3.3 Balanced Trees

- 2-3 trees
- red-black trees
- B-trees

Symbol table review

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	?	yes	`compareTo()`
Goal	log N	log N	log N	log N	log N	log N	yes	`compareTo()`

Challenge. Guarantee performance.

This lecture. 2-3 trees, left-leaning red-black trees, B-trees.

introduced to the world in COS 226, Fall 2007

- 2-3 trees
- red-black trees
- B-trees

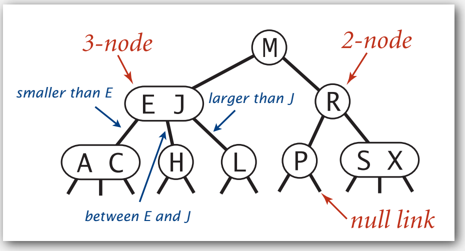
2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

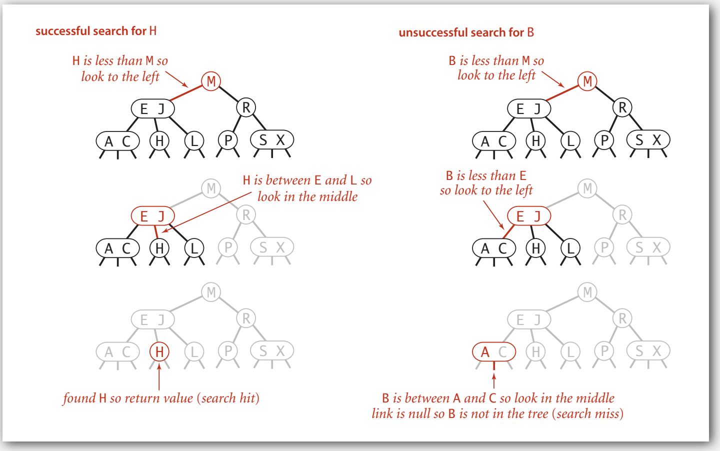
Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.



Search in a 2-3 tree

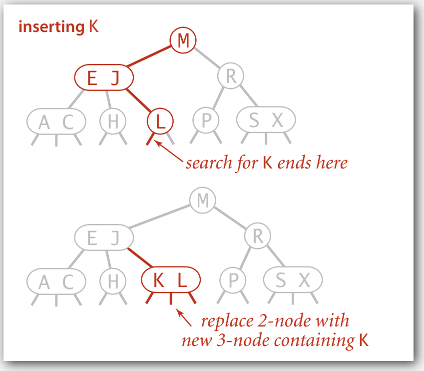
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).



Insertion in a 2-3 tree

Case 1. Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.

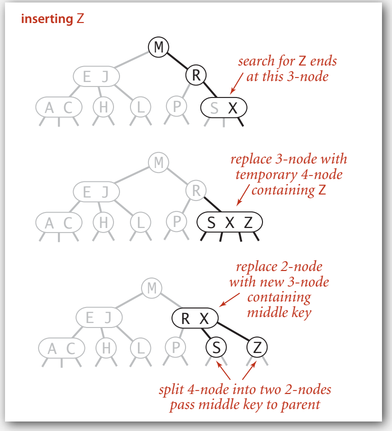


Insertion in a 2-3 tree

Case 2. Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

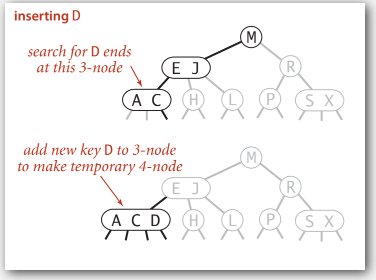
why middle key?

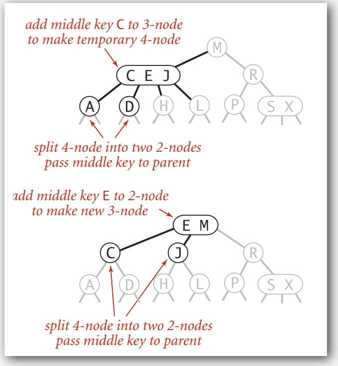


Insertion in a 2-3 tree

Case 2. Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.

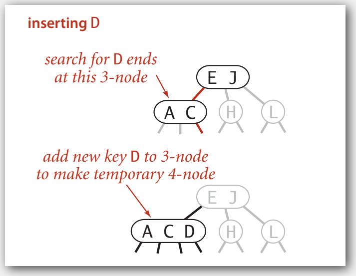


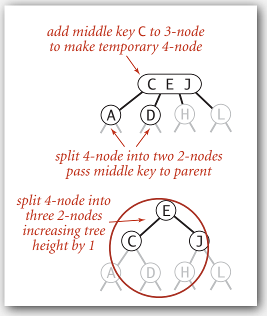


Insertion in a 2-3 tree

Case 2. Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

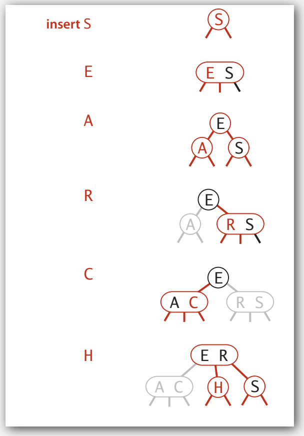


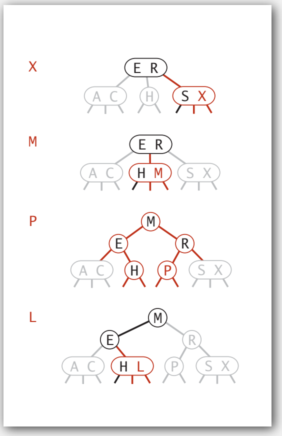


Remark. Splitting the root increases height by 1.

2-3 tree construction trace

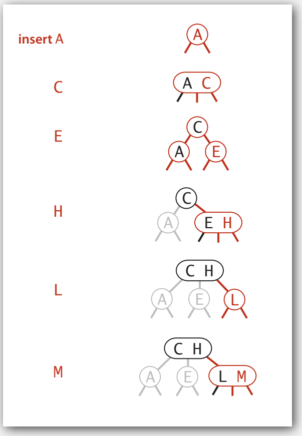
Standard indexing client.

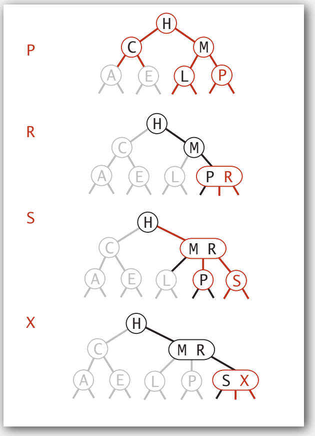




2-3 tree construction trace

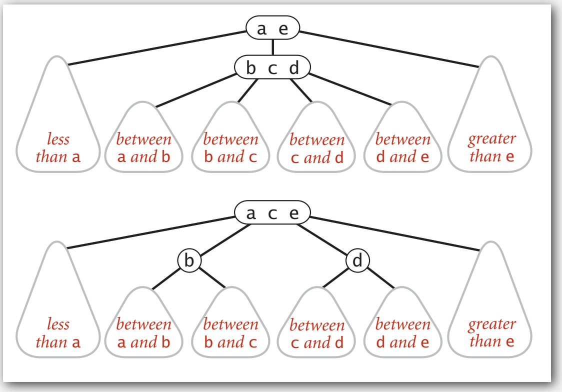
The same keys inserted in ascending order.





Local transformations in a 2-3 tree

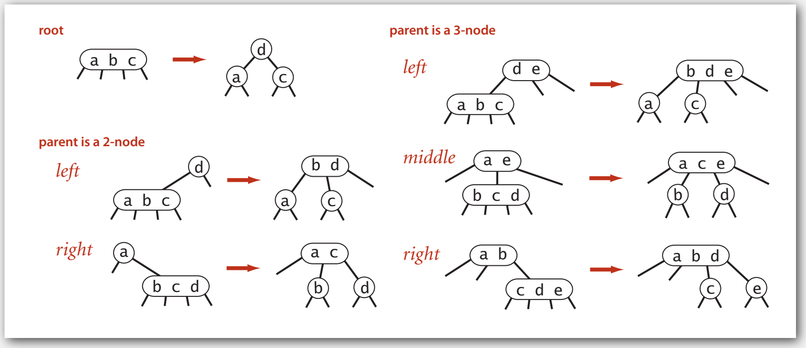
Splitting a 4-node is a local transformation: constant number of operations.



Global properties in a 2-3 tree

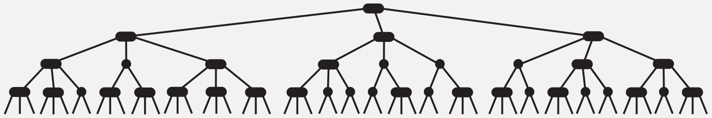
Invariant. Symmetric order.
Invariant. Perfect balance.

Pf. Each transformation maintains order and balance.



2-3 tree: performance

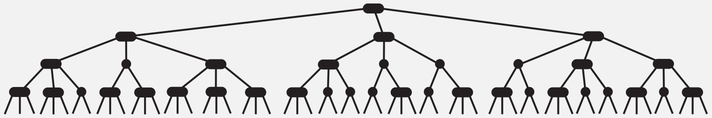
Perfect balance. Every path from root to null link has same length.



Tree height.
- Worst case:
- Best case:

2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.
- Worst case: lg N. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
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BST	N	N	N	1.39 lg N	1.39 lg N	?	yes	`compareTo()`
2-3 tree	c lg N	c lg N	c lg N	c lg N	c lg N	c lg N	yes	`compareTo()`

constants depend upon implementation

2-3 tree: implementation?

Direct implementation is complicated, because:

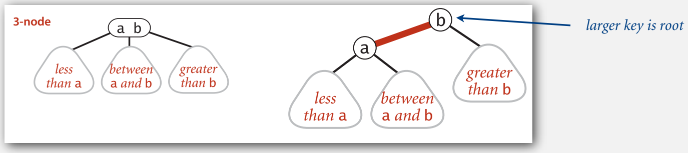
- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

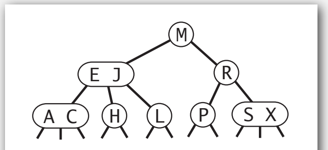
Bottom line. Could do it, but there's a better way.

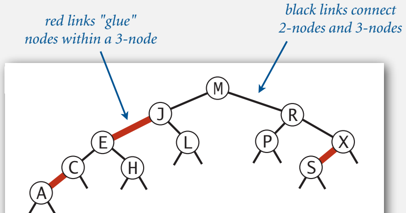
- 2-3-4 trees
- **red-black trees**
- B-trees

Left-leaning red-black trees (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.




2-3 tree



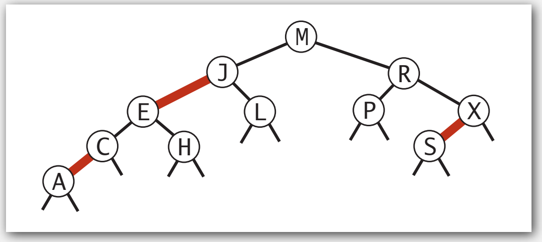
red-black tree

An equivalent definition

A BST such that:

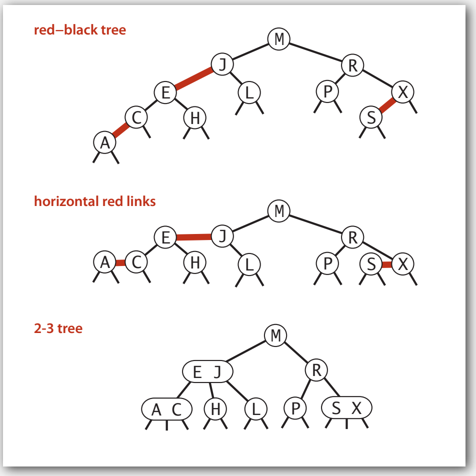
- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

"perfect black balance"



Left-leaning red-black trees: 1-1 correspondence with 2-3 trees

Key property. 1-1 correspondence between 2-3 and LLRB.

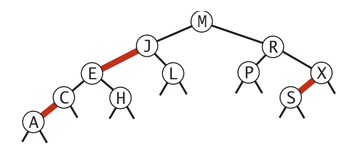


Search implementation for red-black trees

Observation. Search is the same as for elementary BST (ignore color).

but runs faster because of better balance

```
public Val get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if       (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



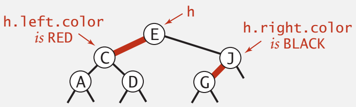
Remark. Many other ops (e.g., ceiling, selection, iteration) are also identical.

Red-black tree representation

Each node is pointed to by precisely one link (from its parent) ⇒ can encode color of links in nodes.

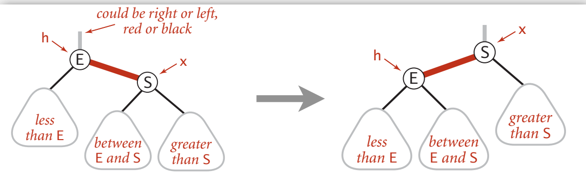
```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;    // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



Elementary red-black tree operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

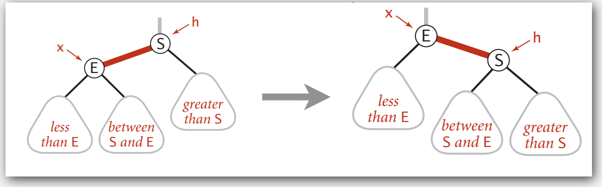


```
private Node rotateLeft(Node h)
{
   assert (h != null) && isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black tree operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

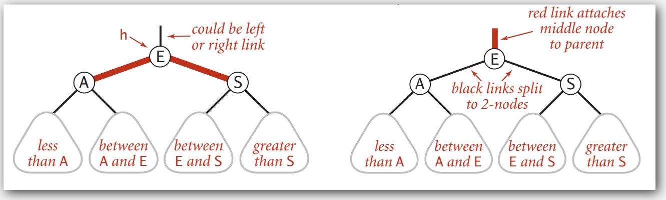


```
private Node rotateRight(Node h)
{
   assert (h != null) && isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black tree operations

Color flip. Recolor to split a (temporary) 4-node.

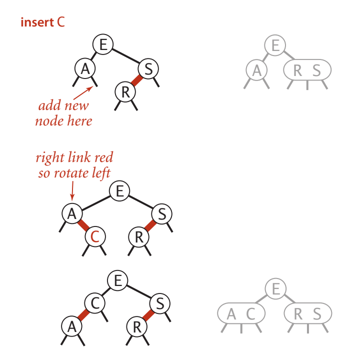


```
private void flipColors(Node h)
{
   assert !isRed(h) && isRed(h.left) && isRed(h.right);

   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

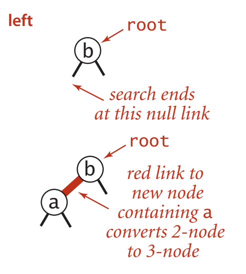
Insertion in a LLRB tree: overview

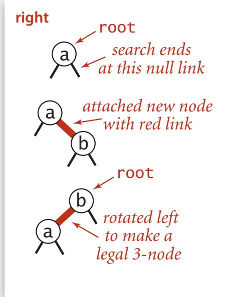
Basic strategy. Maintain 1-1 correspondence with 2-3 trees by applying elementary red-black tree operations



Insertion in a LLRB tree

Warmup 1. Insert into a tree with exactly 1 node.

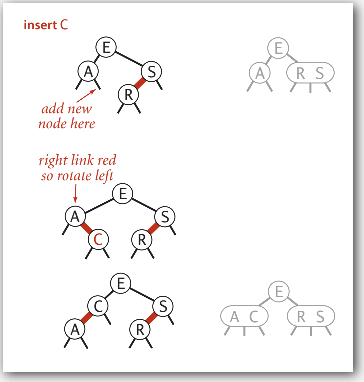




Insertion in a LLRB tree

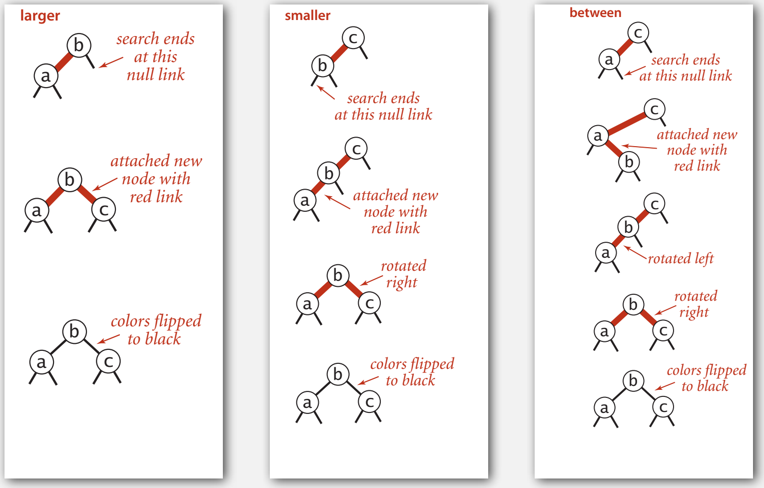
Case 1. Insert into a 2-node at the bottom.

- Do standard BST insert; color new link red.
- If new red link is a right link, rotate left.



Insertion in a LLRB tree

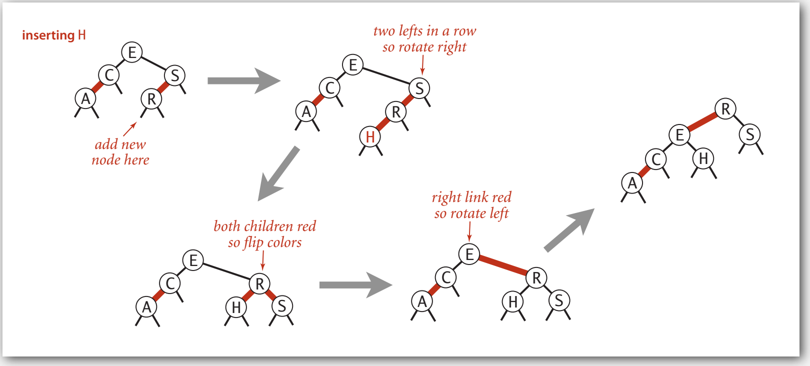
Warmup 2. Insert into a tree with exactly 2 nodes.



Insertion in a LLRB tree

Case 2. Insert into a 3-node at the bottom.

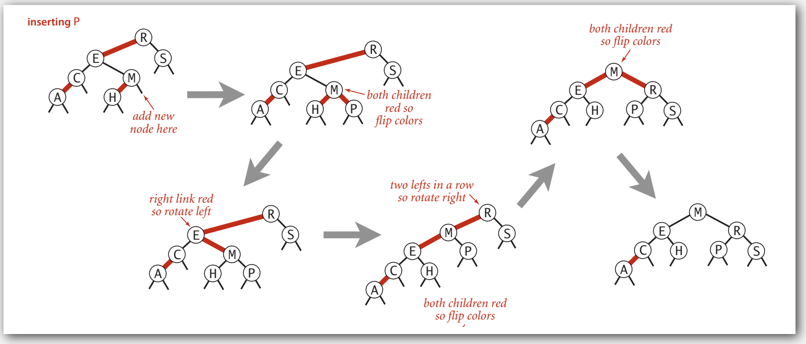
- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).



Insertion in a LLRB tree: passing red links up the tree

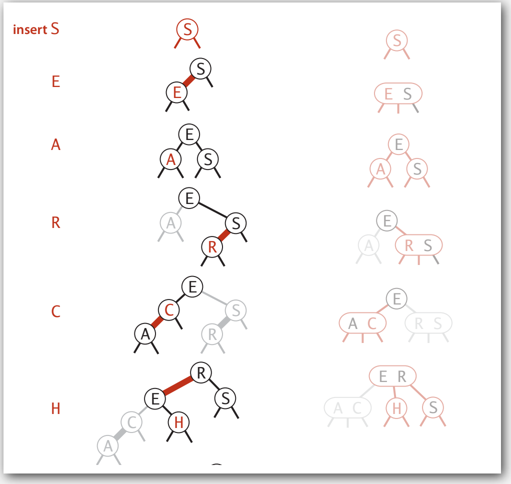
Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat Case 1 or Case 2 up the tree (if needed).



LLRB tree construction trace

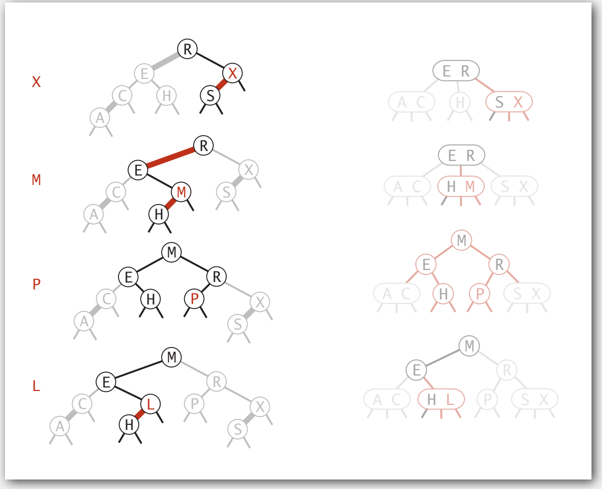
Standard indexing client.



red black tree 2-3 tree

LLRB tree construction trace

Standard indexing client (continued).

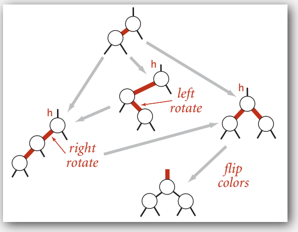


red black tree 2-3 tree

Insertion in a LLRB tree: Java implementation

Same code for both cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.



```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp < 0) h.left  = put(h.left,  key, val);
   else if (cmp > 0) h.right = put(h.right, key, val);
   else h.val = val;

   if (isRed(h.right)  && !isRed(h.left))      h = rotateLeft(h);
   if (isRed(h.left)   && isRed(h.left.left))  h = rotateRight(h);
   if (isRed(h.left)   && isRed(h.right))      h = flipColors(h);

   return h;
}
```

insert at bottom

lean left

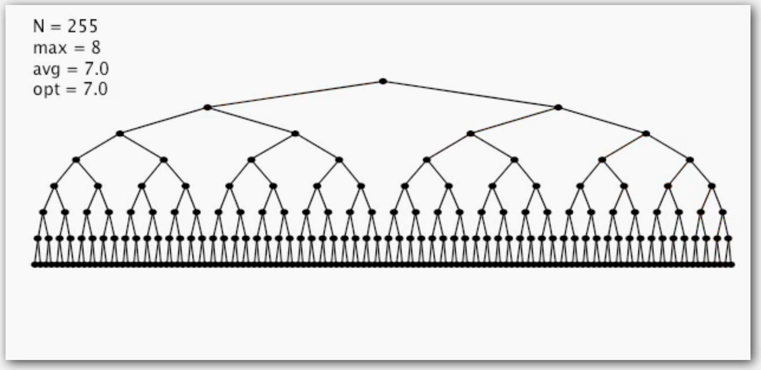
balance 4-node

split 4-node

only a few extra lines of code to provide near-perfect balance

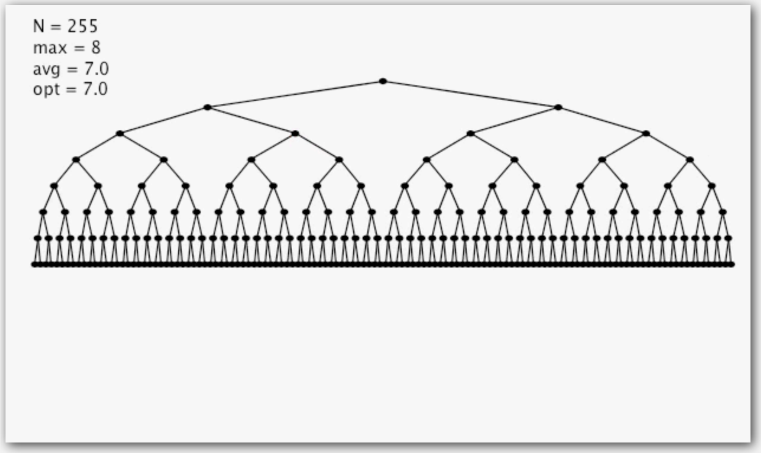
---

Insertion in a LLRB tree: visualization



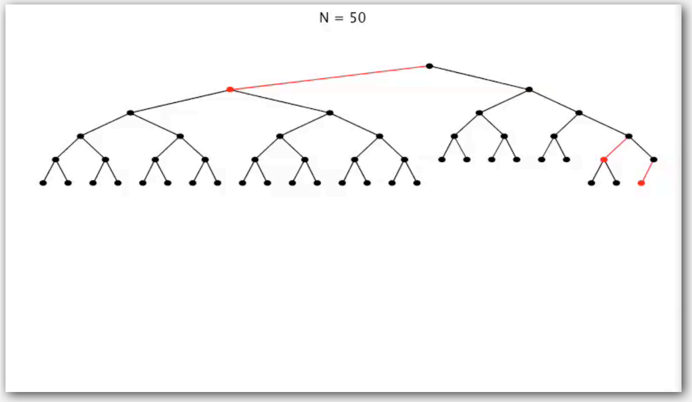
255 insertions in ascending order

Insertion in a LLRB tree: visualization



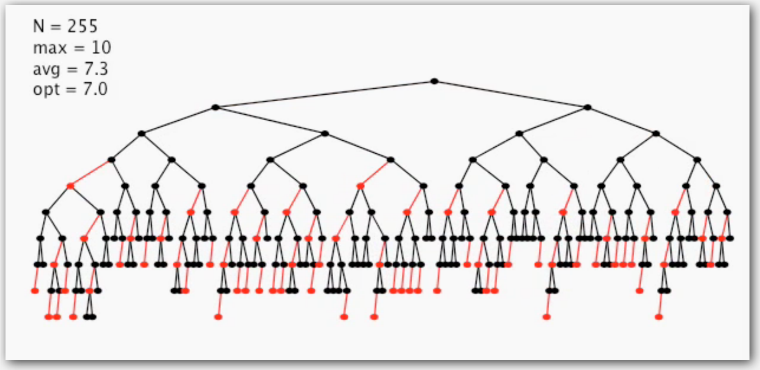
255 insertions in descending order

Insertion in a LLRB tree: visualization



50 random insertions

Insertion in a LLRB tree: visualization



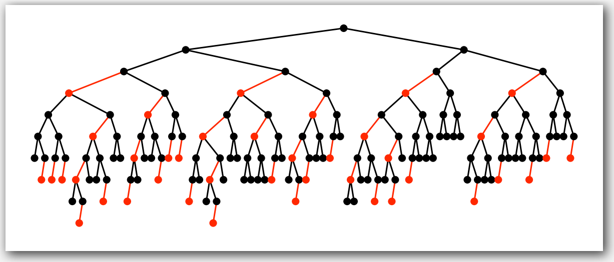
255 random insertions

Balance in LLRB trees

Proposition. Height of tree is ≤ 2 lg N in the worst case.

Pf.

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.

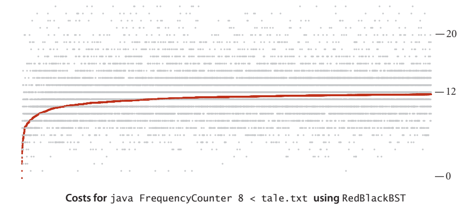


Property. Height of tree is ~ 1.00 lg N in typical applications.

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	?	yes	`compareTo()`
2-3 tree	c lg N	c lg N	c lg N	c lg N	c lg N	c lg N	yes	`compareTo()`
red-black tree	2 lg N	2 lg N	2 lg N	1.00 lg N *	1.00 lg N *	1.00 lg N *	yes	`compareTo()`

* exact value of coefficient unknown but extremely close to 1



Costs for `java FrequencyCounter 8 < tale.txt` using `RedBlackBST`

Why left-leaning trees?

old code (that students had to learn in the past)

```
private Node put(Node x, Key key, Value val, boolean sw)
{
   if (x == null)
      return new Node(key, value, RED);
   int cmp = key.compareTo(x.key);

   if (isRed(x.left) && isRed(x.right))
   {
      x.color = RED;
      x.left.color  = BLACK;
      x.right.color = BLACK;
   }
   if (cmp < 0)
   {
      x.left = put(x.left, key, val, false);
      if (isRed(x) && isRed(x.left) && sw)
         x = rotateRight(x);
      if (isRed(x.left) && isRed(x.left.left))
      {
         x = rotateRight(x);
         x.color = BLACK; x.right.color = RED;
      }
   }
   else if (cmp > 0)
   {
      x.right = put(x.right, key, val, true);
      if (isRed(h) && isRed(x.right) && !sw)
         x = rotateLeft(x);
      if (isRed(h.right) && isRed(h.right.right))
      {
         x = rotateLeft(x);
         x.color = BLACK; x.left.color = RED;
      }
   }
   else x.val = val;
   return x;
}
```

extremely tricky

new code (that you have to learn)

```
public Node put(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);
   int cmp = kery.compareTo(h.key);
   if (cmp < 0)
      h.left  = put(h.left,  key, val);
   else if (cmp > 0)
      h.right = put(h.right, key, val);
   else h.val = val;

   if (isRed(h.right) && !isRed(h.left))
      h = rotateLeft(h);
   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);
   if (isRed(h.left) && isRed(h.right))
      h = flipColors(h);

  return h;
}
```

straightforward
(if you've paid attention)

Why left-leaning trees?

Simplified code.

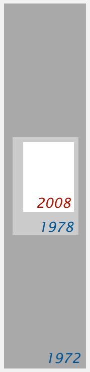
- Left-leaning restriction reduces number of cases.
- Short inner loop.

Same ideas simplify implementation of other operations.

- Delete min/max.
- Arbitrary delete.

Improves widely-used algorithms.

- AVL trees, 2-3 trees, 2-3-4 trees.
- Red-black trees.



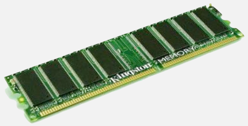
Bottom line. Left-leaning red-black trees are the simplest balanced BST to implement and the fastest in practice.

- 2-3-4 trees
- red-black trees
- **B-trees**

File system model

Page. Contiguous block of data (e.g., a file or 4096-byte chunk).

Probe. First access to a page (e.g., from disk to memory).



Model. Time required for a probe is much larger than time to access data within a page.

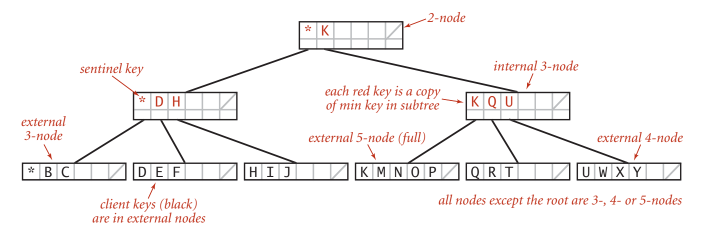
Goal. Access data using minimum number of probes.

B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to M-1 key-link pairs per node.

- At least 2 key-link pairs at root.
- At least M/2 key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

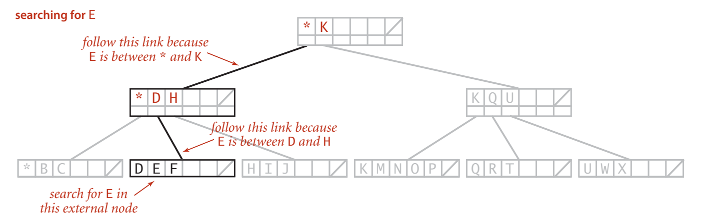
choose M as large as possible so that M links fit in a page, e.g., M = 1000



Anatomy of a B-tree set (M = 6)

Searching in a B-tree

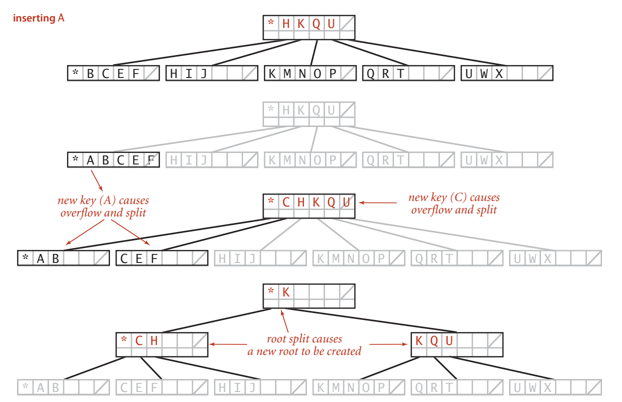
- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.



Searching in a B-tree set (M = 6)

Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with M key-link pairs on the way up the tree.



Inserting a new key into a B-tree set

Balance in B-tree

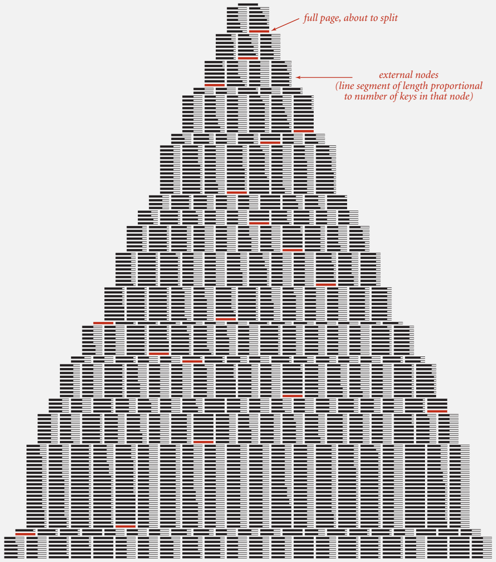
Proposition. A search or an insertion in a B-tree of order M with N keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

Pf. All internal nodes (besides root) have between M/2 and M-1 links.

In practice. Number of probes is at most 4. ← M = 1000; N = 62 billion
$\log_{M/2} N \le 4$

Optimization. Always keep root page in memory.

Building a large B tree



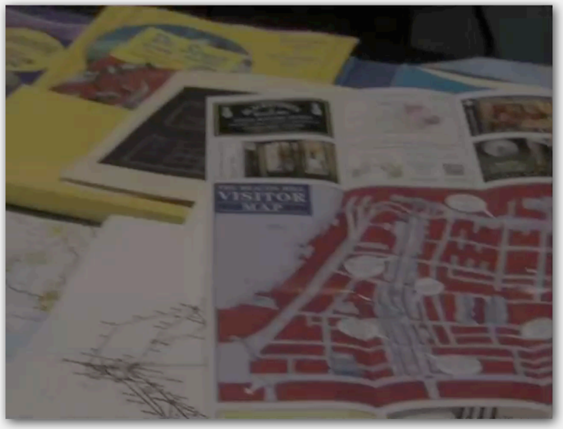
Balanced trees in the wild

Red-black trees are widely used as system symbol tables.
- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.

B-tree variants. B+ tree, B*tree, B# tree, …

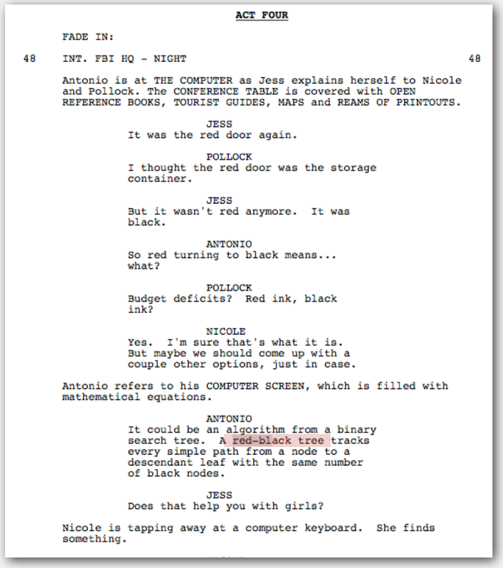
B-trees (and variants) are widely used for file systems and databases.
- Windows: HPFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

Red-black trees in the wild



Common sense. Sixth sense.
Together they're the
FBI's newest team.

Red-black trees in the wild



ACT FOUR

FADE IN:

48 INT. FBI HQ - NIGHT 48

Antonio is at THE COMPUTER as Jess explains herself to Nicole and Pollock. The CONFERENCE TABLE is covered with OPEN REFERENCE BOOKS, TOURIST GUIDES, MAPS and REAMS OF PRINTOUTS.

JESS
It was the red door again.

POLLOCK
I thought the red door was the storage container.

JESS
But it wasn't red anymore. It was black.

ANTONIO
So red turning to black means... what?

POLLOCK
Budget deficits? Red ink, black ink?

NICOLE
Yes. I'm sure that's what it is. But maybe we should come up with a couple other options, just in case.

Antonio refers to his COMPUTER SCREEN, which is filled with mathematical equations.

ANTONIO
It could be an algorithm from a binary search tree. A red-black tree tracks every simple path from a node to a descendant leaf with the same number of black nodes.

JESS
Does that help you with girls?

Nicole is tapping away at a computer keyboard. She finds something.