4.4 Balanced Trees

Reference: Chapter 13, Algorithms in Java, 3rd Edition, Robert Sedgewick.

Symbol Table Review

Symbol table: key-value pair abstraction.
- Insert a value with specified key.
- Search for value given key.
- Delete value with given key.

Randomized BST.
- O(log N) time per op. [unless you get ridiculously unlucky]
- Store subtree count in each node.
- Generate random numbers for each insert/delete op.

This lecture. 2-3-4 trees, red-black trees, B-trees.

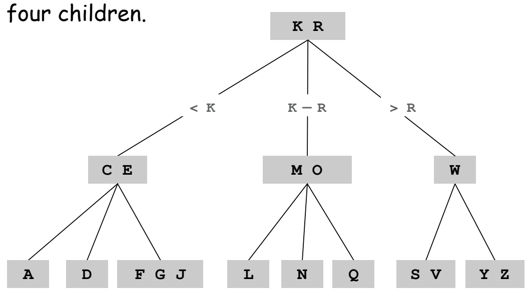
2-3-4 Trees

2-3-4 Tree

2-3-4 tree. Generalize node to allow multiple keys; keep tree balanced.

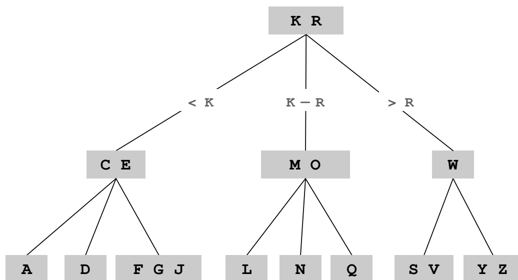
Perfect balance. Every path from root to leaf has same length.

Allow 1, 2, or 3 keys per node.
- 2-node: one key, two children.
- 3-node: two keys, three children.
- 4-node: three keys, four children.



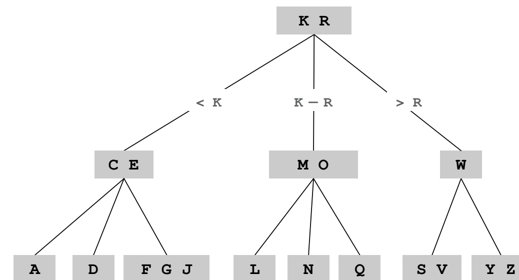
2-3-4 Tree: Search

Search.
- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).



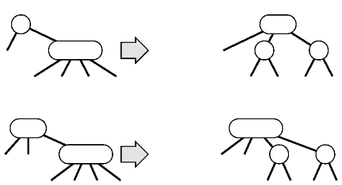
2-3-4 Tree: Insert

Insert.
- Search to bottom for key.
- 2-node at bottom: convert to 3-node.
- 3-node at bottom: convert to 4-node.
- 4-node at bottom: ??



2-3-4 Tree: Splitting Four Nodes

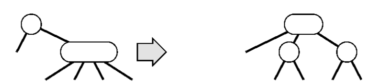
Transform tree on the way down.
- Ensures last node is not a 4-node.
- Local transformation to split 4-nodes:

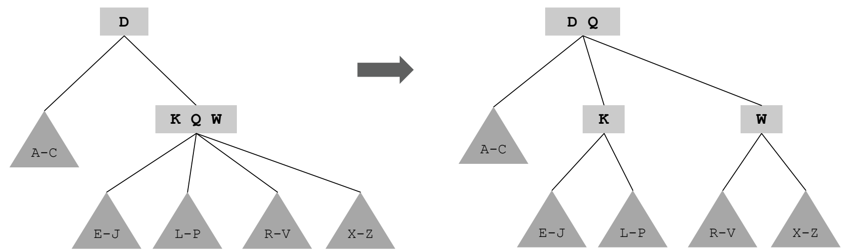


Invariant. Current node is not a 4-node.
Consequence. Insertion at bottom is easy since it's not a 4-node.

2-3-4 Tree: Splitting a Four Node

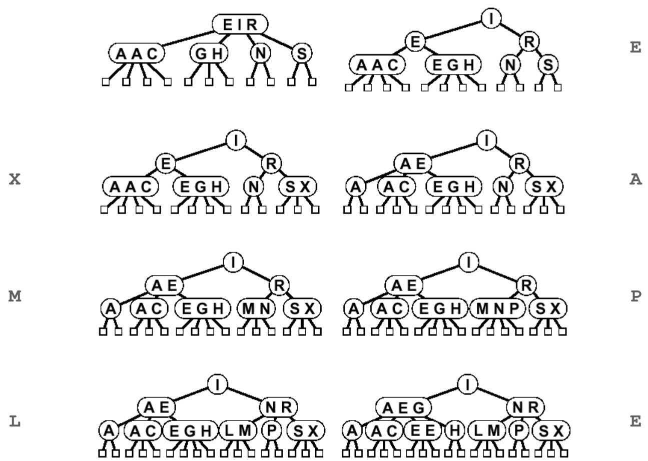
Ex. To split a four node, move middle key up.





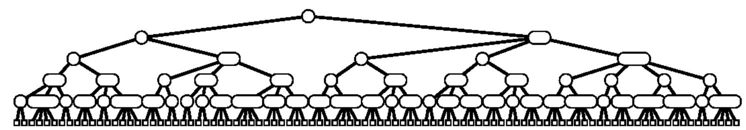
2-3-4 Tree

Tree grows up from the bottom.



2-3-4 Tree: Balance

Property. All paths from root to leaf have same length.



Tree height.

- Worst case: lg N [all 2-nodes]
- Best case: $\log_4 N = 1/2 \lg N$ [all 4-nodes]
- Between 10 and 20 for a million nodes.
- Between 15 and 30 for a billion nodes.

2-3-4 Tree: Implementation?

Direct implementation. Complicated because of:

- Maintaining multiple node types.
- Implementation of `getChild()`.
- Large number of cases for `split()`.

```
private void insert(Key key, Val val) {
   Node x = root;
   while (x.getChild(key) != null) {
      x = x.getChild(key);
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
}
```

fantasy code

Symbol Table: Implementations Cost Summary

	Worst Case			Average Case		
Implementation	Search	Insert	Delete	Search	Insert	Delete
Sorted array	log N	N	N	log N	N	N
Unsorted list	N	1	1	N	1	1
Hashing	N	1	N	1*	1*	1*
BST	N	N	N	log N †	log N †	log N †
Randomized BST	log N ‡	log N ‡	log N ‡	log N	log N	log N
Splay	log N §	log N §	log N §	log N §	log N §	log N §
2-3-4	log N	log N	log N	log N	log N	log N

* assumes hash map is random for all keys
† N is the number of nodes ever inserted
‡ probabilistic guarantee
§ amortized guarantee

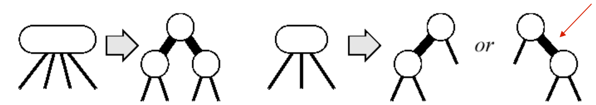
Note. Comparison within nodes not accounted for.

Red-Black Trees

Red-Black Tree

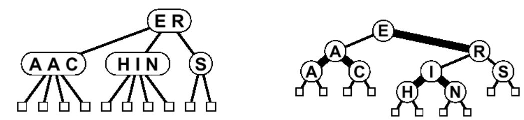
Represent 2-3-4 tree as a BST.

- Use "internal" edges for 3- and 4- nodes.



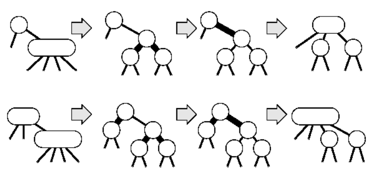
not 1-1 because 3-nodes swing either way.

- Correspondence between 2-3-4 trees and red-black trees.



Red-Black Tree: Splitting Nodes

Two easy cases. Switch colors.



Red-Black Tree: Splitting Nodes

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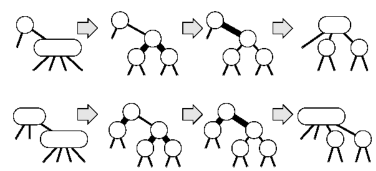
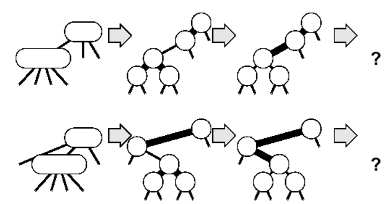


Two hard cases. Use rotations.



do single rotation

do double rotation

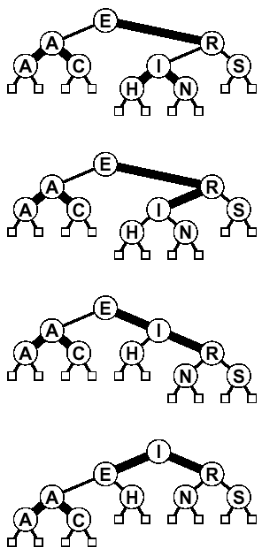
Red-Black Tree: Splitting Nodes

inserting G

change colors

right rotate R →

left rotate E →



Red-Black Tree: Insertion

E

X

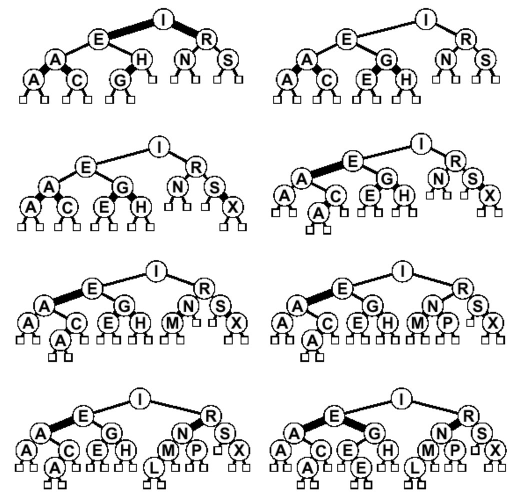
A

M

P

L

E

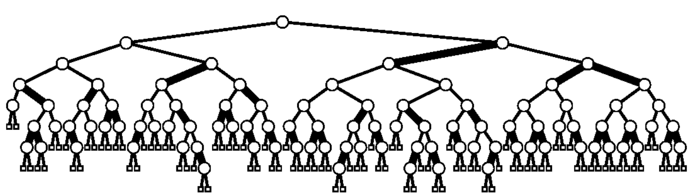


Red-Black Tree: Balance

Property A. Every path from root to leaf has same number of black links.

Property B. At most one red link in-a-row.

Property C. Height of tree is less than 2 lg N + 2.



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Randomized BST	log N ‡	log N ‡	log N ‡	log N	log N	log N
Splay	log N §	log N §	log N §	log N §	log N §	log N §
Red-Black	log N	log N	log N	log N	log N	log N

* assumes hash map is random for all keys
† N is the number of nodes ever inserted
‡ probabilistic guarantee
§ amortized guarantee

Note. Comparison within nodes are accounted for.

Red-Black Trees: Practice

Red-black trees vs. splay trees.
- Fewer rotations than splay trees. ← at most 2 per insertion
- One extra bit per node for color. ← possible to eliminate

Red-black trees vs. hashing.
- Hashing code is simpler and usually faster: arithmetic to compute hash vs. comparison.
- Hashing performance guarantee is weaker.
- BSTs have more flexibility and can support wider range of ops.

In the wild. Red-black trees are widely used as system symbol tables.
- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: `linux/rbtree.h`.

B-Trees

B-Tree

B-Tree. Generalizes 2-3-4 trees by allowing up to M links per node.

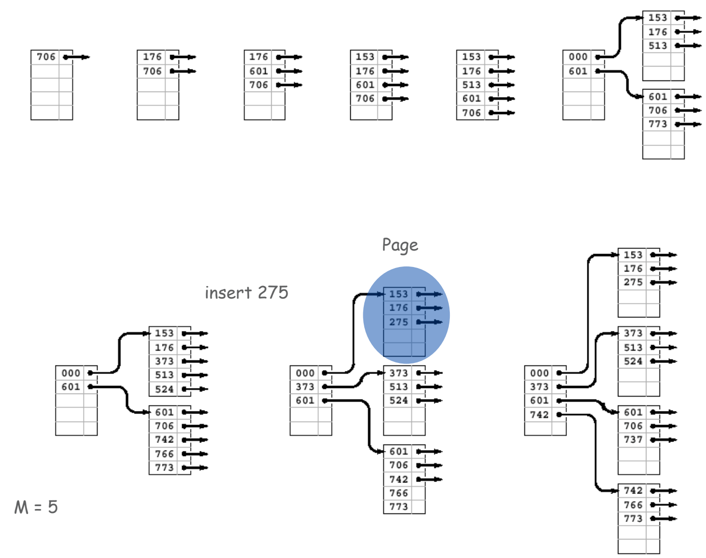
Main application: file systems.
- Reading a page into memory from disk is expensive.
- Accessing info on a page in memory is free.
- Goal: minimize # page accesses.
- Node size M = page size.

Space-time tradeoff.
- M large ⇒ only a few levels in tree.
- M small ⇒ less wasted space.
- Typical M = 1000, N < 1 trillion.

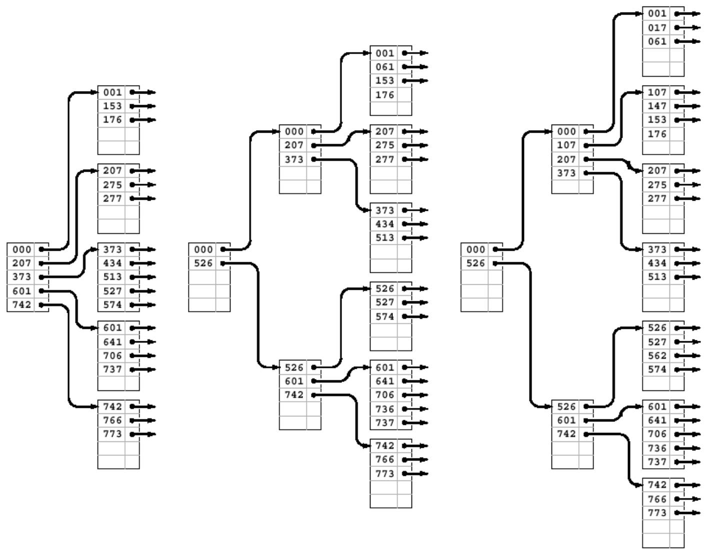
Bottom line. Number of page accesses is $\log_M N$ per op.

3 or 4 in practice!

B-Tree Example



B-Tree Example (cont)



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	Worst Case			Average Case		
Implementation	Search	Insert	Delete	Search	Insert	Delete
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Unsorted list	N	N	N	N	N	N
Hashing	N	1	N	1*	1*	1*
BST	N	N	N	log N †	log N †	log N †
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Splay	log N §	log N §	log N §	log N §	log N §	log N §
Red-Black	log N	log N	log N	log N	log N	log N
B-Tree	1	1	1	1	1	1

page accesses

B-Tree. Number of page accesses is $\log_M N$ per op.

effectively a constant

B-Trees in the Wild

Variants.
- B trees: Bayer-McCreight. [1972, Boeing]
- B+ trees: all data in external nodes.
- B* trees: keeps pages at least 2/3 full.
- R-trees for spatial searching: GIS, VLSI.

File systems.
- Windows: HPFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.

Databases.
- Most common index type in modern databases.
- ORACLE, DB2, INGRES, SQL, PostgreSQL, …

Summary

Goal. ST implementation with log N guarantee for all ops.
- Probabilistic: randomized BST.
- Amortized: splay tree.
- Worst-case: red-black tree.
- Algorithms are variations on a theme: rotations when inserting.

Abstraction extends to give search algorithms for huge files.
- B-tree.

Splay Trees

Splay Trees

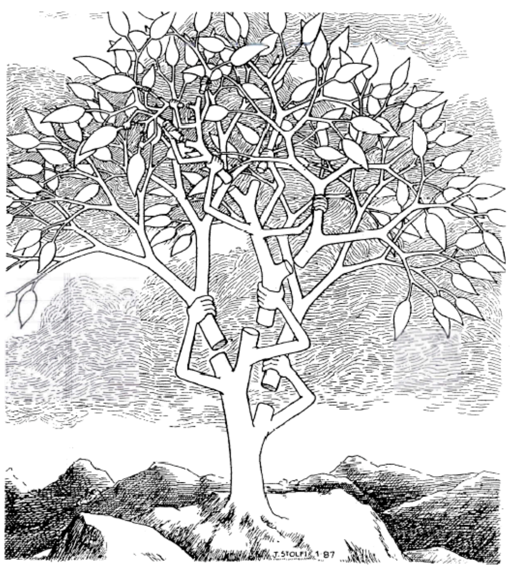
Splay trees = self-adjusting BST.

- Tree automatically reorganizes itself after each op.
- After inserting x or searching for x, rotate x up to root using double rotations.
- Tree remains "balanced" without explicitly storing any balance information.

Amortized guarantee: any sequence of N ops, starting from empty splay tree, takes O(N log N) time.

- Height of tree can be N.
- Individual op can take linear time.

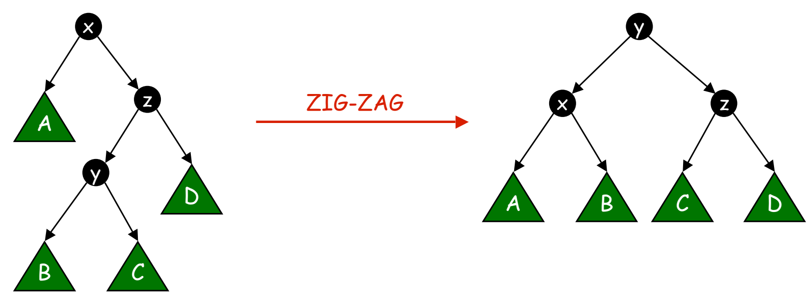
A Self-Adjusting Tree



Splay Trees

Splay.

- Check two links above current node.
- ZIG-ZAG: if orientations differ, same as root insertion.
- ZIG-ZIG: if orientations match, do top rotation first.



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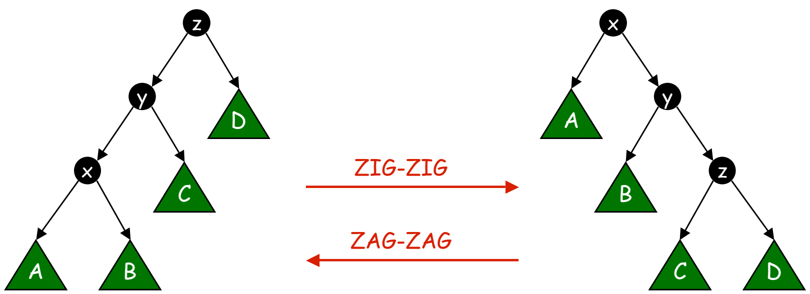

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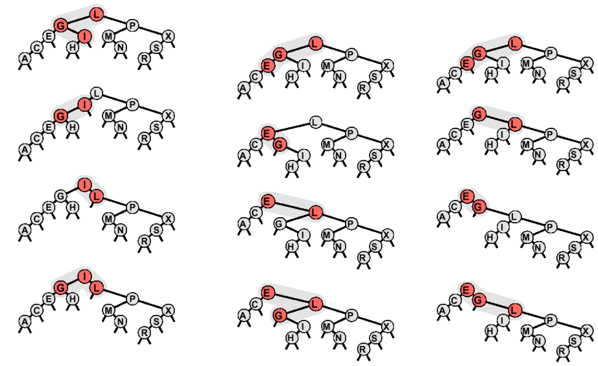
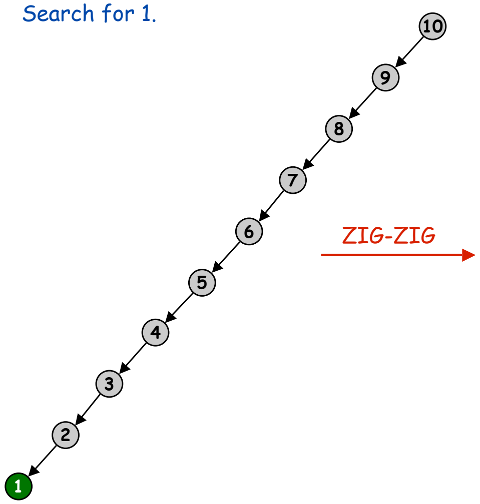


Root = Splay   Root Insertion   Splay Insertion

Splay Example

Search for 1.



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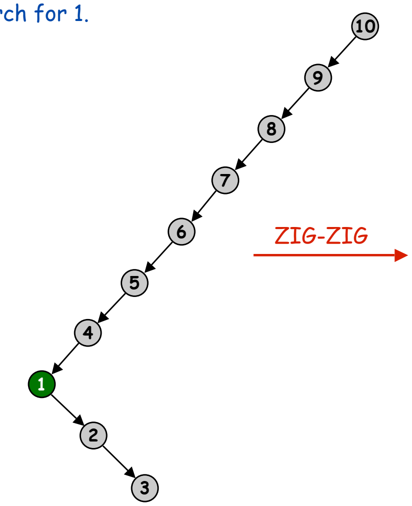

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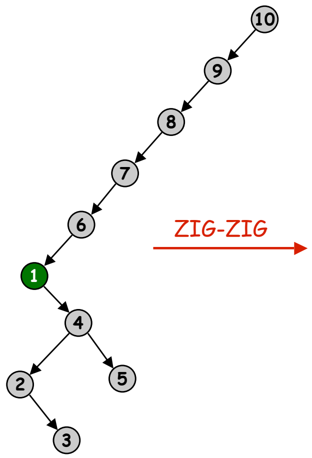

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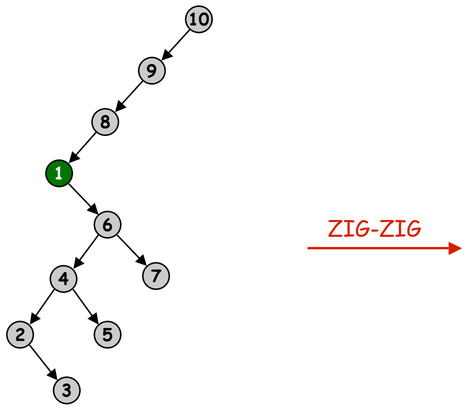

Splay Example

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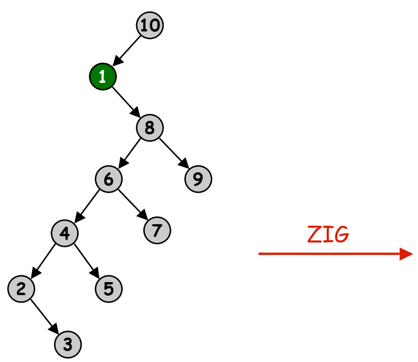

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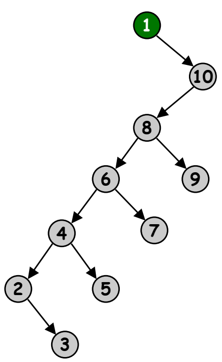
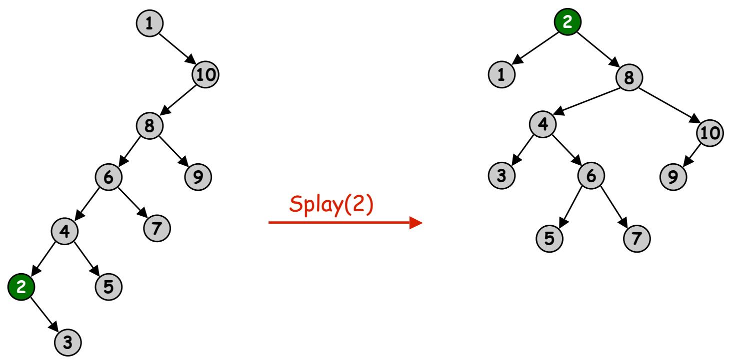


Splay Example

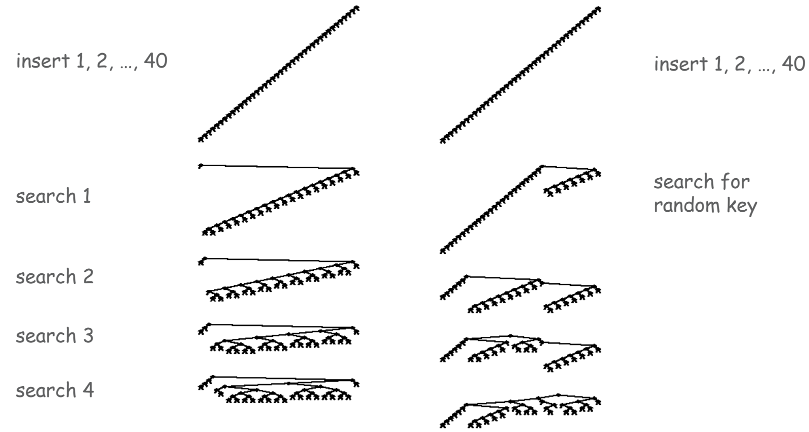
Search for 2.



Splay Trees

Intuition.

- Splay rotations halve search path.
- Reduces length of path for many other nodes in tree.



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BST	N	N	N	log N	log N	sqrt(N) †
Randomized BST	log N ‡	log N ‡	log N ‡	log N	log N	log N
Splay	log N §	log N §	log N §	log N §	log N §	log N §

* assumes we know location of node to be deleted
† if delete allowed, insert/search become sqrt(N)
‡ probabilistic guarantee
§ amortized guarantee

Splay: Sequence of N ops takes linearithmic time.
Ahead: Can we do all ops in log N time guaranteed?