amortize: to put money aside at intervals for gradual payment (of a debt, etc.)

(Webster's)

idea: to average over time.

Motivation for Amortization

In many uses of data structures a <u>sequence</u> of operations (rather than just one) is performed.

We are interested in the <u>total</u> time of the sequence.

Worst-case time per operation may be unduly pessimistic, because of correlated effects of operations on data structure.

Average-case time may be inaccurate since the probabilistic assumptions needed to carry out the analysis may be false.

→ Amortized time (time per operation averaged over a worst-case sequence) is both realistic and robust.

An example: stack manipulation

Unit-time primitives:

push (an item onto the stack)

pop (an item off of the stack)

Operation:

carry out zero or more pops,

followed by a push.

Beginning with an empty stack,

carry out a sequence of n operations

Question: How many pushes and pops total?

Answer: $2n$

Each operation causes one push (immediately) and possibly one pop (later)

(After i operations, there have been $2i-k$ pushes and pops, where k is the stack size)

Applications:

Linear-time string-matching (Knuth, Morris, Pratt)

Planarity testing (Hopcroft, Tarjan)

etc.

How can we formalize this phenomenon

and exploit it in

the design and analysis of algorithms?

A Banker's View of Amortization

Credits: One will pay for a unit-time computation.

Credits can sit in the data structure, representing time saved.

Debits: Each represents an excess unit of time spent.

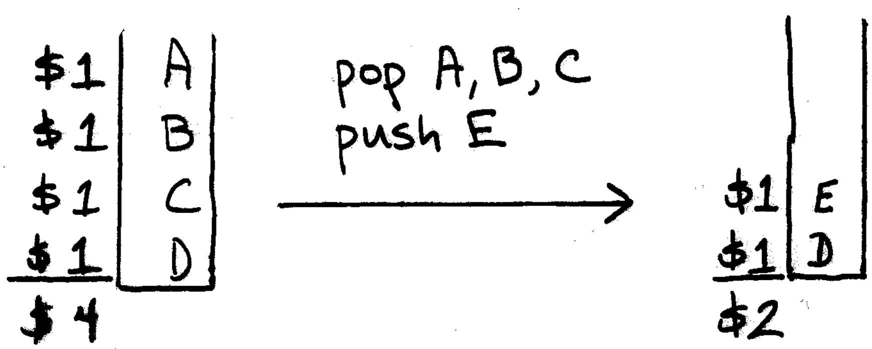
Can also sit in data structure, must be accounted for at end of computation.

A banker's analysis of the stack

Each operation gets two credits.

Number of saved credits equals stack size.

⇒ Each pop paid for by a saved credit.



$$\$4 \quad + \quad \$2 \quad - \quad \$2 \quad = \quad \$4$$

pays for three pops, one push

A Physicist's View of Amortization

Each configuration of data structure has a <u>potential</u> Φ.

a_i (amortized time of operation i) $\equiv$

t_i (actual time of operation i)

$+\ \Phi_i$ (potential *after* operation)

$-\ \Phi_{i-1}$ (potential *before* operation)

$$a_i \equiv t_i + \Phi_i - \Phi_{i-1}$$

$$\sum_{i=1}^{m} t_i = \sum_{i=1}^{m} (a_i + \Phi_{i-1} - \Phi_i)$$

$$= \sum_{i=1}^{m} a_i + \Phi_0 - \Phi_m$$

$$\le \sum_{i=1}^{m} a_i \quad \text{if} \quad \Phi_0 = 0 \text{ and } \Phi_m \ge 0.$$

Definition of potential is arbitrary,

but only a good choice gives

useful results.

A Physicist's analysis of the stack

Potential of stack equals stack size.

$\Rightarrow$ Amortized time of an operation with k pops

$=$ $k+1$ (actual time)

$+$ $s-k+1$ (new potential)

$-$ s (old potential)

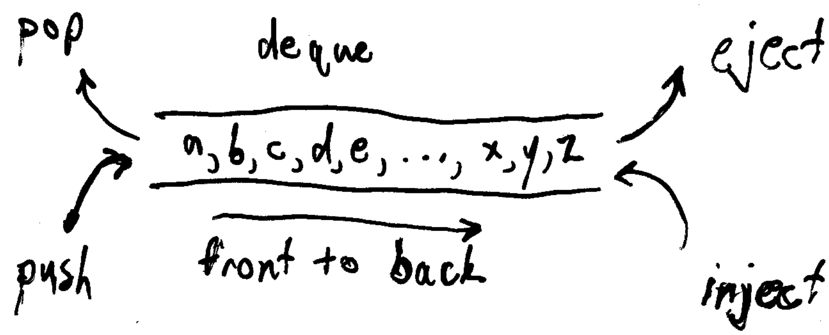
$=$ 2.

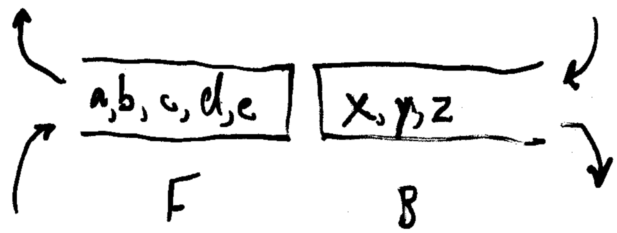
Uses of Amortization

As an analytical tool: obtain new bounds for known algorithms: self-organizing sequential search, disjoint set union, etc.

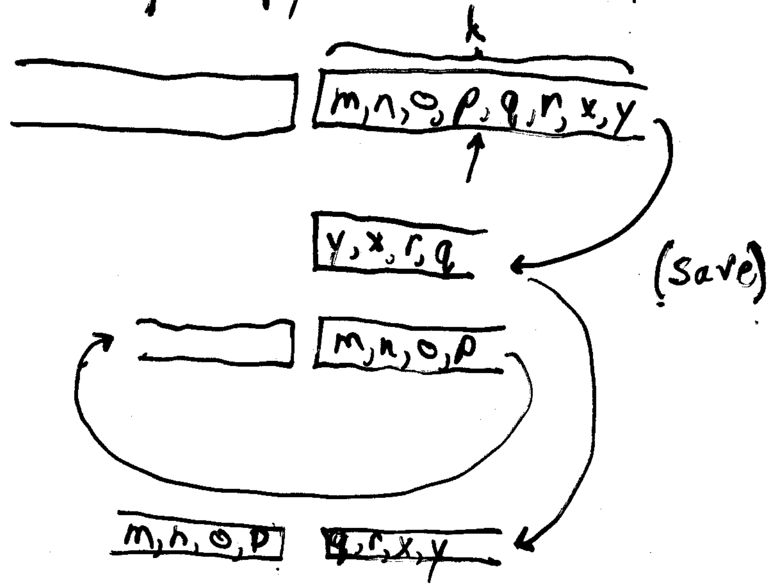
As a design tool: obtain new "self-adjusting" data structures that are simple and have good amortized performance.

Simulation of a deque by 3 stacks (one is scratch)





When F empty, copy half of B into F:



Cost $= \frac{3k}{2}$

Similar when B is empty

Amortized Analysis

$$\Phi = \frac{3}{2}\left| F - B \right|$$

$= 0$ initially

≥ 0 always

"Normal" push/pop/eject/inject

amortized time $\leq$ 1 actual time

$+ \frac{3}{2} \Delta\Phi$

$= 2\frac{1}{2}$

"Abnormal" pop / eject

amortized time $\leq 1 + \frac{3k}{2}$ actual time

$- \frac{3k}{2} + \frac{3}{2} \Delta\Phi$

$|F| = |B| \pm 1$

$= 2\frac{1}{2}$

Competitiveness

On-line vs. off-line algorithms:

How much does knowing the future help?

The skier's dilemma:

Renting skis costs \$$x$ per ski trip

Buying skis costs \$$y$

When to buy?

Goal: minimize the cost ratio as compared to the best policy when the number of trips is known.

Solution: Buy when total rent equals cost of buying

performance ratio (competitive factor) = 2

An on-line algorithm is k-competitive if its performance is within a factor of k of that of the optimum off-line algorithm on any sequence of operations.

Self-organizing linear lists

Data structure: n items stored in a linear list.

Object: perform m access operations.

Cost of accessing i^{th} item = i.

Update primitive: Swap any two <u>adjacent</u> items, at a cost of 1. Can be performed at any time.

$a, b, c, d, e, f, \ldots$

access e

$e, a, b, c, d, f, \ldots$

Move-to-front heuristic (MTF): Move each accessed item to front of list (via $i-1$ swaps)

Total cost to access item $i = 2i-1$.

Single exchange heuristic (SE): Swap accessed item with its predecessor.

Frequency count heuristic (FC): Keep items in decreasing order by access frequency.

Most previous results are average-case:

Fixed access probabilities $P_1, P_2, \ldots, P_n$,

each access is independent.

Optimum algorithm: static list with item

arranged in decreasing access

probability.

Classic result: Average asymptotic access cost
of FC is optimum,
of MTF is within a factor of 2 of

optimum (not counting update cost).

Rivest: SE asymptotically better than

MTF on average.

Various results about different heuristics,

rate of convergence, etc.

Bentley, McGeough: Amortized cost of MTF within a factor of 2 of any static-list algorithm.

Sleator-Tarjan: Amortized cost of MTF within a factor of 2 of any algorithm.

(both results do not count update cost of MTF, increases constant factor to 4).

Experiments (Bentley, McGeough) show that MTF is sometimes <u>better</u> than FC on realistic data.

Potential = 2 × # inversions in MTF list vs. adversary list (A)

$$\le \binom{n}{2} = \frac{n(n-1)}{2}$$

A: . . . i . . . j . . .

MTF: . . . . j . . i . . . .

Access i :

A: 1 2 3 . . . i . . .

MTF: $\underbrace{\ldots\ldots\ldots}_{k}$ i . .

At least $k-i$ items $> i$, $\Phi \downarrow 1$ for each

At most $i-1$ items $< i$, $\Phi \uparrow 1$ for each

Actual cost $= 2k-1$

$$\Delta\Phi \le 2(i-1) - 2(k-i)$$

$$\text{Am. cost (MTF)} = 2k-1 + \Delta\Phi \le 4i-3$$

$$\le 4 \text{ Act. cost (A)}$$

Different cost model:

arbitrary exchanges cost 1

Then off-line can beat on-line by a factor of n (always access last in on-line's list)

Other settings for competitive analysis:

caching, paging, etc.