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An Analysis of the War of Attrition and the All-Pay Auction*

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We study the war of attrition and the all-pay auction when players' signals are affiliated and symmetrically distributed. We (a) find sufficient conditions for the existence of symmetric monotonic equilibrium bidding strategies and (b) examine the performance of these auction forms in terms of the expected revenue accruing to the seller. Under our conditions the war of attrition raises greater expected revenue than all other known sealed-bid auction forms. *Journal of Economic Literature* Classification Numbers: D44, D82. © 1997 Academic Press

1. INTRODUCTION

Since the classic work of Vickrey [15], the ranking of various auction forms in terms of expected revenue has been the central question of auction theory (Milgrom [10] and Wilson [17] provide surveys). When the bidders are risk-neutral and their information about the value of the object is independently and identically distributed, the so-called "revenue equivalence principle" (see, for instance, Myerson [13]) provides a complete answer to the revenue ranking question. When the assumption of independence is relaxed, the answer is less well understood. Utilizing the assumption that the bidders' information is affiliated, Milgrom and Weber [11] develop the most comprehensive set of revenue ranking results to date; however, they restrict attention to "standard" auction forms in which only the winner is required to pay.

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In this paper, we extend the analysis of Milgrom and Weber [11] to auctions with the property that losing bidders also pay positive amounts.[1] Specifically, we examine the performance of two alternative auction forms, the war of attrition and the all-pay auction. These auction forms share the common feature that all losing bidders pay exactly their bids and differ only in the amounts paid by the winning bidder. In the former, the winning bidder pays the second highest bid; whereas in the latter, the winner pays his own bid. As such, they are analogous to the standard second-price and first-price sealed-bid auctions, respectively.[2] We identify circumstances in which the expected revenue from the auctions considered here exceeds that from the corresponding auctions in which only the winner pays. Thus, the war of attrition yields greater revenue than the second-price auction, and the all-pay auction yields greater revenue than the first-price auction. We also show that the war of attrition outperforms the all-pay auction and thus yields higher revenue than all known sealed-bid auctions.

All of our results require that the equilibrium strategies in the two all-pay auctions be increasing in the signals received by the bidders. The conditions for this are stronger than those needed for equilibria to be increasing in the standard auctions and are identified below. Roughly, these conditions amount to assuming that the bidders' signals are not too strongly affiliated.

Our reasons for examining these alternative, "non-standard" auction forms are threefold: First, from a mechanism design standpoint, the restriction to allocation schemes which require only the person receiving the object to pay seems unwarranted. For instance, the prevalence of lotteries as allocation mechanisms serves to highlight the restrictiveness of this assumption. Second, although these forms may not be widely used in an auction setting, the underlying games are natural models of conflict in many situations. For instance, the war of attrition has been used as a model for conflict among animals and insects [8, 2] and the struggle for survival among firms [7], while the all-pay auction has been used to model the arms race [14] and rent-seeking activity, such as lobbying [4, 3]. Our third reason for considering these forms is that, as our results indicate, they raise greater revenue than the forms previously considered and hence are worthy of attention as auctions *per se.*[3]

This paper is organized as follows: Section 2 briefly describes the model used in the analysis. Since this is the same as the model in [11], we eschew

[1] Of course, in an auction with an entry fee each bidder would also pay a positive amount. However, the amount would be fixed by the seller and, unlike the auctions we consider, not depend on the bid itself. We examine this issue in more detail in Section 5.7 below.

[2] Indeed, the "war of attrition" is perhaps better described as a "second-price all-pay auction" and the "all-pay auction" as a "first-price all-pay auction." We have chosen, however, to retain existing terminology.

[3] While it is known that in common value settings it is possible to extract nearly all the surplus [6, 9], the mechanisms that do this depend on the underlying distribution of signals.

a detailed description. As in [11], our major assumption is that bidders' signals are affiliated. In Section 3, we find sufficient conditions for the existence of a symmetric and increasing equilibrium in the war of attrition. Section 4 contains a parallel development for the all-pay auction. In both the war of attrition and the all-pay auction, bidders' signals being not "too affiliated" is sufficient to guarantee the existence of symmetric, increasing equilibria. In Section 5, we develop revenue comparisons between these auctions and the standard forms. These comparisons form the basis for the results reported above. We also show, by means of an example, that in constrast to the standard auction forms the war of attrition may extract all the surplus from buyers and hence may be optimal. Section 6 concludes.

2. PRELIMINARIES

We follow the model and notation of Milgrom and Weber [11] exactly. There is a single object to be auctioned and there are n bidders. Each bidder i receives a real valued *signal*, X_i, prior to the auction that affects the value of the object. Let $S=(S_1, S_2, ..., S_m)$ be other random variables that influence the value but are not observed by any bidder. The *value* of the object to bidder i is then:

$$V_i = u(S, X_i, \{X_j\}_{j \neq i}),$$

where u is non-negative, continuous, and increasing[4] in its variables. It is assumed that $E[V_i] < \infty$.

The random variables $S_1, S_2, ..., S_m, X_1, X_2, ..., X_n$ have a joint density $f(S, X_1, X_2, ..., X_n)$, and the function f is symmetric in the bidders' signals. We suppose that f satisfies the *affiliation* inequality, that is,

$$f(z \vee z')\, f(z \wedge z') \geqslant f(z)\, f(z'),$$

where $z \vee z'$ denotes the component-wise maximum of z and z' and $z \wedge z'$ denotes the component-wise minimum of z and z'. Intuitively, this means that a high value of one of the variables, S_j or X_i, makes it more likely that the other variables also take on high values.

In what follows, the random variable $Y_1 = \max\{X_j\}_{j \neq 1}$. Let $f_{Y_1}(\cdot \mid x)$ denote the conditional density of Y_1 given that $X_1 = x$, and let $F_{Y_1}(\cdot \mid x)$ denote the corresponding cumulative distribution function. Finally, let

$$\lambda(y \mid x) = \frac{f_{Y_1}(y \mid x)}{1 - F_{Y_1}(y \mid x)}$$

denote the *hazard rate* of the distribution $F_{Y_1}(\cdot \mid x)$.

[4] Throughout the paper, the term "increasing" is synonymous with "strictly increasing."

The variables X_1 and Y_1 are also affiliated (see [11]), and the following simple facts about the conditional distribution, $F_{Y_1}(\cdot \mid x)$, will be useful for our analysis.[5]

Fact 1. $F_{Y_1}(y \mid x)/f_{Y_1}(y \mid x)$ is non-increasing in x.

Fact 2. $\lambda(y \mid x)$ is non-increasing in x.

Fact 3. $F_{Y_1}(y \mid x)$ is non-increasing in x.

Define $v(x, y) = E[V_1 \mid X_1 = x, Y_1 = y]$. Since X_1 and Y_1 are affiliated, $v(x, y)$ is a non-decreasing function of its arguments. As in [11], we assume that it is, in fact, increasing.

The reader should consult [11] for details.

3. EQUILIBRIUM IN THE WAR OF ATTRITION

We model the war of attrition as an auction in the following manner. Prior to the start of the auction each bidder, *i*, receives a signal, X_i, which gives him or her some information about the value of the object. Each bidder submits a sealed bid of b_i, and the payoffs are:

$$W_i = \begin{cases} V_i - \max_{j \neq i} b_j & \text{if} \quad b_i > \max_{j \neq i} b_j \\ -b_i & \text{if} \quad b_i < \max_{j \neq i} b_j \\ \dfrac{1}{\#\{k : b_k = b_i\}} V_i - b_i & \text{if} \quad b_i = \max_{j \neq i} b_j, \end{cases}$$

where $i \neq j$. We have assumed that if $b_i = \max_{j \neq i} b_j$, the prize goes to each winning bidder with equal probability.

The analogy with the classic war of attrition model of conflict among two animals (Bishop, *et al.* [2]) should be clear. The derivation of a symmetric equilibrium with independent private values, that is, when $X_i = V_i$ and the X_i's are independently and identically distributed, is well known (see, for instance, [12]).

We begin with a heuristic derivation of the symmetric equilibrium strategy.

Suppose that bidders $j \neq 1$ follow the symmetric and increasing equilibrium strategy β. Suppose bidder 1 receives a signal, $X_1 = x$, and "bids" b. Then bidder 1's expected payoff is:

$$\Pi(b, x) = \int_{-\infty}^{\beta^{-1}(b)} (v(x, y) - \beta(y)) \, f_{Y_1}(y \mid x) \, dy - [1 - F_{Y_1}(\beta^{-1}(b) \mid x)] \, b. \quad (1)$$

[5] A proof of Fact 1 may be found in [11]. Fact 2 may be proved in a similar manner. Fact 3 then follows from Facts 1 and 2.

Maximizing (1) with respect to b yields the first-order condition

$$v(x, \beta^{-1}(b))\, f_{Y_1}(\beta^{-1}(b) \mid x) \frac{1}{\beta'(\beta^{-1}(b))} - [1 - F_{Y_1}(\beta^{-1}(b) \mid x)] = 0. \quad (2)$$

At a symmetric equilibrium, $b = \beta(x)$, and thus (2) yields

$$\begin{aligned} \beta'(x) &= v(x, x) \frac{f_{Y_1}(x \mid x)}{1 - F_{Y_1}(x \mid x)} \\ &= v(x, x)\, \lambda(x \mid x) \end{aligned} \quad (3)$$

and thus

$$\beta(x) = \int_{-\infty}^{x} v(t, t)\, \lambda(t \mid t)\, dt,$$

where, as before, $\lambda(\cdot \mid t)$ denotes the hazard rate of the conditional distribution $F_{Y_1}(\cdot \mid t)$.

The derivation of β is only heuristic because (3) is merely a necessary condition, and the optimality of $\beta(x)$ against β has not been established. Theorem 1 below provides a sufficient condition for β to be a symmetric equilibrium.

DEFINITION 1. *Let* $\varphi : \mathbf{R}^2 \to \mathbf{R}$ *be defined by*: $\varphi(x, y) = v(x, y)\lambda(y \mid x)$.

THEOREM 1. *Suppose that, for all* y, $\varphi(\cdot, y)$ *is an increasing function. A symmetric equilibrium in the war of attrition is given by the function* β *defined as*

$$\beta(x) = \int_{-\infty}^{x} v(t, t)\, \lambda(t \mid t)\, dt. \quad (4)$$

Proof. Let $\bar{x}$ denote the supremum of the support of Y_1. If bidders $j \neq 1$ use the strategy β, then it is never profitable for bidder 1 to bid more than $\beta(\bar{x})$.

If bidder 1 bids an amount, $\beta(z)$, when the signal is x, his or her payoff is

$$\begin{aligned} \Pi(z, x) &= \int_{-\infty}^{z} (v(x, y) - \beta(y))\, f_{Y_1}(y \mid x)\, dy - [1 - F_{Y_1}(z \mid x)]\, \beta(z) \\ &= \int_{-\infty}^{z} v(x, y) f_{Y_1}(y \mid x)\, dy - \beta(z)\, F_{Y_1}(z \mid x) \\ &\quad + \int_{-\infty}^{z} \beta'(y)\, F_{Y_1}(y \mid x)\, dy - [1 - F_{Y_1}(z \mid x)]\, \beta(z) \\ &= \int_{-\infty}^{z} v(x, y)\, f_{Y_1}(y \mid x)\, dy + \int_{-\infty}^{z} \beta'(y)\, F_{Y_1}(y \mid x)\, dy - \beta(z). \end{aligned}$$

Using (3) and (4) we obtain

$$\begin{aligned}\Pi(z, x) &= \int_{-\infty}^{z} v(x, y)\, f_{Y_1}(y \mid x)\, dy - \int_{-\infty}^{z} v(y, y)\, \lambda(y \mid y)[1 - F_{Y_1}(y \mid x)]\, dy \\ &= \int_{-\infty}^{z} [v(x, y)\, \lambda(y \mid x)\, dy - v(y, y)\, \lambda(y \mid y)][1 - F_{Y_1}(y \mid x)]\, dy \\ &= \int_{-\infty}^{z} [\varphi(x, y) - \varphi(y, y)][1 - F_{Y_1}(y \mid x)]\, dy.\end{aligned}$$

Since, by assumption, $\varphi(\cdot, y)$ is an increasing function, for all $y < x$, $[\varphi(x, y) - \varphi(y, y)] > 0$, and, for all $y > x$, $[\varphi(x, y) - \varphi(y, y)] < 0$. Thus, $\Pi(z, x)$ is maximized by choosing $z = x$.

Finally, observe that the equilibrium payoff to a bidder who receives a signal of x is

$$\Pi(x, x) = \int_{-\infty}^{x} [\varphi(x, y) - \varphi(y, y)][1 - F_{Y_1}(y \mid x)]\, dy \geqslant 0 \tag{5}$$

and thus each bidder is willing to participate in the auction. ∎

Notice that $\varphi(x, y)$ is the product of $v(x, y)$ and $\lambda(y \mid x)$ where the former is increasing in x, while the latter is decreasing in x. As a result, $\varphi(\cdot, y)$ is increasing if the affiliation between X_1 and Y_1 is not so strong that it overwhelms the increase in the expected value of the object, $v(\cdot, y)$, resulting from a higher signal, x. Of course, the assumption is automatically satisfied if bidders' signals are independent. On the other hand, when affiliation is strong, small changes in a bidder's signal may have a relatively large marginal effect on his expectation of the value of the object. That is, circumstances in which $\lambda(y \mid \cdot)$ is steeply decreasing might at the same time lead to a situation in which $v(\cdot, y)$ is steeply increasing. The latter may compensate for the former and $\varphi(\cdot, y)$ may be increasing even with strongly affiliated signals.

The assumption that $\varphi(\cdot, y)$ is increasing can be replaced by the weaker condition that, for all x and y,

$$(x - y) \times [\varphi(x, y) - \varphi(y, y)] > 0$$

since this is all that is needed to ensure that $\Pi(z, x)$ is maximized at $z = x$.

As an example of a situation where the conditions of Theorem 1 are satisfied, suppose $n = 2$, and let X (resp., Y) be the random variable that denotes bidder 1's (resp., 2's) signal. Let

$$f(x, y) = \tfrac{4}{5}(1 + xy) \qquad \text{on} \quad [0, 1] \times [0, 1].$$

Then $f_{Y_1}(y \mid x) = 2(1 + xy)/(2 + x)$, and $\lambda(y \mid x) = 2(1 + xy)/(2 + x - 2y - xy^2)$. If $v(x, y) = x$, then $\varphi(x, y) = 2x(1 + xy)/(2 + x - 2y - xy^2)$, and it may be verified that $\varphi(\cdot, y)$ is an increasing function.

Next, let supp f_X denote the common support of the bidders' signals, and let $\underline{x}$ and $\bar{x}$ denote the infimum and supremum of supp f_X, respectively. Of course, it may be that $\underline{x} = -\infty$ or $\bar{x} = \infty$ or both.

Two features of the equilibrium strategy (4), deserve to be highlighted. First, a bidder receiving the lowest possible signal, $\underline{x}$, bids zero. This is true even if $v(\underline{x}, \underline{x})$ is strictly positive. Second, as a bidder's signal approaches $\bar{x}$, his bid becomes unbounded. Again, this holds even if the expected value of the object at $\bar{x}$, $v(\bar{x}, \bar{x})$ is finite.

PROPOSITION 1. *Suppose that, for all* y, $\varphi(\cdot, y)$ *is an increasing function. Then* (i) $\lim_{x \to \underline{x}} \beta(x) = 0$ *and* (ii) $\lim_{x \to \bar{x}} \beta(x) = \infty$.

Proof. (i) follows directly from (4).

To verify (ii), choose z such that $v(z, z) > 0$. From (4), we can write

$$\begin{aligned}
\beta(x) &= \int_{-\infty}^{x} v(y, y)\, \lambda(y \mid y)\, dy \\
&= \int_{-\infty}^{z} v(y, y)\, \lambda(y \mid y)\, dy + \int_{z}^{x} v(y, y)\, \lambda(y \mid y)\, dy \\
&\geqslant \int_{-\infty}^{z} v(y, y)\, \lambda(y \mid y)\, dy + \int_{z}^{x} v(z, y)\, \lambda(y \mid z)\, dy \\
&\geqslant \int_{-\infty}^{z} v(y, y)\, \lambda(y \mid y)\, dy + \int_{z}^{x} v(z, z)\, \lambda(y \mid z)\, dy,
\end{aligned} \tag{6}$$

where the first inequality follows from the fact that $\varphi(\cdot, y) = v(\cdot, y)\, \lambda(y \mid \cdot)$ is increasing and the second from the fact that $v(z, \cdot)$ is increasing.

But now observe that for all y

$$\lambda(y \mid z) = -\frac{d}{dy} (\ln[1 - F_{Y_1}(y \mid z)])$$

and thus

$$\int_{z}^{x} \lambda(y \mid z)\, dy = \ln\left(\frac{1 - F_{Y_1}(z \mid z)}{1 - F_{Y_1}(x \mid z)}\right). \tag{7}$$

Using (7) in (6) we obtain

$$\beta(x) \geqslant \int_{-\infty}^{z} v(y, y)\, \lambda(y \mid y)\, dy + v(z, z) \ln\left(\frac{1 - F_{Y_1}(z \mid z)}{1 - F_{Y_1}(x \mid z)}\right).$$

As $x \to \bar{x}$, $F_{Y_1}(x|z) \to 1$. This completes the proof. ∎

4. EQUILIBRIUM IN THE ALL-PAY AUCTION

In an all-pay auction each bidder submits a sealed bid of b_i and the payoffs are:

$$W_i = \begin{cases} V_i - b_i & \text{if} \quad b_i > \max_{j \neq i} b_j \\ -b_i & \text{if} \quad b_i < \max_{j \neq i} b_j \\ \dfrac{1}{\#\{k : b_k = b_i\}} V_i - b_i & \text{if} \quad b_i = \max_{j \neq i} b_j, \end{cases}$$

where $i \neq j$. As before, we have assumed that if $b_i = \max_{j \neq i} b_j$, the prize goes to each winning bidder with equal probability.

Once again it is useful to begin with a heuristic derivation.

Suppose bidders $j \neq 1$ follow the symmetric (increasing) equilibrium strategy α. Suppose bidder 1 receives a signal, $X_1 = x$, and bids b. Then bidder 1's expected payoff is:

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y)\, f_{Y_1}(y \mid x)\, dy - b. \tag{8}$$

Maximizing (8) with respect to b yields the first-order condition

$$v(x, \alpha^{-1}(b))\, f_{Y_1}(\alpha^{-1}(b) \mid x)\, \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 = 0. \tag{9}$$

At a symmetric equilibrium, $\alpha(x) = b$ and thus (9) becomes

$$\alpha'(x) = v(x, x)\, f_{Y_1}(x \mid x) \tag{10}$$

and thus

$$\alpha(x) = \int_{-\infty}^{x} v(t, t)\, f_{Y_1}(t \mid t)\, dt.$$

Once again, the derivation is heuristic since (10) is only a necessary condition.

DEFINITION 2. Let $\psi : \mathbf{R}^2 \to \mathbf{R}$ be defined by $\psi(x, y) = v(x, y)\, f_{Y_1}(y \mid x)$.

THEOREM 2. *Suppose that, for all* y, $\psi(\cdot, y)$ *is an increasing function. A symmetric equilibrium in the all-pay auction is given by the function* α *defined as*:

$$\alpha(x) = \int_{-\infty}^{x} v(t, t)\, f_{Y_1}(t \mid t)\, dt. \tag{11}$$

Proof. Let $\bar{x}$ denote the supremum of the support of Y_1. If bidders $j \neq 1$ use the strategy α, then clearly it cannot be a best response for bidder 1 to bid more than $\alpha(\bar{x})$.

If bidder 1 bids an amount $\alpha(z)$ when the signal is x, his or her payoff is

$$\begin{aligned}\Pi(z, x) &= \int_{-\infty}^{z} v(x, y)\, f_{Y_1}(y \mid x)\, dy - \alpha(z)\\ &= \int_{-\infty}^{z} v(x, y)\, f_{Y_1}(y \mid x)\, dy - \int_{-\infty}^{z} v(y, y)\, f_{Y_1}(y \mid y)\, dy\\ &= \int_{-\infty}^{z} [\psi(x, y) - \psi(y, y)]\, dy.\end{aligned}$$

Since, by assumption, $\psi(\cdot, y)$ is an increasing function, for all $y < x$, $[\psi(x, y) - \psi(y, y)] > 0$, and, for all $y > x$, $[\psi(x, y) - \psi(y, y)] < 0$. Thus, $\Pi(z, x)$ is maximized by choosing $z = x$.

Finally, observe that the equilibrium payoff to a bidder who receives a signal of x is

$$\Pi(x, x) = \int_{-\infty}^{x} [\psi(x, y) - \psi(y, y)]\, dy \geqslant 0$$

and thus each bidder is willing to participate in the auction. ∎

It is useful to compare some qualitative features of the equilibrium strategy (11) in the all-pay auction to the equilibrium strategy (4) of the war of attrition. It is still the case that a bidder receiving the lowest possible signal, $\underline{x}$, bids zero. This is true even if $v(\underline{x}, \underline{x})$ is strictly positive. However, in an all-pay auction, as a bidder's signal approaches $\bar{x}$, his bid is bounded if the expected value of the object at $\bar{x}$, $v(\bar{x}, \bar{x})$ is finite. This is in contrast to the unboundedness of the bids in the war of attrition.

PROPOSITION 2. *Suppose that, for all* y, $\psi(\cdot, y)$ *is an increasing function. Then* (i) $\lim_{x \to \underline{x}} \alpha(x) = 0$ *and* (ii) $\lim_{x \to \bar{x}} \alpha(x) \leqslant \lim_{x \to \bar{x}} v(x, x)$.

Proof. (i) follows immediately from (11).

To verify (ii) notice that

$$\alpha(x) \leqslant \int_{-\infty}^{x} v(x, y)\, f_{Y_1}(y \mid x)\, dy \leqslant v(x, x) \int_{-\infty}^{x} f_{Y_1}(y \mid x)\, dy \leqslant v(x, x). \quad ∎$$

5. REVENUE COMPARISONS

In this section we examine the performance of the war of attrition and the all-pay auction in terms of the expected revenue accruing to the seller. As a benchmark, recall from Milgrom and Weber [11] that, with affiliation, the expected revenue from a second-price auction is greater than the expected revenue from a first-price auction.

5.1. *War of Attrition versus Second-Price Auction*

Our first result is that, under the condition that $\varphi(x, y)$ is increasing in x, the revenue from the war of attrition is greater than that from a second-price auction.

THEOREM 3. *Suppose $\varphi(\cdot, y)$ is increasing. Then the expected revenue from a war of attrition is greater than or equal to the expected revenue from a second-price auction.*

Proof. In a second-price auction, the equilibrium bid by a bidder who receives a signal of x is $v(x, x)$ (see [11], pp. 1100–1101), and thus the expected payment by such a bidder is:

$$e^{II}(x) = \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid x) dy. \tag{12}$$

In a war of attrition, the expected payment in equilibrium by a bidder who receives a signal of x is

$$\begin{aligned}
e^{W}(x) &= \int_{-\infty}^{x} \beta(y)\, f_{Y_1}(y \mid x)\, dy + [1 - F_{Y_1}(x \mid x)]\, \beta(x) \\
&= \beta(x)\, F_{Y_1}(x \mid x) - \int_{-\infty}^{x} \beta'(y)\, F_{Y_1}(y \mid x)\, dy + [1 - F_{Y_1}(x \mid x)]\, \beta(x) \\
&= \beta(x) - \int_{-\infty}^{x} \beta'(y)\, F_{Y_1}(y \mid x)\, dy \\
&= \int_{-\infty}^{x} v(y, y)\, \lambda(y \mid y)\, dy - \int_{-\infty}^{x} v(y, y)\, \lambda(y \mid y)\, F_{Y_1}(y \mid x)\, dy \\
&= \int_{-\infty}^{x} v(y, y)\, \lambda(y \mid y)\, f_{Y_1}(y \mid x) \left[\frac{1 - F_{Y_1}(y \mid x)}{f_{Y_1}(y \mid x)} \right] dy \\
&= \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)} \right] dy.
\end{aligned} \tag{13}$$

For $y \leqslant x$, Fact 2 implies that $\lambda(y \mid y) \geqslant \lambda(y \mid x)$ and thus

$$\begin{aligned} e^W(x) &= \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)} \right] dy \\ &\geqslant \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid x)\, dy \\ &= e^{II}(x) \end{aligned}$$

using (12). ∎

5.2. *All-Pay Auction versus First-Price Auction*

Our second result is that, under the condition that $\psi(x, y)$ is increasing in x, the revenue from an all-pay auction is greater than that from a first-price auction.

THEOREM 4. *Suppose $\psi(\cdot, y)$ is increasing. Then the expected revenue from an all-pay auction is greater than or equal to the expected revenue from a first-price sealed-bid auction.*

Proof. Let $b^*(x)$ denote the equilibrium bid in a first-price auction of a bidder who receives a signal of x (see [11], p. 1107). Then the expected payment is:

$$\begin{aligned} e^I(x) &= F_{Y_1}(x \mid x)\, b^*(x) \\ &= F_{Y_1}(x \mid x) \int_{-\infty}^{x} v(y, y)\, dL(y \mid x) \qquad (14) \end{aligned}$$

where

$$L(y \mid x) = \exp\left(-\int_{y}^{x} \frac{f_{Y_1}(t \mid t)}{F_{Y_1}(t \mid t)}\, dt \right).$$

Equation (14) can be rewritten as

$$e^I(w) = \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid y) \left[\frac{F_{Y_1}(x \mid x)}{F_{Y_1}(y \mid y)} \right] \exp\left(-\int_{y}^{x} \frac{f_{Y_1}(t \mid t)}{F_{Y_1}(t \mid t)}\, dt \right) dy. \qquad (15)$$

We now argue that, for all $y < x$:

$$\exp\left(-\int_{y}^{x} \frac{f_{Y_1}(t \mid t)}{F_{Y_1}(t \mid t)}\, dt \right) \leqslant \left[\frac{F_{Y_1}(y \mid y)}{F_{Y_1}(x \mid x)} \right]. \qquad (16)$$

To see this, note that

$$-\int_y^x \frac{f_{Y_1}(t \mid t)}{F_{Y_1}(t \mid t)}\, dt \leqslant -\int_y^x \frac{f_{Y_1}(t \mid y)}{F_{Y_1}(t \mid y)}\, dt$$
$$= \ln F_{Y_1}(y \mid y) - \ln F_{Y_1}(x \mid y)$$
$$\leqslant \ln F_{Y_1}(y \mid y) - \ln F_{Y_1}(x \mid x),$$

where the first inequality follows from Fact 1 in Section 2 and the second inequality follows from Fact 3. Taking the exponent of both sides yields (16).

Using (16) in (15),

$$e^I(x) \leqslant \int_{-\infty}^x v(y, y)\, f_{Y_1}(y \mid y)\, dy$$
$$= \alpha(x)$$
$$= e^A(x),$$

the expected payment in an all-pay auction. ∎

Weber [16] reports a result similar to Theorem 4, as do Amann and Leininger [1]. The proof in the latter paper utilizes the "linkage principle" [11] and is very different from the one presented here. Also, it directly assumes the existence of an increasing equilibrium in the all-pay auction.

5.3. *War of Attrition versus All-Pay Auction*

Our next result compares the expected revenue from the war of attrition to that from the all-pay auction.

We first establish that the sufficient condition identified in Section 3 implies that there is an increasing equilibrium in the all-pay auction also.

PROPOSITION 3. *Suppose that $\varphi(\cdot, y)$ is an increasing function of x. Then $\psi(\cdot, y)$ is an increasing function of x.*

Proof. Let $x < x'$. Then since $\varphi(\cdot, y)$ is an increasing function, we have that

$$v(x, y)\left[\frac{f_{Y_1}(y \mid x)}{1 - F_{Y_1}(y \mid x)}\right] < v(x', y)\left[\frac{f_{Y_1}(y \mid x')}{1 - F_{Y_1}(y \mid x')}\right].$$

By Fact 3 in Section 2, $F_{Y_1}(y \mid x) \geqslant F_{Y_1}(y \mid x')$ and thus

$$v(x, y)\, f_{Y_1}(y \mid x) < v(x', y)\, f_{Y_1}(y \mid x'),$$

which completes the proof. ∎

We now show that if $\varphi(\cdot, y)$ is increasing, the war of attrition outperforms the all-pay auction.

THEOREM 5. *Suppose $\varphi(\cdot, y)$ is increasing. Then the expected revenue from a war of attrition is greater than or equal to the expected revenue from an all-pay auction.*

Proof. In an all-pay auction, the expected payment in equilibrium by a bidder who receives a signal of x is

$$e^A(x) = \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid y)\, dy.$$

From (13) in a war of attrition, the expected payment in equilibrium is

$$\begin{aligned} e^W(x) &= \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)}\right] dy \\ &= \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid y) \left[\frac{1 - F_{Y_1}(y \mid x)}{1 - F_{Y_1}(y \mid y)}\right] dy \\ &\geqslant \int_{-\infty}^{x} v(y, y)\, f_{Y_1}(y \mid y)\, dy \\ &= e^A(x), \end{aligned}$$

since, by Fact 3, for all $y < x$, $F_{Y_1}(y \mid x) \leqslant F_{Y_1}(y \mid y)$. ∎

Analogous to the result that the equilibrium bids in the second-price auction exceed those in the first-price auction, we have the result that the equilibrium bids in the war of attrition exceed those in the all-pay auction. To see this, observe that

$$\begin{aligned} \beta(x) &= \int_{-\infty}^{x} v(t, t) \left[\frac{f_{Y_1}(t \mid t)}{1 - F_{Y_1}(t \mid t)}\right] dt \\ &\geqslant \int_{-\infty}^{x} v(t, t)\, f_{Y_1}(t \mid t)\, dt \\ &= \alpha(x). \end{aligned}$$

5.4. *All-Pay Auction versus Second-Price Auction*

It remains to compare the expected revenues from an all-pay auction to that from a second-price auction. We now show by means of two examples that no general ranking of the two auctions is possible.

Let:

$$f(x, y) = \tfrac{4}{5}(1 + xy) \qquad \text{on} \quad [0, 1] \times [0, 1].$$

The density function f satisfies the affiliation inequality.

If $v(x, y) = x$, $\psi(x, y)$ is an increasing function of x so that the conditions of Theorem 2 are satisfied. Routine calculations show that, for all $x > 0$, $e^A(x) > e^{II}(x)$, and thus, in this case, the expected revenue from an all-pay auction is *greater* than that from a second-price auction.

On the other hand, if $v(x, y) = x^4$, we have that for $x^* \simeq 0.707$:

$$e^A(x) > e^{II}(x) \qquad \text{if} \quad x < x^*$$

$$e^A(x) < e^{II}(x) \qquad \text{if} \quad x > x^*$$

and furthermore

$$e^A = \int_0^1 e^A(x)\, f_X(x)\, dx < \int_0^1 e^{II}(x)\, f_X(x)\, dx = e^{II}$$

so that, in this case, the expected revenue from an all-pay auction is *less* than that from a second-price auction.

5.5. *Summary of Revenue Comparisons*

The relationships between the expected revenues from *increasing* equilibria of the various auctions may be summarized as follows:

War of
Attrition

$\nearrow \qquad \nwarrow$

Second-Price $\qquad$ All-Pay

$\nwarrow \qquad \nearrow$

First-Price

Why is it that making all bidders pay increases expected revenues? To understand the reason for this it is worthwhile to recall why the standard second-price auction performs better than the first-price auction [11]. By establishing a *linkage* between the price paid by the winning bidder and the second highest signal, a second-price auction induces a feedback between a bidder's own signal and his expected payment. For a fixed bid, if a bidder's own signal is higher, he expects that the second highest signal is also higher, and hence, so is the price he expects to pay. Thus the expected payment function in a second-price auction is steeper in a bidder's own signal than in a first-price auction. Since the expected payment of a bidder with the lowest possible signal is zero in both cases, the expected payment in a second-price auction is greater than in a first-price auction. We will refer to this as the "second-price effect."

In both the war of attrition and the all-pay auction losing bidders are required to pay their own bids. This induces a similar *linkage*. For a fixed bid, if a bidder's own signal is higher, he expects that the other signals are also higher, thus raising the probability that he will lose the auction. This increases his or her expected payment relative to that in a standard auction. Once again, the expected payment function in, say, an all-pay auction is steeper in a bidder's own signal than in a first-price auction. As before, the expected payment in an all-pay auction is also greater than in a first-price auction. This may be referred to as the "losing bid effect."

To see this formally, let $e^M(z, x)$ be the expected payment by a bidder with signal x who bids as if his or her signal were z in an auction mechanism M. For the first-price and all-pay auctions we have

$$e^I(z, x) = F_{Y_1}(z \mid x)\, b^*(z)$$

$$e^A(z, x) = F_{Y_1}(z \mid x)\, \alpha(z) + [1 - F_{Y_1}(z \mid x)]\, \alpha(z).$$

As in Milgrom and Weber [11] the key to the ranking of auctions is the derivative of $e^M(z, x)$ with respect to the second argument, that is, $e_2^M(z, x)$ evaluated at $z = x$. The following is a simple version of the linkage principle [11] which suffices for our purposes.

PROPOSITION 4 (Linkage Principle). *Suppose L and M are two auction mechanisms with symmetric increasing equilibria such that the expected payment of a bidder with signal $\underline{x}$ is 0. If for all x, $e_2^M(x, x) \geqslant e_2^L(x, x)$ then for all x, $e^M(x, x) \geqslant e^L(x, x)$.*

Proof. First, observe that the expected profit of a bidder with signal x who bids as if his or her signal were z in an auction mechanism M can be written as

$$\Pi^M(z, x) = R(z, x) - e^M(z, x),$$

where

$$R(z, x) = \int_{-\infty}^{z} v(x, y)\, f_{Y_1}(y \mid x)\, dy$$

is the expected gain of winning. Observe that this expression is the same for both auction forms as long as each has a symmetric increasing equilibrium so that the circumstances in which a bidder wins is the same in both auctions.

In equilibrium, it is optimal to choose $z = x$ and the resulting first-order conditions imply that

$$e_1^M(x, x) = e_1^L(x, x) \tag{17}$$

where $e_1^M(x, x)$ is the derivative of $e^M(z, x)$ with respect to z evaluated at $z = x$.

Now write $\Delta(x) = e^M(x, x) - e^L(x, x)$ and using (17) we have that

$$\Delta'(x) = e_2^M(x, x) - e_2^L(x, x),$$

which is non-negative by assumption. Since $\Delta(\underline{x}) = 0$, for all x, $\Delta(x) \geqslant 0$. ∎

To see that adding an all-pay component to the standard auctions leads to an increase in $e_2(x, x)$, notice that in a first-price auction

$$e_2^I(z, x) = \frac{\partial}{\partial x}\left[F_{Y_1}(z \mid x)\, b^*(z)\right] < 0$$

because of Fact 3, whereas in an all-pay auction

$$e_2^A(z, x) = \frac{\partial}{\partial x}\left[F_{Y_1}(z \mid x)\, \alpha(z)\right] + \frac{\partial}{\partial x}\left[(1 - F_{Y_1}(z \mid x))\, \alpha(z)\right] = 0.$$

The second term in this expression is what we have called the "losing bid effect."

The revenue ranking results of this section are driven by the combination of the two effects, the "second-price effect" and the "losing bid effect." As shown above, an all-pay auction is superior to a first-price auction because of the addition of the losing bid effect. Of course, the second-price auction is superior to the first-price auction because of the second-price effect. The war of attrition incorporates both the second-price and the losing bid effects and thus dominates the other three forms.[6] The comparison between the second-price auction and the all-pay auction is ambiguous because each incorporates only one of the two effects and either effect may dominate.

Our results require that the equilibrium strategies are increasing in the signals received by the bidders and, implicitly, that the affiliation between bidders' signals is not too strong. Nonetheless, if the value of the object increases rapidly, then even strongly affiliated signals may result in a symmetric, increasing equilibrium.

[6] There is a danger of oversimplification here. Since the second-price effect is present in both the second-price auction and the war of attrition, their relative sizes may matter and there is no *a priori* reason to believe that the war of attrition will be superior. The combination of the second-price and the losing bid effects in the war of attrition could potentially be smaller than the pure second-price effect in the second-price auction. Of course, Theorem 3 shows otherwise.

Weber [16] has constructed the following simple example in which an increasing equilibrium does not exist. Suppose that the density function f is uniform over $[0, 1]^2 \cup [1, 2]^2$. Then the support of $f_{Y_1}(\cdot \mid x)$ is $[0, 1]$ if $x \in [0, 1]$ and is $[1, 2]$ if $x \in [1, 2]$. For this example, the affiliation is rather strong, and in both the war of attrition and the all-pay auction, the symmetric equilibrium strategies are not increasing in the signals. This is because for small $\delta > 0$, a bidder with signal $1 - \delta$ is almost sure to win and bids high, whereas a bidder with signal $1 + \delta$ is almost sure to lose and bids nearly zero.

A comparison of auction forms when symmetric equilibrium strategies are not increasing presents many difficulties, both technical and conceptual. Standard methods of finding equilibrium strategies rely essentially on these being monotonic; when the equilibrium is non-monotonic, it cannot be derived as the solution to a differential equation. Even if non-monotonic equilibrium strategies could be determined, the comparison of auctions solely on the basis of revenue would be problematic. This is because when equilibria are not increasing, it is not necessary that the bidder with the highest signal would win the object. In these circumstances, different auction forms may lead to completely different allocations of the object and thus the *total* surplus (the sum of the seller's expected revenue and the bidders' expected payoff) may also differ across auctions. Thus, while the ranking of auctions in terms of expected revenue may still be useful from the perspective of a seller, the possible inefficiencies resulting from the seller's preferred auction may result in a different ranking from a social perspective.

5.6. *Full Extraction of Surplus*

The auction forms we have considered may, in certain circumstances, extract *all* the surplus from the bidders; that is, the bidders' expected payoff in equilibrium may be zero. This is in marked contrast to the standard auction forms for which the expected payoff is always positive.

PROPOSITION 5. *A bidder's equilibrium payoff in a second-price auction (and hence in a first-price auction) is strictly positive.*

Proof. Recall that the equilibrium payoff of a bidder with a signal $x > \underline{x}$ in a second-price auction may be written as

$$\Pi^{II}(x, x) = \int_{-\infty}^{x} [v(x, y) - v(y, y)] \, f_{Y_1}(y \mid x) \, dy$$

$$> 0,$$

since $v(x, y)$ is assumed to be (strictly) increasing in x.[7] ∎

[7] It is necessary here that the variables X_1 and Y_1 are not perfectly correlated. This follows from our assumption that the joint distribution of these variables admits a density.

To see that the war of attrition may extract all the surplus, consider the following example. Suppose $n=2$ and let X (resp., Y) be the random variable that denotes bidder 1's (resp., 2's) signal. Let:

$$f(x, y) = \tfrac{1}{4}(x+y)^{-1/2} \exp(-(x+y)^{1/2}) \qquad \text{on} \quad (0, \infty) \times (0, \infty).$$

It is routine to verify that f satisfies the affiliation inequality. Then

$$f_{Y_1}(y \mid x) = \tfrac{1}{2}(x+y)^{-1/2} \exp(-(x+y)^{1/2} + x^{1/2})$$

and

$$\lambda(y \mid x) = \tfrac{1}{2}(x+y)^{-1/2}.$$

If $v(x, y) = (x+y)^{1/2}$, then $\varphi(x, y) = \frac{1}{2}$. Even though $\varphi(x, y)$ is not strictly increasing in x, a symmetric monotonic equilibrium of the war of attrition exists and the equilibrium strategy is strikingly simple:

$$\beta(x) = \frac{x}{2}.$$

Using (5), the equilibrium payoff is

$$\begin{aligned} \Pi^W(x, x) &= \int_{-\infty}^{x} [\varphi(x, y) - \varphi(y, y)][1 - F_{Y_1}(y \mid x)]\, dy \\ &= 0. \end{aligned}$$

From the work of Crémer and McLean [6] and McAfee *et al.* [9], we know that it is possible to extract (almost) all the surplus in common value settings. In general, however, the optimal mechanism depends on the distribution of signals and may be rather complicated. We have shown that, in certain circumstances, the war of attrition may actually be the optimal mechanism.

In a similar vein, Bulow and Klemperer [5] have constructed an example in which the all-pay auction extracts all the surplus from the bidders. Our results imply that if the all-pay auction extracts all the surplus, then there cannot be a monotonic equilibrium in the war of attrition; indeed this is the case in Bulow and Klemperer's example.

5.7. *Entry Fees*

Consider a situation where all bidders wishing to participate in a second-price sealed-bid auction are required to pay a fixed (small) entry fee, ε, at the time bids are submitted. Thus, all bidders, including *losing* bidders, pay positive amounts to the seller. This introduces an all-pay aspect to a standard second-price auction, making it appear, at least superficially, like the war of attrition. Since the all-pay aspect is crucial to the revenue ranking results obtained previously, one might wonder whether the beneficial effect of having losers pay their bids is tempered or perhaps even neutralized by the introduction of these fees. It may, however, be shown that all of the revenue ranking results of this section continue to hold in the presence of entry fees.

6. CONCLUSION

The war of attrition and the all-pay auction are rather simple modifications of the second-price and first-price auctions, respectively. We have identified conditions under which these generate higher expected revenue than their standard counterparts by incorporating an additional revenue enhancing linkage, the "losing bid effect."

One might reasonably ask why such auction forms are not observed in practice. The possible reasons are numerous: The presence of risk-averse bidders is known to affect the revenue ranking results of the standard auction forms, and there is no reason to suppose that their all-pay analogues would be immune to this effect. Simultaneous or sequential auctions for goods whose values are related in some way would also affect the revenue ranking results. The assumption of a purely non-cooperative one-shot framework might likewise be suspect if bidders know one another or expect to meet again in the future. Finally, generating higher revenues, by itself, need not imply that an auction form would be adopted since, at the same time, bidders would be less inclined to participate in such auctions. Institutional or cultural constraints as well as competition from other sellers may then preclude a given seller from choosing a war of attrition or an all-pay auction despite its desirable revenue characteristics.

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