COS 423 Problem Set 9 Due Thurs. 4/23/09
Spring 2009

Collaboration Allowed

1. (Equivalence of flow problems) The *transportation problem* is defined as follows. Given is a directed bipartite graph containing two sets of vertices, *supply vertices* and *demand vertices*; every arc leads from a supply vertex to a demand vertex. Each vertex also has a non-negative number, which in the case of a supply vertex is interpreted as a supply of material, in the case of a demand vertex is interpreted as a demand of material. The problem to be solved is to determine whether all the demand vertices can have their demands satisfied by shipping material from the supply vertices along the arcs of the network. The constraint is that the total amount sent from any supply vertex cannot exceed its supply. Note that the arcs of the network do not have capacity constraints, although the amount sent along an arc cannot exceed the minimum of the supply at one end and the demand at the other end. If the demands can be satisfied, one would like a constructive solution; that is, for each arc, the amount of material to be moved along it.

(a) Show how to solve the transportation problem by converting it into a maximum flow problem and solving the maximum flow problem. Your transformation should be as efficient as possible.

(b) Consider the yes-no version of the maximum flow problem in which a capacitated network with a source and sink is given along with a target flow value, and the problem is to determine whether the given amount of flow can be sent from the source to the sink. Show how to solve this problem by converting it into a transportation problem and solving the transportation problem. Your transformation should be as efficient as possible.

2. In a maximum flow problem, a *positive* augmenting path is a path from s to t such that on each arc (v,w) of the path, the flow is strictly less than the capacity: $f(v,w) < c(v,w)$. Note that an augmenting path need not be a positive path, since in general we allow flow reduction as well as flow increase on such a path.

(a) Show that, starting from the zero flow, there is always a way to build a maximum flow in m or fewer steps of the following form: find a positive augmenting path and send some additional flow along it. (See Exercise 26.2-8 in CLRS.)

(b) Give an example of a network and a flow such that the flow is not maximum but there is no positive augmenting path.

3. (Extra credit) Professor D proposes the following method to find a maximum flow: Start with the zero flow. Find a positive augmenting path. Send as much flow along it as possible. (The amount of flow increase is $\min\{c(v,w) - f(v,w) \mid (v,w) \text{ is on the path}\}$. Repeat until there is no positive augmenting path. Either prove that, for any network there is some order in which to choose paths such that this method will give a maximum flow, or exhibit a network on which this method cannot produce a maximum flow, no matter how flow paths are chosen.