Algorithms, Complexity, and Combinatorics

<u>Objects of this area of study:</u>

<u>Develop</u> good algorithms

<u>Analyze</u> the complexity of algorithms

Provide <u>lower bounds</u> on the complexity of problems

(we will deal only with <u>sequential</u>, not <u>parallel</u>, algorithms)

<u>Historical approach</u>:

Algorithm development

Empirical study

Theoretical analysis rare

Lower bounds non-existent

Algorithm: Step-by-step
problem solving method

Although "algorithms" existed thousands of years ago, the birth of the computer was necessary and sufficient to power their study.

Techniques are borrowed from

logic:

simulation, diagonalization

This work occurred just before the advent of computers (1930's).

Hilbert, at the turn of the century, was aware of the issue:

Hilbert's 10^{th} problem, proved undecidable in 1970 by Matijasevic, building on work of Davis, Robinson

The next questions:

How efficient is an algorithm?

How efficient can algorithms for a given problem be?

Complexity Theory

Complexity measures:

program length } function only of the problem

(sequential) running time
storage space

parallel running time
number of processors

} function of the input

Possible Complexity measures

__Static__ (data independent)

1. Program size (number of instructions)

__Dynamic__ (data dependent)

1. Running time as a function of data size.
2. Storage space as a function of data size.

__Data for dynamic measures__

1. Worst case.
2. Representative (average) case.

__Special measures for lower bounds__

1. Tests in decision tree.
2. Arithmetic operations in straight-line program.
3. Memory accesses.

Program length and
programming time:
a digression

Programming, from at least one point of view (Dijkstra's) is a rigorous, logical activity

akin to theorem-proving and equally demanding of correctness.

Our Complexity Measure

Worst-case running time

as a function of input size.

Constant-time operations:

Accessing a single cell or node.

Performing a single arithmetic or logical operation.

Asymptotic analysis:

We ignore constant factors and concentrate on large problem sizes.

N

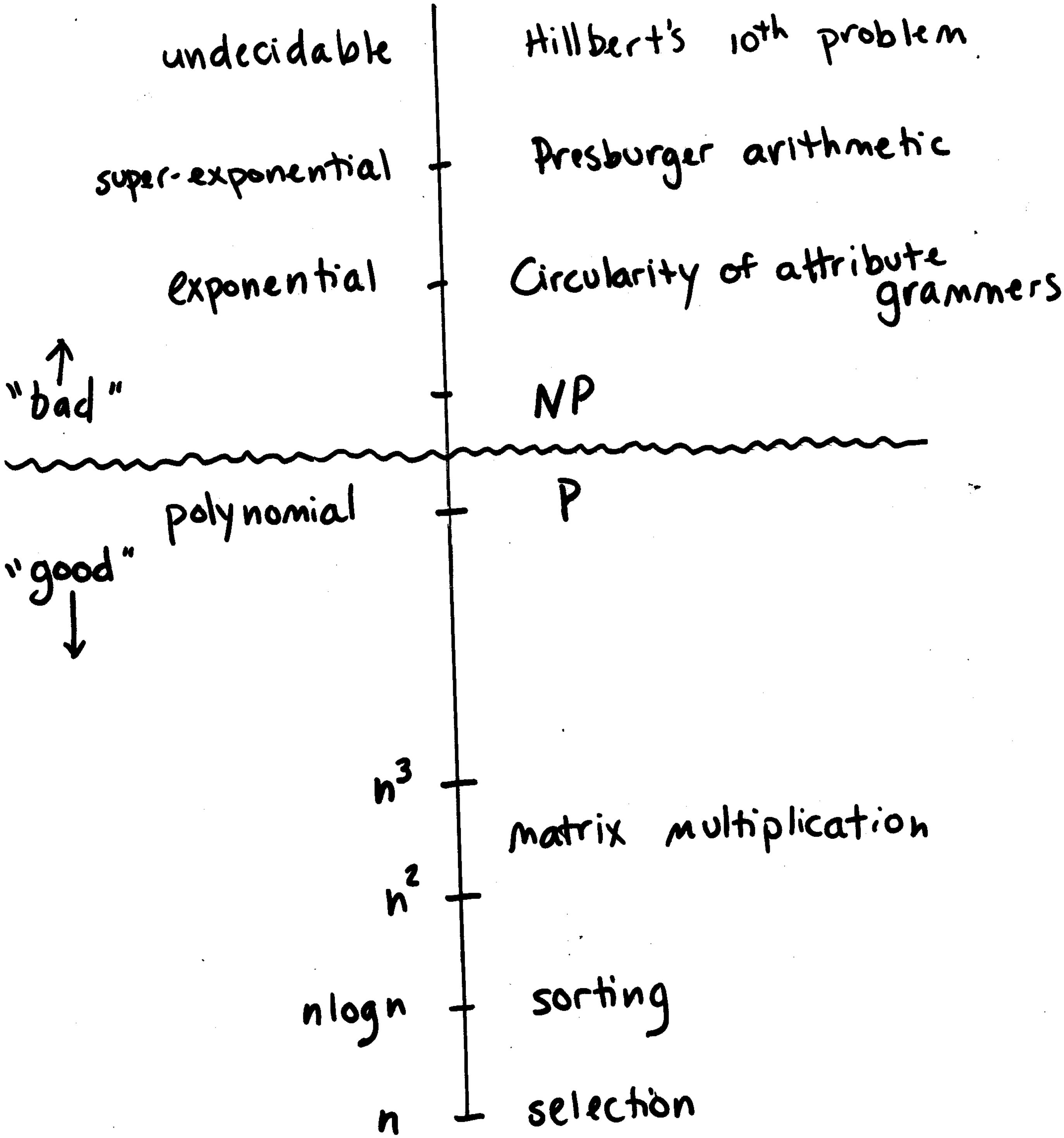
size / omplexity	20	50	100	200	500	1000
1000 N	.02 sec	.05 sec	.1 sec	.2 sec	.5 sec	1 sec
000 N log N	.09 sec	.3 sec	.6 sec	1.5 sec	4.5 sec	10 sec
$100\ N^2$	.04 sec	.25 sec	1 sec	4 sec	25 sec	2 min
$10\ N^3$	.02 sec	1 sec	10 sec	1 min	21 min	2.7 hr
$N^{\log N}$	.4 sec	1.1 hr	220 days	125 cent	$5 \cdot 10^8$ cent	
$2^{N/3}$	.0001 sec	.1 sec	2.7 hr	$3 \cdot 10^4$ cent		
2^N	1 sec	35 yr	$3 \cdot 10^4$ cent			
3^N	58 min	$2 \cdot 10^9$ cent				

Running time estimates:

one step = one microsecond

logarithms are base two

The spectrum of Computational Complexity



High-Level Complexity vs. Low-Level Complexity

High-Level Complexity		Low-Level Complexity
Ignorance of e.g. polynomial functions	•	Ignorance of Constant Factr (maybe)
Emphasis on Lower bounds	•	Emphasis on upper bounds
Techniques are those of logic	•	Techniques are eclectic

simulation
diagonalization
quantifier eliminization
(to get algorithms)

High order complexity

(What can't we do?)

Undecidability Turing's halting problem

↓

Hilbert's tenth problem
(solution of Diophantine equations)

Good (polynomial time) vs **bad** (exponential time) algorithms

Exponential or **super-exponential** lower bounds

Equivalence of extended regular expressions

2^{2^n} Validity in Presburger arithmetic

Circularity in semantic definitions for context-free languages

NP-complete problems

P is the class of problems solvable in polynomial time.

NP is the class of problems whose solution can be checked in polynomial time.

NP-complete problems

Hardest problems in NP; if <u>any</u> has a polynomial time algorithm, they <u>all</u> do.

Examples

Validity in propositional calculus

Travelling salesman problem

Maximum independent set problem

Graph coloring

High order complexity results are machine independent;

complexity in all machine models is polynomially related.

Turing machine is usually used.

Key ideas

simulation (reducibility, transformability)

diagonalization

These are not powerful enough to resolve the

$P = NP?$

question.

Low order complexity

(What <u>can</u> we do?)

Instead of lower bounds, emphasis is on developing fast algorithms.

(Lower bounds almost non-existent)

Better-and-better polynomial upper bounds

Low-Level Complexity

Lower bounds are based on problem-specific computation models

count only the relevant or dominant operations

eg. comparisons (sorting)

multiplications (matrix multiplication)

model "natural" algorithms

Fast algorithms rely on

algorithmic techniques:
recursion, dynamic
programming, divide-
and-conquer, graph
search, etc.

data structures: lists,
stacks, queues,
trees, etc.

Analysis of algorithms and
related questions requires
eclectic mathematics

Techniques

Recursion
- Dynamic programming
- Divide and Conquer

Data structures
- Linear lists
  - stack, queue, deque
  - list of lists (radix sorting)
  - partitioned stack (planarity testing)
- Trees
  - compressed trees
  - heaps (priority queues) } basic
  - search trees
  - dynamic trees } advanced
  - permutable trees

Graph search

Depth-first (maze traversal (Tremaux): connectivity problems)

Breadth-first (network flow)

Shortest-first (shortest paths)

Oldest-last (maze traversal (Tarry): Eulerian cycles)

Maximum cardinality

Lexicographic

Optimization

Greed

Augmentation

The Role of Theory in Algorithm Design

Whereas improvements in hardware and in coding can produce constant factor improvements, theoretical insights can lead to asymptotic improvements and gains in simplicity and generality (and correctness)

Key Points

As computers become faster and as computer memories become larger, theoretical analysis yielding asymptotic complexity becomes more, not less important.

More "room" is available for the efficiency of clever algorithms to show up.

Often, the key to solving a problem efficiently is to use the right data structure.

Algorithmic questions are often at heart data structure questions.