Maximum Flow Problem

Network $G = (V, E)$, source s, sink t

edge capacities $u(v,w)$ for $(v,w) \in E$

$|V| = n \qquad |E| = m \qquad U = \max |u(v,w)|$

Assume network is symmetric:

$(v,w) \in E$ iff $(w,v) \in E$

Flow $f : E \to \mathbb{R}$

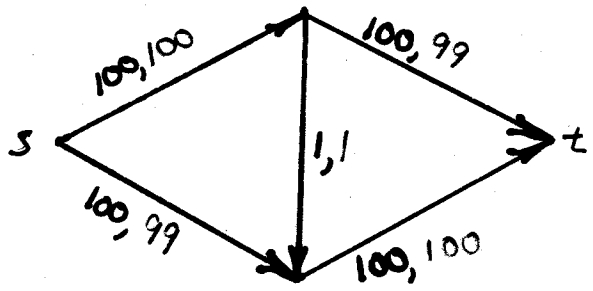
$f(v,w) \leq u(v,w)$

$f(v,w) = - f(w,v)$

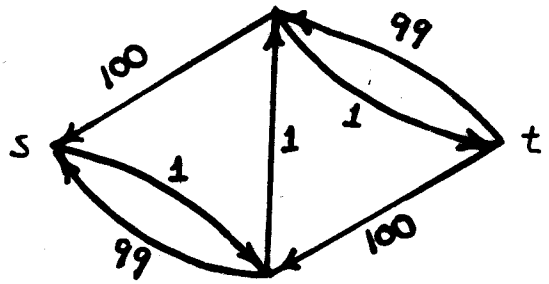
$e(w) = \sum_{v}^{1} f(v,w) = 0 \qquad \forall w \notin \{s,t\}$

Objective: maximize $e(t)$ $(= - e(s))$

Network



Residual Network

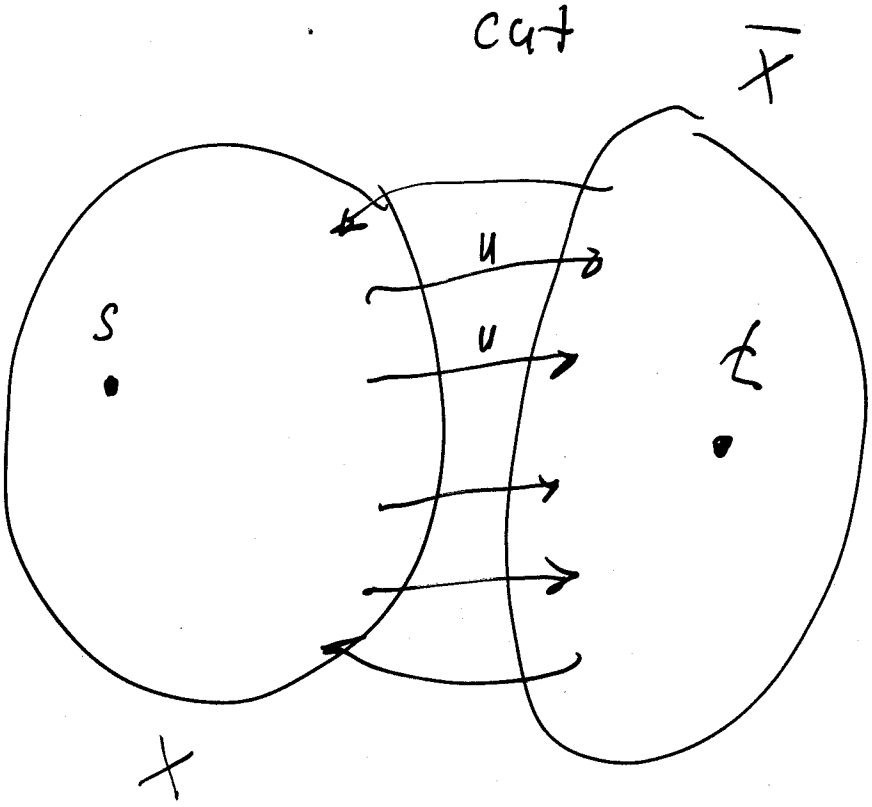


Ford-Fulkerson method:

repeat { find an augmenting path
augment flow

Time: $O(nmU)$ (not polynomial, need not terminate if capacities are irrational)

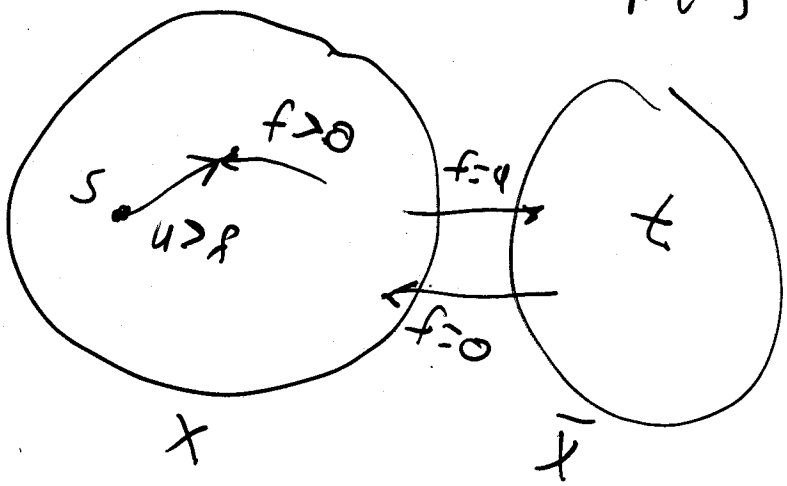
cut



$cap(X, \bar{X}) = \sum u$ across cut

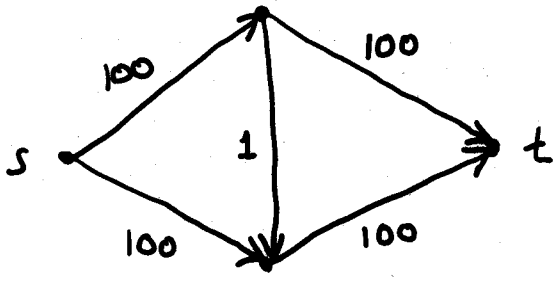
For any flow $f(X, \bar{X}) = \sum f$ from X to $\bar{X}$
$- \sum f$ from $\bar{X}$ to X

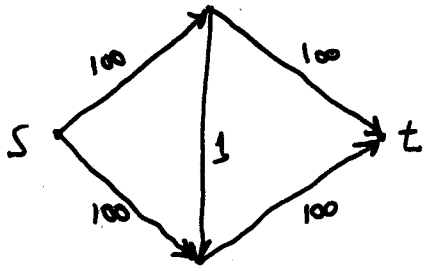
flow value $= f(X, \bar{X}) \leq cap(X, \bar{X})$

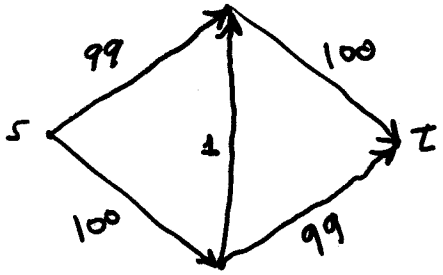


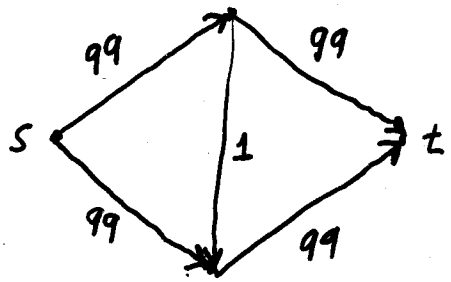
max flow = min cut

(Bad) Example









etcetera

Edmonds & Karp: always augment along a shortest (fewest edges) path:

$O(m)$ time per path $\times$ $O(m)$ paths per length

$\times O(n)$ path lengths $= O(nm^2)$ time

Dinic: find all augmenting paths of a given length at once, in a phase:

$O(n)$ time per path $\times O(nm)$ paths

$+ O(m)$ time per phase $\times O(n)$ phases $=$

$O(n^2m)$ time

Classical Algorithms

Date	Discoverer	Time
1956	Ford & Fulkerson	$O(nmU)$
1969	Edmonds & Karp	$O(nm^2)$
1970	Dinic	$O(n^2m)$
1974	Karzanov (same bound by several others later)	$O(n^3)^*$
1977	Cherkasky	$O(n^2m^{1/2})^*$
1978	Galil	$O(n^{5/3}m^{2/3})^*$
1978	Galil & Naamad; Shiloach	$O(nm(\log n)^2)$
1980	Sleator & Tarjan	$O(nm\log n)$
1983	Gabow	$O(nm\log U)$

*Forerunners of preflow push method

Preflow Push Approach (Goldberg)

Two ideas:

Make the basic steps in the computation smaller

(relax the flow conservation requirement)

Use a less global, more distributed approach to

do the preprocessing associated with each

phase

Main effect: simpler algorithm?

Our Approach

Preflow (Karzanov): like a flow except that the total flow into a vertex can exceed the total flow out.

A vertex $\neq t$ with extra incoming flow is <u>active</u>. The net incoming flow $e(v)$ is the <u>excess</u> of vertex v.

Idea: move flow excess toward sink along estimated shortest paths. Move excess that cannot reach the sink back to the source, also along estimated shortest paths.

To estimate path lengths: a <u>valid labeling</u> is an integer function d on vertices such that:

(i) $d(t) = 0$
(ii) $d(s) = n$
(iii) $d(v) \leq d(w) + 1$ if $u_f(v,w) > 0$

$d(v)$ is a lower bound on the minimum of distance to t, n + distance to s

Algorithm

1. Saturate all edges leaving s. Choose initial d.
2. Repeat push and relabel steps in any order until no vertex is active.

push (v,w):

if v is active, $u_f(v,w) > 0$, and $d(v) = d(w) + 1$ then move $\min\{e(v), u_f(v,w)\}$ units of flow from v to w (the push is <u>saturating</u> if $u_f(v,w)$ units are moved)

relabel (v):

if v is active and for all (v,w), $u_f(v,w) = 0$ or $d(v) \leq d(w)$ then let $d(v) = \min\{d(w)+1 \mid u_f(v,w) > 0\}$

Bounds

Every active vertex has a label of at most $2n-1$:

there is always a residual path to s.

$\Rightarrow O(n^2)$ relabelings, taking $O(nm)$ time.

Between saturating pushes through the same edge, ends of edge must be relabeled

$\Rightarrow O(nm)$ saturating pushes.

The heart of the analysis is in bounding the number of nonsaturating pushes.

Generic Bound : $O(n^2 m)$

Pf. Define $\Phi = \sum_{v \text{ active}} d(v)$.

$0 \leq \Phi \leq 2n^2$. A nonsaturating push decreases Φ by one.

Increases to Φ: $O(n^2)$ in total due to relabelings.

$O(n^2 m)$ due to saturating pushes:

$O(n)$ per saturating push.

$\Rightarrow O(n^2 m)$ nonsaturating pushes.

FIFO Method

Maintain a queue of active vertices.

Always push from the vertex on the front of the queue.

Add newly active vertices to the rear of the queue.

Analysis

Phases: phase 1 = processing of vertices originally on queue.

phase i+1 = processing of vertices added to queue during phase i.

Only one nonsaturating push per vertex per phase:

such a push reduces the excess to zero and removes the vertex from the queue.

$O(n^2)$ bound on # phases

Define $\Phi = \max_{v \text{ active}} d(v)$. $\quad 0 \le \Phi \le 2n$.

A phase reduces Φ by one unless a relabeling occurs.

All increase in Φ is due to relabelings, totals $O(n^2)$.

The number of phases in which Φ doesn't change is also $O(n^2)$.

$\Rightarrow$ $O(n^2)$ total phases.

$\Rightarrow$ $O(n^3)$ nonsaturating pushes.

Ahuja-Orlin Excess Scaling

Maintain Δ, an upper bound on max excess

Maintain integrality of flow.

After each phase, replace Δ by $\Delta/2$.

Stop when $\Delta < 1$.

Push from a vertex v of smallest $d(v)$ with

$e(v) > \Delta/2$.

When pushing from v to $w \neq t$, move

$\min \{e(v), u_f(v,w), \Delta - e(w)\}$

Analysis

Each nonsaturating push moves at least

$\Delta/2$ units of flow.

Let $\Phi = \sum_{v \text{ active}} e(v)\, d(v)/\Delta$

$0 \leq \Phi \leq 2n^2$

Each nonsaturating push decreases Φ by $\geq 1/2$.

Increases in Φ: $O(n^2)$ associated with relabeling.

$O(n^2)$ per phase from change in Δ.

$O(\log U)$ phases $\Rightarrow$

$O(n^2 \log U)$ nonsaturating pushes

saturating pushes = $O(nm)$

nonsaturating pushes = $O(n^2 \log U)$

Can these estimates be balanced?

Yes: change algorithm: make <u>all</u> pushes large enough by retaining enough excess to immediately saturate very-small-capacity edges.

pushes = $O(n^{3/2} m^{1/2} (\log U)^{1/2})$

Cheriyan - Mehlhorn

What about relabeling time??

Max flow

"Best" known bound

$$O\left(\min\{n^{2/3}, m^{1/2}\}\, m \log(n^2/m) \log U\right)$$

Goldberg + Rao, 1997

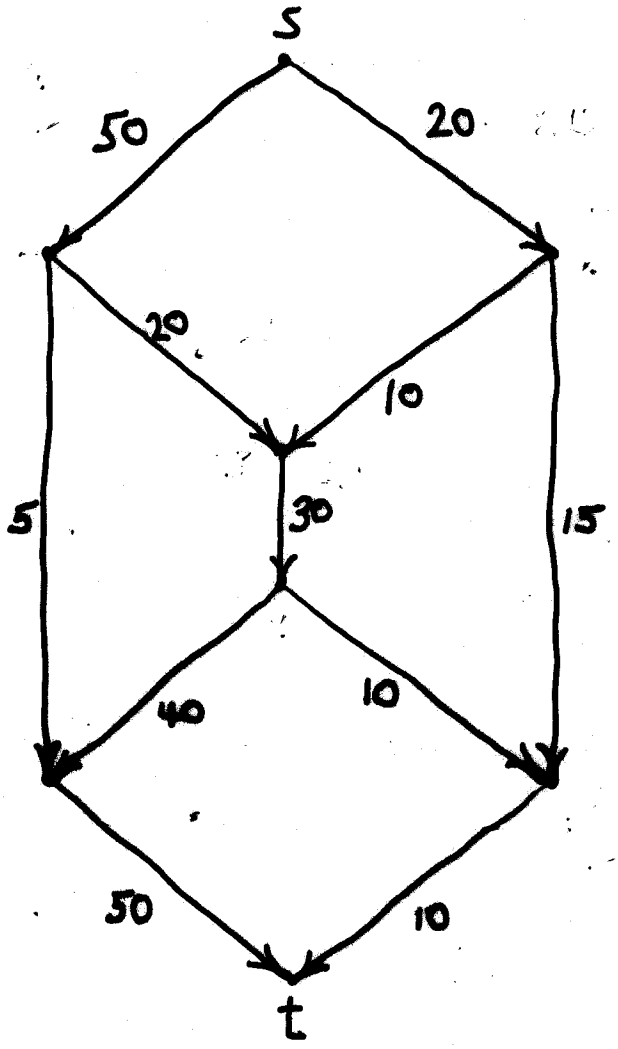
Practice

Appropriate versions of the preflow push method are easy to implement and very fast in practice: 4-14 times faster than Dinic on reasonable classes of graphs.

Important heuristic: periodically compute tight distance labels using breadth-first search. (Otherwise the relabeling time is too high.)

The FIFO algorithm can be parallelized: push from all active vertices at once. It seems to give drastic speedups in practice.

Whether dynamic trees help on very large graphs has not yet been studied.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