COS 423 Problem Set 10 (Final Take-Home Exercise) Due Monday 5/18/09
Spring 2009

No collaboration

1. Consider multivariate polynomials with integer coefficients, such as $x^2y + 5y - 10$. We restrict values of the variables to 1 and –1 and the degree of the polynomial to cubic: no term contains more than three variables, including duplicates. (The polynomial above is cubic; $x^2y^2 - 55x$ is not.) Prove that the problem of determining whether a given cubic polynomial evaluates to zero for some assignment of 1 and –1 to the variables is NP-complete.

2. Consider a Boolean formula that consists of a disjunction of terms, each term being a binary combination of two literals by one of the binary operators “and”, “or”, “implies”, “equals”, “nor”, and “nand”, e.g. “($\neg x$ and y) or ($\neg z$ nand $\neg w$) or (y implies z)…” Give an efficient algorithm (as efficient as you can) to test whether such a formula is a tautology; that is, it is true for all truth assignments to the variables. (Given an appropriate representation of the input, there is a linear-time algorithm for this problem.) Note: “nand” means “not and”; “nor” means “not or”; “x implies y” is equivalent to “$\neg x$ or y”.

3. Recall the definition of a rank-balanced binary tree: each node has a rank, leaves have rank zero, missing nodes have rank –1, and every node other than the root has rank difference one or two. (The *rank difference* of a node is its parent’s rank minus its own rank.)

(a) Give a linear-time algorithm to determine whether a given binary tree can be assigned ranks to make it rank-balanced. Hint: Process the tree bottom-up. For each node, compute the range of possible ranks it can have.

(b) A *lopsided node* in a red-black tree is a node from which there is a path of all black nodes to a leaf node, and a (different) path of nodes alternating in color to a missing node. Prove that ranks can be assigned to the nodes of a red-black tree to make it rank-balanced if and only if it does not contain a lopsided node. (This problem is the repaired other half of a problem from earlier in the semester.) Hint: Use ideas from your solution to (a).

4. Make up your own problem on any subject we have covered in class, and provide a solution to your problem. Explain why you have chosen your problem: what concepts does it illustrate? You may choose a problem from the book and solve it, but I encourage you to design your own problem.