Auctions vs. Negotiations: We'll restrict ourselves to I.P.V. model, risk neutral.

Simple example: Seller wants to sell 1 object, value = 0.
\( v^* \)'s uniform on \([0, 1]\), (always \( v = 1 \)).

Case I: Sells to one buyer. Optimal auction is achieved if \( v^*_x \) is chosen so that \( M(v^*_x) = 0 \).
Marginal revenue:
\[
M(v) = v - \frac{1 - F}{f} = 2v - 1 \implies v^*_x = 1/2
\]
With one buyer, this is optimal mechanism of any sort.

\[
E[\text{revenue}] = \int_{v^*}^{1} (2v-1) dv = \int_{\frac{1}{2}}^{1} (2v-1) dv = v^2 \bigg|_{\frac{1}{2}}^{1} = \frac{1}{4}
\]

Case II: Sells to two buyers (Same \( F \), no reserve; that is, reserve = 0.

\[
E[\text{revenue}] = \int_{0}^{1} (2v-1) dv = \int_{0}^{1} (2v-1) 2\delta dv
\]
\[
= \frac{4}{3} v^3 \bigg|_{0}^{1} - v^2 \bigg|_{0}^{1} = \frac{1}{3} > \frac{1}{4}
\]

So: The extra bidder is worth more than setting the reserve optimally.

To generalize:
Some facts

1. \( \int_0^1 M(v) \, dF^n(v) = E \{ \max [M(v_1), M(v_2), \ldots, M(v_n)] \} \)

   Proof: \( F^n \) is distr. func. of \( \max \) of \( n \) ind. draws

2. \( \int_{V^*_n} M(v) \, dF^n(v) = E \{ \max [M(v_1), M(v_2), \ldots, M(v_n), 0] \} \)

   Proof: integrate only where \( M(v) > 0 \), \( V^*_n = M^{-1}(0) \), \( M \) mono \( \uparrow \)

3. \( E \{ M(v) \} = 0 \)

   Proof: expected revenue with one bidder, no reserve = 0.
   - or, check
     \[
     \int_0^1 \left[ v - \frac{1-F}{F} \right] \, dF = \int_0^1 (v + 1 + F) \, dF
     \]
     \[
     = \int_0^1 v \, dF + \int_0^1 F \, dF = \int_0^1 v \, dF + 1 - \int_0^1 F \, dF = 0
     \]

Consider now the comparison between an optimal auction with \( n \) bidders, and a no-reserve auction with \( n+1 \) bidders:

\[
\Delta = E \{ \text{revenue with } n+1 \} - E \{ \text{revenue with } n \} = \int_0^1 M \, dF^{n+1} - \int_{V^*_n} M \, dF^n
\]

\[
eq E \{ \max [M(v_1), M(v_2), \ldots, M(v_n)] \} - E \{ \max [M(v_1), \ldots, M(v_n), 0] \}
\]
Case 1: Suppose in some realization \( v_1, \ldots, v_n \)

\[ R_n = \max \left[ M(v_1), \ldots, M(v_n) \right] \geq 0 \]

Then

\[ \Delta = E \left\{ \max \left[ R_n, M(v_{n+1}) \right] \right\} - E \left\{ \max \left[ R_n, 0 \right] \right\} \]

\[ = E \left\{ \max \left[ R_n, M(0_{n+1}) \right] \right\} - E \left\{ \max \left[ R_n \right] \right\} \]

\( > 0 \)

Case 2: Suppose \( R_n < 0 \)

\[ \Delta = E \left\{ \min \left[ R_n, M(v_{n+1}) \right] \right\} - 0 > 0 \]

\[ \text{recall } E[M] \neq 0 \]