

Reductions

How do we show that a problem
is easy?

Reduce each instance to one (or more)
instances of a known easy problem.

$$I_1 \in P_1 \quad R(I_1) = I_2 \in P_2$$

To solve I_1 in P_1 :

1. Apply R to turn I_1 into $I_2 \in P_2$
2. Apply algorithm for P_2 to I_2 .

Cost = cost of applying R plus
cost of applying P_2 algorithm.

Examples

Reduction to some form of matrix multiplication:

Transitive closure

Context-free language recognition

Reduction to linear programming

Reduction to network flow

etc.

Cost of reduction?

linear time, quadratic time, ...

Positive use of reduction:

? \Rightarrow easy

Reduce a problem of unknown complexity
to an easy problem

"Negative" use of reduction:

How do we show that a problem of
unknown complexity is hard?

Reduce a hard problem to it

hard \Rightarrow ?

If the questionable problem were easy, so would be
the hard problem XX

Reductions in both directions show
computational equivalence (up to the
cost of the reduction)

$$P_1 \Leftrightarrow P_2$$

both are easy or both are hard

Transitivity of reductions

$$\begin{array}{ccccc} P_1 & \Rightarrow & P_2 & \Rightarrow & P_3 \\ & & R_1 & & R_2 \end{array}$$

Gives a reduction from P_1 to P_3

Cost is cost of $R_2(R_1())$

both p-time, overall p-time
linear linear

$$\begin{array}{cc}
 R_1 & R_2 \\
 n & a_n \quad b_n
 \end{array}$$

$$\begin{array}{cc}
 a_n & (ab)_n
 \end{array}$$

$$\begin{array}{cc}
 n & a_n^2 \quad b_n^2
 \end{array}$$

$$\begin{array}{cc}
 a_n^2 & b(a_n^2)^2 = ba^2 n^4
 \end{array}$$

$$\begin{array}{cc}
 a_n^k & b n^l
 \end{array}$$

$$(b a n^k)^l = b a^l n^{\underline{\underline{kl}}}$$

Satisfiability: Is a Boolean (logical) function true for some choice of variable assignments?

$$(x \vee y) \wedge (\bar{x} \vee \bar{y}) \quad \text{sat: } x=1, y=0$$

\wedge and

\vee or

$-$ not

x variable

x, \bar{x} literal

$(x \vee \bar{y} \vee z)$ clause: disjunction ("or") of literals

$(x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$ conjunctive normal form:
conjunct ("and") of clauses

$x \vee \bar{x}$

tautology: true for all choices of variables

F is sat iff \bar{F} is not a tautology:
can be falsified

Reduction of CNF sat to 3-CNF sat

(at most 3 literals/clause)

$$(x \vee y \vee z \vee w \vee u \vee v) \Rightarrow$$

$$(x \vee y \vee a) \wedge (\bar{a} \vee z \vee b) \wedge (\bar{b} \vee w \vee c)$$

$$\wedge (\bar{c} \vee u \vee v)$$

Needs $k-3$ extra vars per clause of length k .

Graph coloring reducible to sat,

and vice-versa (p-time reductions)

Must phrase graph coloring as a yes-no question: can graph G be colored with k colors?

G : n vertices, m edges

F : nk variables x_{ij} , one per vertex per color

x_{ij} true iff vertex i colored color j

Classes:

Each vertex colored

$$(x_{i1} \vee x_{i2} \vee \dots \vee x_{ik}) \quad i \in V \quad \dots \quad n$$

[No vertex colored twice

$$(\bar{x}_{ij} \vee \bar{x}_{il}) \quad i \in V, \quad j \neq l \text{ colors} \quad \dots \quad n \binom{k}{2}]$$

No adjacent vertices the same color

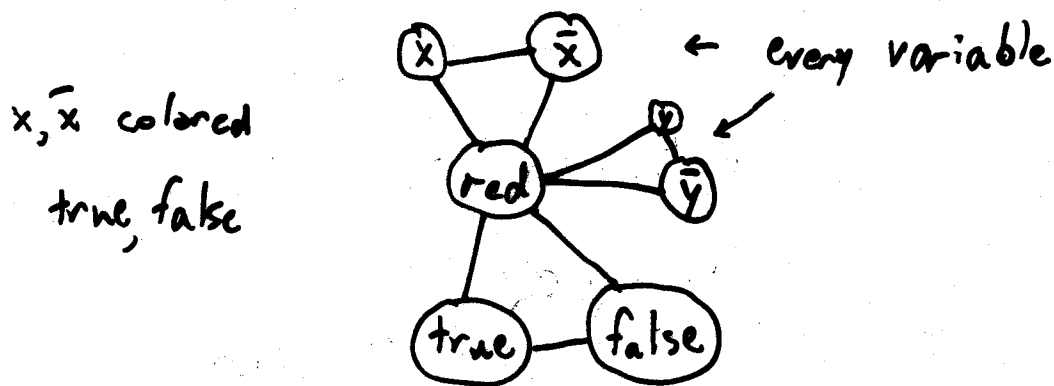
$$(\bar{x}_{il} \vee \bar{x}_{jl}) \quad (i,j) \in E, \quad l \text{ a color} \quad \dots \quad mk$$

$$\# \text{ literals} = nk + 2n \binom{k}{2} + 2mk$$

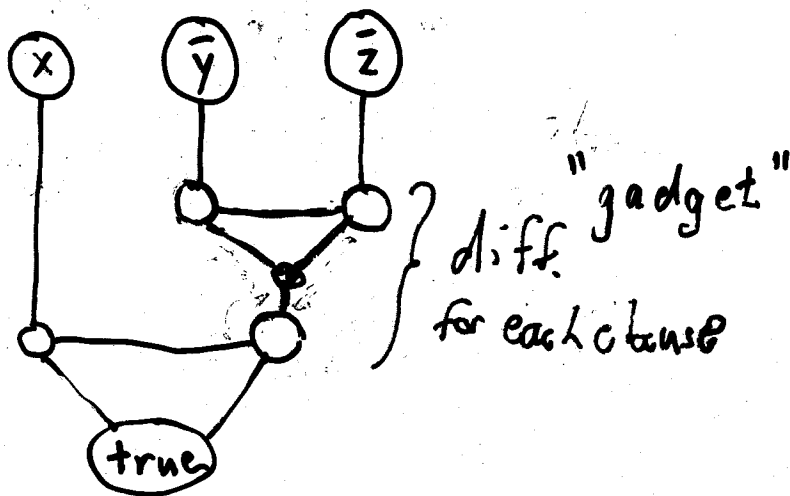
Vice-versa: (3-sat)

Reduction to 3-coloring

Vertices: $x, \bar{x}, \text{true}, \text{false}, \text{red}$, 5 per clause



Clause $x \vee \bar{y} \vee \bar{z}$



Colorable iff formula satisfiable

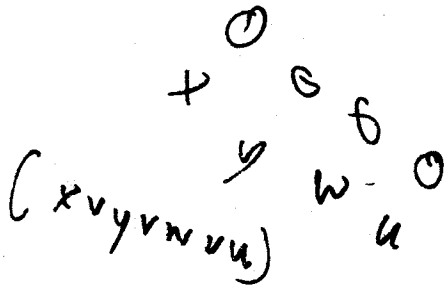
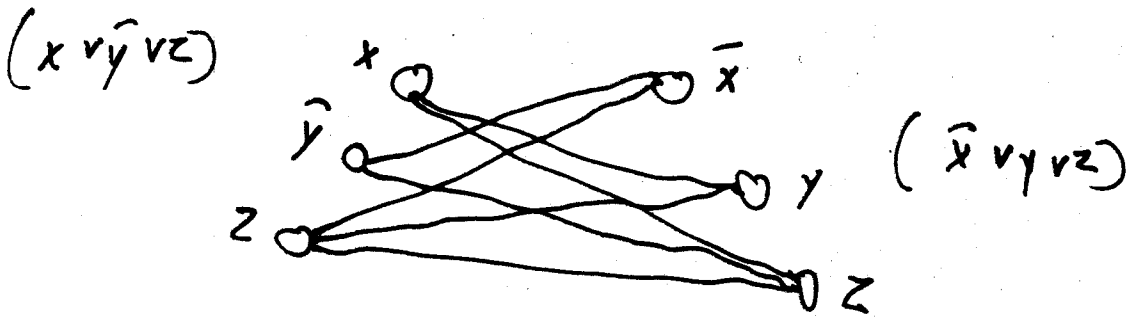
Sat \Rightarrow Clique

Given a graph, are there k pairwise adjacent vertices?

One vertex per literal occurrence,

two vertices in different clauses joined by an edge if compatible (not x, \bar{x})

$k = \#$ clauses



Clique \Rightarrow Sat

$$x_{ij} \quad i \in V, 1 \leq j \leq k$$

x_{ij} true = vertex i is j 'th in clique

$i, b \in V$ not adj.

$$(\bar{x}_{ij} \vee \bar{x}_{kj'}) \quad \forall j, j'$$

$$(\bar{x}_{ij} \vee \bar{x}_{i'j}) \quad i \neq i' \quad \forall j$$

$$(x_{1j} \vee x_{2j} \vee \dots \vee x_{nj}) \quad \forall j$$

Clique \Leftrightarrow Independent set

Are there k pairwise nonadjacent vertices?

Complement graph

Clique \Leftrightarrow Vertex cover

Are there k vertices "covering" all edges

S a vertex cover in G iff

$V-S$ is an independent set in G iff

$V-S$ is a clique in \bar{G}