_{23} \le 1$
	$x_{24} \le 1$
	$x_{3t} \le 2$
	$x_{4t} \le 3$
equilibrium constraints	$x_{s1} = x_{13} + x_{14}$
	$x_{s2} = x_{23} + x_{24}$
	$x_{13} + x_{23} = x_{3t}$
	$x_{14} + x_{24} = x_{4t}$
	all $x_{ij} \ge 0$

capacity constraints

interpretation: x_{ij} = flow in edge i-j



add dummy edge from t to s

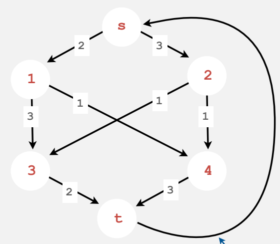


Maxflow problem reduces to LP

One variable per edge.
One inequality per edge, one equality per vertex.

maximize	$x_{3t} + x_{4t}$	
subject to the constraints	$x_{s1} \le 2$	capacity constraints
	$x_{s2} \le 3$	
	$x_{13} \le 3$	
	$x_{14} \le 1$	
	$x_{23} \le 1$	
	$x_{24} \le 1$	
	$x_{3t} \le 2$	
	$x_{4t} \le 3$	
equilibrium constraints	$x_{s1} = x_{13} + x_{14}$	
	$x_{s2} = x_{23} + x_{24}$	
	$x_{13} + x_{23} = x_{3t}$	
	$x_{14} + x_{24} = x_{4t}$	
	all $x_{ij} \ge 0$	

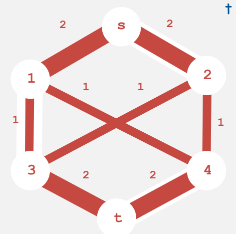
interpretation:
x_{ij} = flow in edge i-j



add dummy edge from t to s

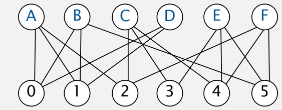
solution

$x_{s1} = 2$
$x_{s2} = 2$
$x_{13} = 1$
$x_{14} = 1$
$x_{23} = 1$
$x_{24} = 1$
$x_{3t} = 2$
$x_{4t} = 2$



Maximum cardinality bipartite matching problem

Bipartite graph. Two sets of vertices; edges connect vertices in one set to the other.



Matching. Set of edges with no vertex appearing twice.

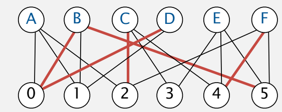
Goal. Find a maximum cardinality matching.

Interpretation. Mutual preference constraints.

- Ex: people to jobs.
- Ex: Medical students to residence positions.
- Ex: students to writing seminars.
- [many other examples]

Alice	Adobe
Adobe, Apple, Google	Alice, Bob, Dave
Bob	Apple
Adobe, Apple, Yahoo	Alice, Bob, Dave
Carol	Google
Google, IBM, Sun	Alice, Carol, Frank
Dave	IBM
Adobe, Apple	Carol, Eliza
Eliza	Sun
IBM, Sun, Yahoo	Carol, Eliza, Frank
Frank	Yahoo
Google, Sun, Yahoo	Bob, Eliza, Frank

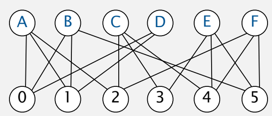
job offers



Maximum cardinality bipartite matching reduces to LP

LP formulation.

- One variable per edge, one equality per vertex.
- Interpretation: an edge is in matching iff $x_i = 1$.



maximize	$x_{A0} + x_{A1} + x_{A2} + x_{B0} + x_{B1} + x_{B5} + x_{C2} + x_{C3} + x_{C4} + x_{D0} + x_{D1} + x_{E3} + x_{E4} + x_{E5} + x_{F2} + x_{F4} + x_{F5}$	
subject to the constraints	$x_{A0} + x_{A1} + x_{A2} = 1$	$x_{A0} + x_{B0} + x_{D0} = 1$
	$x_{B0} + x_{B1} + x_{B5} = 1$	$x_{A1} + x_{B1} + x_{D1} = 1$
	$x_{C2} + x_{C3} + x_{C4} = 1$	$x_{A2} + x_{C2} + x_{F2} = 1$
	$x_{D0} + x_{D1} = 1$	$x_{C3} + x_{E3} = 1$
	$x_{E3} + x_{E4} + x_{E5} = 1$	$x_{C4} + x_{E4} + x_{F4} = 1$
	$x_{F2} + x_{F4} + x_{F5} = 1$	$x_{B5} + x_{E5} + x_{F5} = 1$
	all $x_{ij} \ge 0$	

constraints on top vertices (left) and bottom vertices (right)

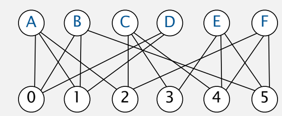
crucial point: not always so lucky!

Theorem. [Birkhoff 1946, von Neumann 1953]
All extreme points of the above polyhedron have integer (0 or 1) coordinates.
Corollary. Can solve bipartite matching problem by solving LP.

Maximum cardinality bipartite matching reduces to LP

LP formulation.

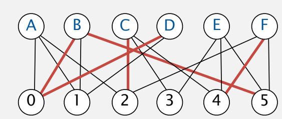
- One variable per edge, one equality per vertex.
- Interpretation: an edge is in matching iff $x_i = 1$.



maximize	$x_{A0} + x_{A1} + x_{A2} + x_{B0} + x_{B1} + x_{B5} + x_{C2} + x_{C3} + x_{C4} + x_{D0} + x_{D1} + x_{E3} + x_{E4} + x_{E5} + x_{F2} + x_{F4} + x_{F5}$	
subject to the constraints	$x_{A0} + x_{A1} + x_{A2} = 1$	$x_{A0} + x_{B0} + x_{D0} = 1$
	$x_{B0} + x_{B1} + x_{B5} = 1$	$x_{A1} + x_{B1} + x_{D1} = 1$
	$x_{C2} + x_{C3} + x_{C4} = 1$	$x_{A2} + x_{C2} + x_{F2} = 1$
	$x_{D0} + x_{D1} = 1$	$x_{C3} + x_{E3} = 1$
	$x_{E3} + x_{E4} + x_{E5} = 1$	$x_{C4} + x_{E4} + x_{F4} = 1$
	$x_{F2} + x_{F4} + x_{F5} = 1$	$x_{B5} + x_{E5} + x_{F5} = 1$
	all $x_{ij} \ge 0$	

solution

$x_{A1} = 1$
$x_{B5} = 1$
$x_{C2} = 1$
$x_{D0} = 1$
$x_{E3} = 1$
$x_{F4} = 1$
all other $x_{ij} = 0$



Linear programming perspective

Got an optimization problem?

Ex. Shortest paths, maximum flow, matching, ….

Approach 1. Use a specialized algorithm to solve it.

- Algorithms in Java.
- Vast literature on complexity.
- Performance on real problems not always well-understood.

Approach 2. Reduce to a LP model; use a commercial solver.

- A direct mathematical representation of the problem often works.
- Immediate solution to the problem at hand is often available.
- Might miss faster specialized solution, but might not care.

Got an LP solver? Learn to use it!

```
% ampl
AMPL Version 20010215 (SunOS 5.7)
ampl: model maxflow.mod;
ampl: data maxflow.dat;
ampl: solve;
CPLEX 7.1.0: optimal solution;
objective 4;
```


- designing algorithms
- **establishing lower bounds**
- intractability

Bird's-eye view

Goal. Prove that a problem requires a certain number of steps.

Ex. $\Omega(N \log N)$ lower bound for sorting.

```
1251432
2861534
3988818
4190745
13546464
89885444
43434213
```

argument must apply to all conceivable algorithms

Bad news. Very difficult to establish lower bounds from scratch.

Good news. Can spread $\Omega(N \log N)$ lower bound to Y by reducing sorting to Y.

assuming cost of reduction is not too high

Linear-time reductions

Def. Problem X linear-time reduces to problem Y if X can be solved with:

- Linear number of standard computational steps.
- Constant number of calls to Y.

Ex. Almost all of the reductions we've seen so far. [Which one wasn't?]

Establish lower bound:

- If X takes $\Omega(N \log N)$ steps, then so does Y.
- If X takes $\Omega(N^2)$ steps, then so does Y.

Mentality.

- If I could easily solve Y, then I could easily solve X.
- I can't easily solve X.
- Therefore, I can't easily solve Y.

Lower bound for convex hull

Proposition. In quadratic decision tree model, any algorithm for sorting N integers requires $\Omega(N \log N)$ steps.

allows quadratic tests of the form:
$x_i < x_j$ or $(x_j - x_i)(x_k - x_i) - (x_j)(x_j - x_i) < 0$

Proposition. Sorting linear-time reduces to convex hull.
Pf. [see next slide]

```
1251432
2861534
3988818
4190745
13546464
89885444
43434213
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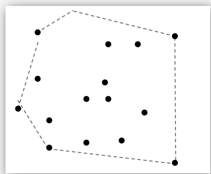
sorting



convex hull

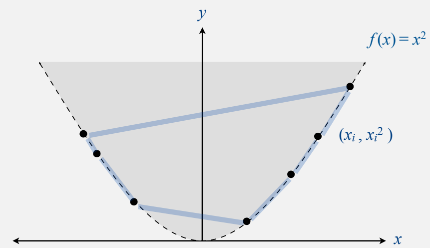
a quadratic test

Implication. Any ccw-based convex hull algorithm requires $\Omega(N \log N)$ ccw's.

Sorting linear-time reduces to convex hull

Proposition. Sorting linear-time reduces to convex hull.

- Sorting instance: $x_1, x_2, \ldots, x_N$.
- Convex hull instance: $(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2)$.



Pf.

- Region $\{x : x^2 \geq x\}$ is convex $\Rightarrow$ all points are on hull.
- Starting at point with most negative x, counter-clockwise order of hull points yields integers in ascending order.

Lower bound for 3-COLLINEAR

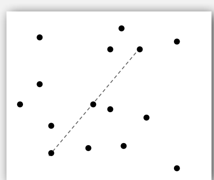
3-SUM. Given N distinct integers, are there three that sum to 0?

3-COLLINEAR. Given N distinct points in the plane, are there 3 that all lie on the same line?

recall Assignment 3

```
1251432
-2861534
3988818
-4190745
13546464
89885444
-43434213
```

3-sum



3-collinear

Lower bound for 3-COLLINEAR

3-SUM. Given N distinct integers, are there three that sum to 0?

3-COLLINEAR. Given N distinct points in the plane, are there 3 that all lie on the same line?

Proposition. 3-SUM linear-time reduces to 3-COLLINEAR.
Pf. [see next 2 slide]

Conjecture. Any algorithm for 3-SUM requires $\Omega(N^2)$ steps.
Implication. No sub-quadratic algorithm for 3-COLLINEAR likely.

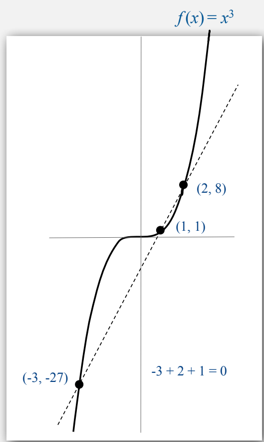
your $N^2 \log N$ algorithm was pretty good

3-SUM linear-time reduces to 3-COLLINEAR

Proposition. 3-SUM linear-time reduces to 3-COLLINEAR.

- 3-SUM instance: $x_1, x_2, \ldots, x_N$.
- 3-COLLINEAR instance: $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

Lemma. If a, b, and c are distinct, then $a + b + c = 0$ if and only if (a, a^3), (b, b^3), and (c, c^3) are collinear.



3-SUM linear-time reduces to 3-COLLINEAR

Proposition. 3-SUM linear-time reduces to 3-COLLINEAR.

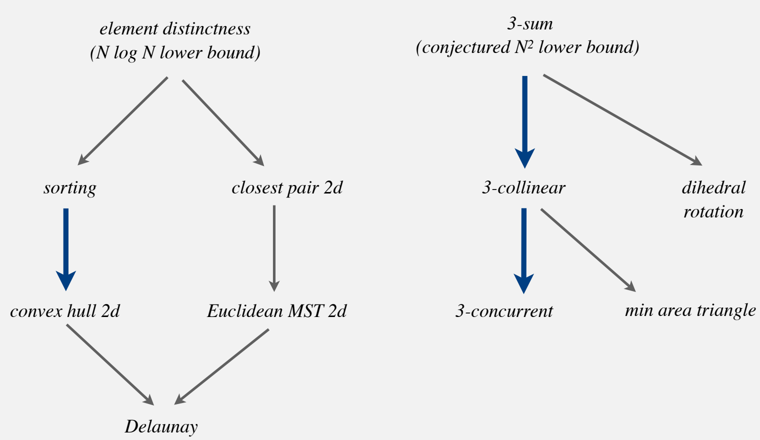
- 3-SUM instance: $x_1, x_2, \ldots, x_N$.
- 3-COLLINEAR instance: $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

Lemma. If a, b, and c are distinct, then $a + b + c = 0$ if and only if (a, a^3), (b, b^3), and (c, c^3) are collinear.

Pf. Three distinct points (a, a^3), (b, b^3), and (c, c^3) are collinear iff:

$$\begin{aligned} 0 &= \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} \\ &= a(b^3 - c^3) - b(a^3 - c^3) + c(a^3 - b^3) \\ &= (a - b)(b - c)(c - a)(a + b + c) \end{aligned}$$

More linear-time reductions and lower bounds



Establishing lower bounds: summary

Establishing lower bounds through reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself no linear-time convex hull algorithm exists?
A1. [hard way] Long futile search for a linear-time algorithm.
A2. [easy way] Linear-time reduction from sorting.

Q. How to convince yourself no sub-quadratic 3-COLLINEAR algorithm exists.
A1. [hard way] Long futile search for a sub-quadratic algorithm.
A2. [easy way] Linear-time reduction from 3-SUM.

- designing algorithms
- establishing lower bounds
- **intractability**

Bird's-eye view

Def. A problem is intractable if it can't be solved in polynomial time.

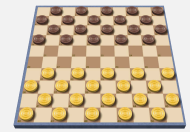
Desiderata. Prove that a problem is intractable.

Two problems that require exponential time.

input size = c + lg K

- Given a constant-size program, does it halt in at most K steps?
- Given N-by-N checkers board position, can the first player force a win?

using forced capture rule



Frustrating news. Few successes.

3-satisfiability

Literal. A boolean variable or its negation. x_i or $\neg x_i$

Clause. An *or* of 3 distinct literals. $C_1 = (\neg x_1 \vee x_2 \vee x_3)$

Conjunctive normal form. An *and* of clauses. $\Phi = (C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5)$

3-SAT. Given a CNF formula Φ consisting of k clauses over n literals, does it have a satisfying truth assignment?

$$\Phi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_2 \vee x_3 \vee x_4)$$

yes instance

x_1	x_2	x_3	x_4
T	T	F	T

$$(\neg T \vee T \vee F) \wedge (T \vee \neg T \vee F) \wedge (\neg T \vee \neg T \vee \neg F) \wedge (\neg T \vee \neg T \vee T) \wedge (\neg T \vee F \vee T)$$

Applications. Circuit design, program correctness, ...

3-satisfiability is believed intractable

Q. How to solve an instance of 3-SAT with n variables?

A. Exhaustive search: try all 2^n truth assignments.

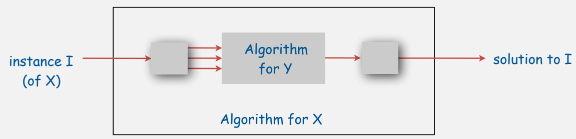
Q. Can we do anything substantially more clever?

Conjecture (P ≠ NP). 3-SAT is intractable (no poly-time algorithm).

Polynomial-time reductions

Def. Problem X poly-time (Cook) reduces to problem Y if X can be solved with:

- Polynomial number of standard computational steps.
- Polynomial number of calls to Y.



Establish intractability. If 3-SAT poly-time reduces to Y, then Y is intractable. (assuming 3-SAT is intractable)

Mentality.

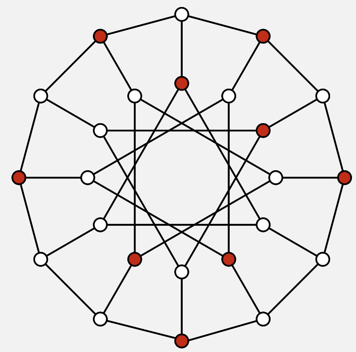
- If I could solve Y in poly-time, then I could also solve 3-SAT in poly-time.
- 3-SAT is believed to be intractable.
- Therefore, so is Y.

Independent set

Def. An independent set is a set of vertices, no two of which are adjacent.

IND-SET. Given a graph G and an integer k, find an independent set of size k.

k = 9



Applications. Scheduling, computer vision, clustering, ...

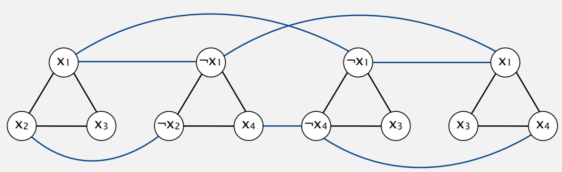
3-satisfiability reduces to independent set

Proposition. 3-SAT poly-time reduces to IND-SET.

Pf. Given an instance Φ of 3-SAT, create an instance G of IND-SET:

- For each clause in Φ, create 3 vertices in a triangle.
- Add an edge between each literal and its negation.

k = 4



$$\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_3 \lor x_4)$$

3-satisfiability reduces to independent set

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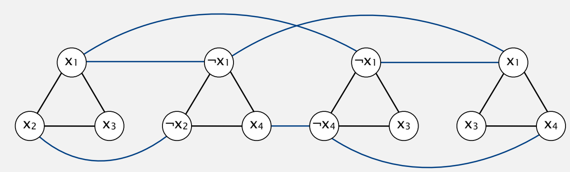


$$\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_3 \lor x_4)$$

- G has independent set of size k $\Rightarrow$ Φ satisfiable.

set literals corresponding to vertices in independent to true;
set remaining literals in consistent manner

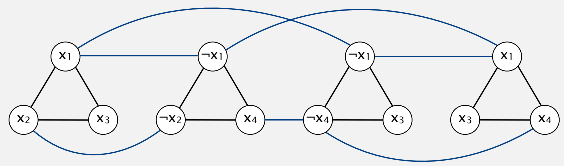
3-satisfiability reduces to independent set

Proposition. 3-SAT poly-time reduces to IND-SET.

Pf. Given an instance Φ of 3-SAT, create an instance G of IND-SET:

- For each clause in Φ, create 3 vertices in a triangle.
- Add an edge between each literal and its negation.

k = 4



$$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_3 \vee x_4)$$

- G has independent set of size $k \Rightarrow \Phi$ satisfiable.
- Φ satisfiable $\Rightarrow G$ has independent set of size k.

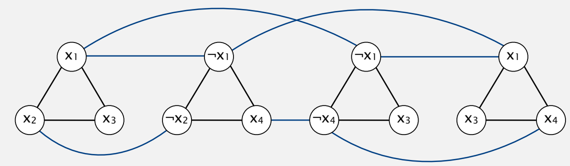
for each clause, take vertex corresponding to one true literal

3-satisfiability reduces to independent set

Proposition. 3-SAT poly-time reduces to IND-SET.

Implication. Assuming 3-SAT is intractable, so is IND-SET.

k = 4



$$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_3 \vee x_4)$$

Integer linear programming

ILP. Given a system of linear inequalities, find an integral solution.

$$3x_1 + 5x_2 + 2x_3 + x_4 + 4x_5 \geq 10$$
$$5x_1 + 2x_2 + 4x_4 + 1x_5 \leq 7$$
$$x_1 + x_3 + 2x_4 \leq 2$$
$$3x_1 + 4x_3 + 7x_4 \leq 7$$
$$x_1 + x_4 \leq 1$$
$$x_1 + x_3 + x_5 \leq 1$$
$$\text{all } x_i = \{ 0, 1 \}$$

linear inequalities

integer variables

Context. Cornerstone problem in operations research.

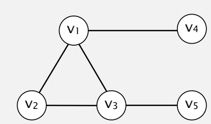
Remark. Finding a real-valued solution is tractable (linear programming).

Independent set reduces to integer linear programming

Proposition. IND-SET poly-time reduces to ILP.

Pf. Given an instance G, k of IND-SET, create an instance of ILP as follows:

Intuition. $x_i = 1$ if and only if vertex v_i is in independent set.



is there an independent set of size 3 ?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3$$
$$x_1 + x_2 \leq 1$$
$$x_2 + x_3 \leq 1$$
$$x_1 + x_3 \leq 1$$
$$x_1 + x_4 \leq 1$$
$$x_3 + x_5 \leq 1$$
$$\text{all } x_i = \{ 0, 1 \}$$

number of vertices selected

at most one vertex selected from each edge

binary variables

is there a feasible solution?

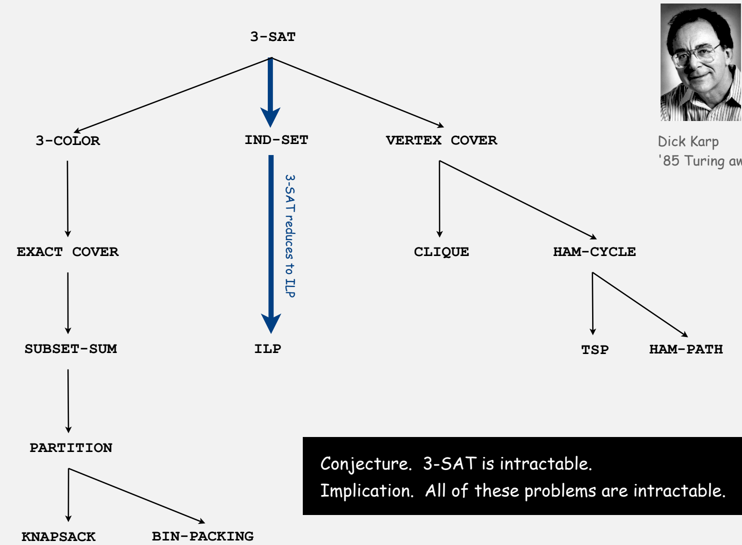
3-satisfiability reduces to integer linear programming

Proposition. 3-SAT poly-time reduces to IND-SET.
Proposition. IND-SET poly-time reduces to ILP.

Transitivity. If X poly-time reduces to Y and Y poly-time reduces to Z, then X-poly-time reduces to Z.

Implication. Assuming 3-SAT is intractable, so is ILP.

More poly-time reductions from 3-satisfiability



Dick Karp
'85 Turing award

Conjecture. 3-SAT is intractable.
Implication. All of these problems are intractable.

Implications of poly-time reductions from 3-satisfiability

Establishing intractability through poly-time reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself that a new problem is (probably) intractable?
A1. [hard way] Long futile search for an efficient algorithm (as for 3-SAT).
A2. [easy way] Reduction from 3-SAT.

Caveat. Intricate reductions are common.

Search problems

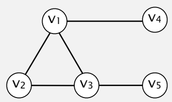
Search problem. Problem where you can check a solution in poly-time.

Ex 1. 3-SAT.

$$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_3 \vee x_4)$$

$x_1 = true,\ x_2 = true,\ x_3 = true,\ x_4 = true$

Ex 2. IND-SET.



$k = 3$

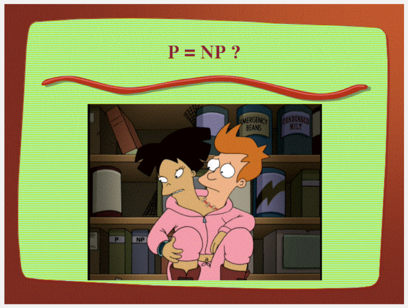
$\{ v_2 ,\ x_4,\ v_5 \}$

P vs. NP

P. Set of search problems solvable in poly-time.
Importance. What scientists and engineers can compute feasibly.

NP. Set of search problems.
Importance. What scientists and engineers aspire to compute feasibly.

Fundamental question.



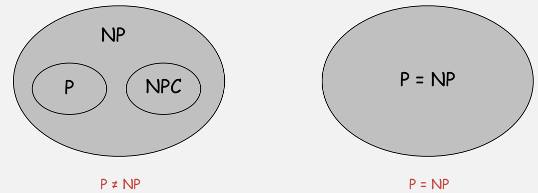
Consensus opinion. No.

Cook's theorem

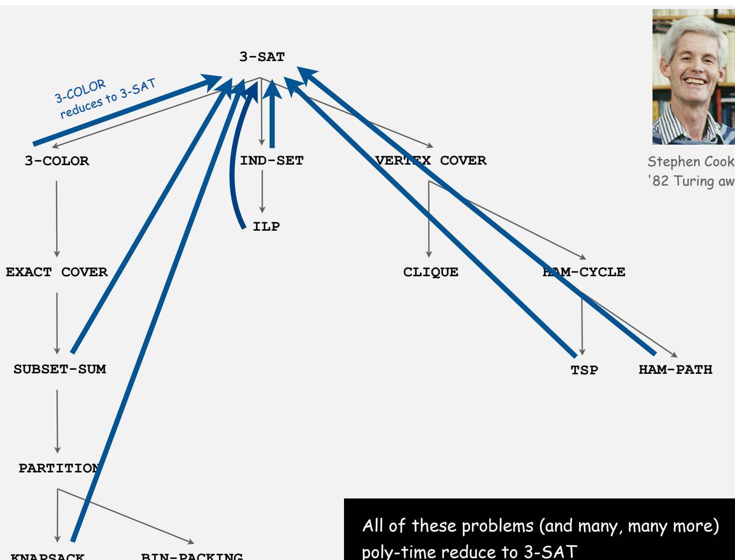
Def. An NP is NP-complete if all problems in NP poly-time to reduce to it.

Cook's theorem. 3-SAT is NP-complete.
Corollary. 3-SAT is tractable if and only if P = NP.

Two worlds.



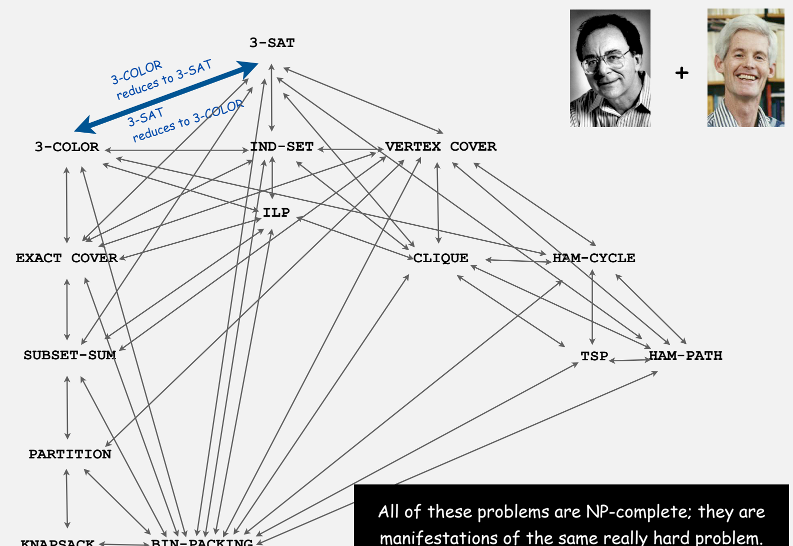
Implications of Cook's theorem



Stephen Cook
'82 Turing award

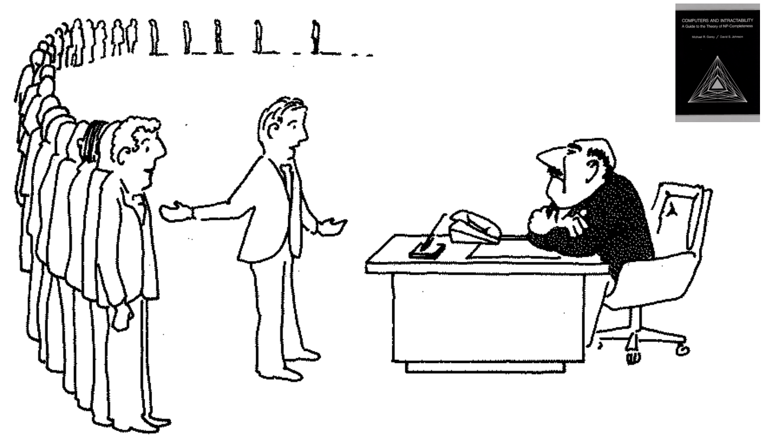
All of these problems (and many, many more) poly-time reduce to 3-SAT

Implications of Karp + Cook



All of these problems are NP-complete; they are manifestations of the same really hard problem.

Implications of NP-completeness



"I can't find an efficient algorithm, but neither can all these famous people."

Birds-eye view: review

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, …
linearithmic	N log N	sorting, convex hull, closest pair, farthest pair, …
quadratic	N^2	???
	…	
exponential	c^N	???

Frustrating news. Huge number of problems have defied classification.

Birds-eye view: revised

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, …
linearithmic	N log N	sorting, convex hull, closest pair, farthest pair, …
3-SUM complete	probably N^2	3-SUM, 3-COLLINEAR, 3-CONCURRENT, …
	…	
NP-complete	probably c^N	3-SAT, IND-SET, ILP, …

Good news. Can put problems in equivalence classes.

Summary

Reductions are important in theory to:

- Establish tractability.
- Establish intractability.
- Classify problems according to their computational requirements.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - stack, queue, priority queue, symbol table, set, graph
  - sorting, regular expression, Delaunay triangulation
  - minimum spanning tree, shortest path, maximum flow, linear programming
- Determine difficulty of your problem and choose the right tool.
  - use exact algorithm for tractable problems
  - use heuristics for intractable problems