The Minimum Spanning Tree Problem

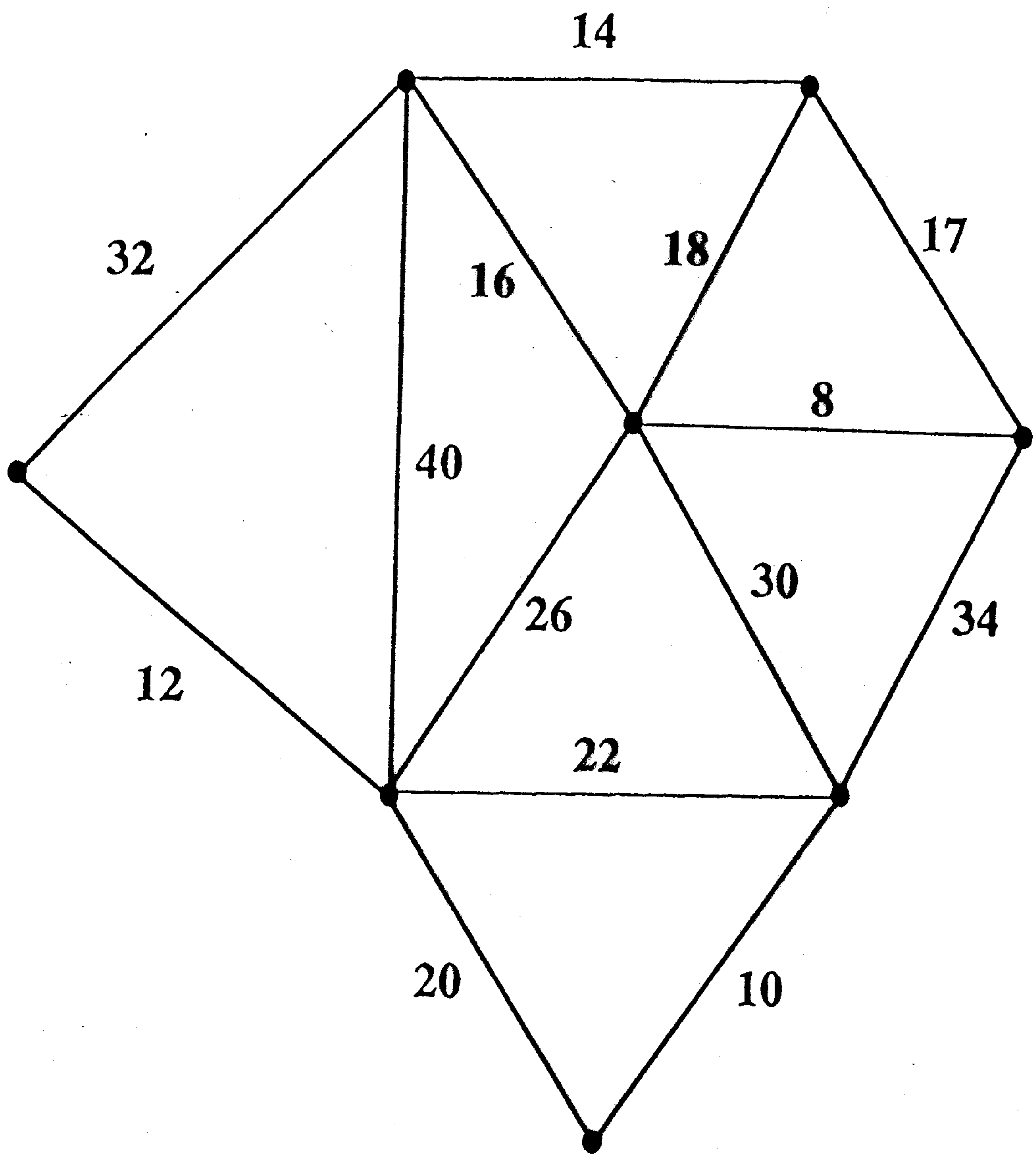
Given a connected graph, find a spanning tree of minimum total edge cost.

where,

n = the number of vertices

m = the number of edges

$$n-1 \leq m \leq \binom{n}{2}$$



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Applications

Network Construction

Clustering

Minimum Tour Relaxation (Held-Karp 1-trees)

A Simple Solution From the 80's

(with apologies to Oliver Stone)

Gorden Gecko: "Greed is Good"

Repeatedly select the cheapest unselected edge and add it to the tree under construction if it connects two previously disconnected pieces.

Kruskal, 1956

The greedy method generalizes to matroids.

We shall generalize the method rather than the domain of application.

Generalized Greedy Method

Beginning with all edges uncolored,

sequentially color edges

blue (accepted) or red (rejected).

Blue Rule:

> **Color blue any minimum-cost uncolored edge crossing a cut with no blue edges crossing.**

Red Rule:

> **Color red any maximum-cost uncolored edge on a cycle with no red edges.**

∃ MST with all blue, no red

Jarnik's Algorithm

Grow a tree from a single start vertex.

At each step add a cheapest edge with exactly one end in the tree.

Boruvka's Algorithm

Repeat the following step until all vertices are connected:

> **For each blue component, select a cheapest edge connecting to another component; color all selected edges blue.**

For correctness, a tie-breaking rule is needed.

Henceforth, assume all edge costs are distinct.

Then there is a unique spanning tree.

"Classical" Algorithms

(before algorithm analysis)

Kruskal's algorithm, 1956

$O(m \log n)$ **time**

Jarnik's algorithm, 1930

$O(n^2)$ **time**

also Prim, Dijkstra

Boruvka's algorithm, 1926

$O(\min\{m \log n, n^2\})$ **time**

and many others

The greedy algorithm runs to completion:

Let e be an uncolored edge.

If e's ends are in different blue trees, apply blue rule.

If e's ends are in the same blue tree, apply red rule.

Correctness of blue rule:

e = min uncolored edge across cut.

T = min tree containing all blue edges, no red ones.

If e not in T: find path in T connecting ends of e, edge e' on path crossing cut.

Swap e and e' to give T'

$c(e) \le c(e')$ by blue rule,

$c(e) \ge c(e')$ by minimality of T

T' satisfies invariant after e is blue

(swapping can only occur if equal costs.)

Correctness of red rule:

e = max uncolored edge on cycle.

T = min tree containing all blue edges, no red ones

If e in T, delete e from T, find edge e' on cycle (other than e) reconnecting two parts, form T' by swapping e' for e in T.

$c(e) \geq c(e')$ by red rule

$c(e) \leq c(e')$ by minimality of T

Selected History

Boruvka, 1926	$O(\min\{m \log n, n^2\})$
Jarnik, 1930 **Prim, 1957** **Dijkstra, 1959**	$O(n^2)$
Kruskal, 1956	$O(m \log n)$
Williams, Floyd, 1964 **heaps**	$O(m \log n)$
Yao, 1975 **packets in Boruvka's algorithm**	$O(m \log\log n)$
Fredman, Tarjan, 1984 **F-heaps in:**	
Jarnik's algorithm	$O(n \log n + m)$
a hybrid Jarnik-Boruvka algorithm	$O(m \log^* n)$
Gabow, Galil, Spencer, 1984 **Packets in F-T algorithm**	$O(m \log\log^* n)$

$$\log^* n = \min\{i \mid \log\log\log \ldots \log n \leq 1\}$$

where the logarithm is iterated i times

Models of Computation

We assume comparison of the two edge costs takes unit time, and no other manipulation of edge costs is allowed.

Another model:

bit manipulation of the binary representations of edge costs is allowed.

In this model,

Fredman-Willard, 1990, achieved O(m) time.

(fast small heaps by bit manipulation)