Priority Queues

- API
- elementary implementations
- binary heaps
- heapsort
- event-driven simulation

References: Algorithms in Java, Chapter 9
Intro to Algs and Data Structs, Chapter 3

Priority Queues

Data. Items that can be compared.

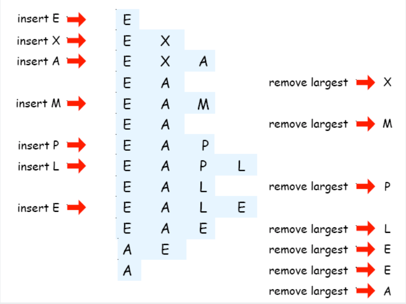
Basic operations.

- Insert.
- Remove largest.

defining ops

- Copy.
- Create.
- Destroy.
- Test if empty.

generic ops



insert E →	E				
insert X →	E	X			
insert A →	E	X	A		
	E	A			remove largest → X
insert M →	E	A	M		
	E	A			remove largest → M
insert P →	E	A	P		
insert L →	E	A	P	L	
	E	A	L		remove largest → P
insert E →	E	A	L	E	
	E	A	E		remove largest → L
	A	E			remove largest → E
	A				remove largest → E
					remove largest → A

API

elementary implementations

binary heaps

heapsort

event-driven simulation

Priority Queue Applications

- Event-driven simulation.	[customers in a line, colliding particles]
- Numerical computation.	[reducing roundoff error]
- Data compression.	[Huffman codes]
- Graph searching.	[Dijkstra's algorithm, Prim's algorithm]
- Computational number theory.	[sum of powers]
- Artificial intelligence.	[A* search]
- Statistics.	[maintain largest M values in a sequence]
- Operating systems.	[load balancing, interrupt handling]
- Discrete optimization.	[bin packing, scheduling]
- Spam filtering.	[Bayesian spam filter]

Generalizes: stack, queue, randomized queue.

Priority queue client example

Problem: Find the largest M of a stream of N elements.

- Fraud detection: isolate $$ transactions.
- File maintenance: find biggest files or directories.

Constraint. Not enough memory to store N elements.
Solution. Use a priority queue.

Operation	time	space
sort	N lg N	N
elementary PQ	M N	M
binary heap	N lg M	M
best in theory	N	M

```
MinPQ<String> pq = new MinPQ<String>();

while(!StdIn.isEmpty())
{
   String s = StdIn.readString();
   pq.insert(s);
   if (pq.size() > M)
      pq.delMin();
}

while (!pq.isEmpty())
   System.out.println(pq.delMin());
```

Priority queue: unordered array implementation

```
public class UnorderedPQ<Item extends Comparable>
{
   private Item[] pq;   // pq[i] = ith element on PQ
   private int N;       // number of elements on PQ

   public UnorderedPQ(int maxN)
   {  pq = (Item[]) new Comparable[maxN];  }

   public boolean isEmpty()
   {  return N == 0; }

   public void insert(Item x)
   {  pq[N++] = x;  }

   public Item delMax()
   {
      int max = 0;
      for (int i = 1; i < N; i++)
         if (less(max, i)) max = i;
      exch(max, N-1);
      return pq[--N];
   }
}
```

no generic array creation

API
elementary implementations
binary heaps
heapsort
event-driven simulation

Priority queue elementary implementations

Implementation	Insert	Del Max
unordered array	1	N
ordered array	N	1

worst-case asymptotic costs for PQ with N items

	unordered	ordered
insert P	P	P
insert Q	P Q	P Q
insert E	P Q E	E P Q
delmax (Q)	P E	E P
insert X	P E X	E P X
insert A	P E X A	A E P X
insert M	P E X A M	A E M P X
delmax (X)	P E M A	A E M P

Challenge. Implement both operations efficiently.

API

elementary implementations

binary heaps

heapsort

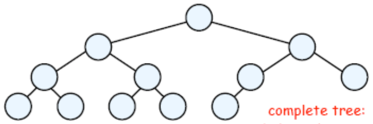
event-driven simulation

Binary Heap

Heap: Array representation of a heap-ordered complete binary tree.

Binary tree.

- Empty or
- Node with links to left and right trees.



complete tree: balanced except for bottom level

Binary Heap

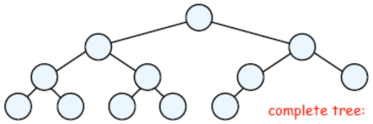
Heap: Array representation of a heap-ordered complete binary tree.

Binary tree.

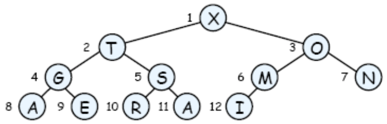
- Empty or
- Node with links to left and right trees.

Heap-ordered binary tree.

- Keys in nodes.
- No smaller than children's keys.



complete tree: balanced except for bottom level



Binary Heap

Heap: Array representation of a heap-ordered complete binary tree.

Binary tree.

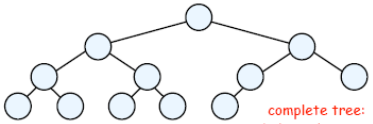
- Empty or
- Node with links to left and right trees.

Heap-ordered binary tree.

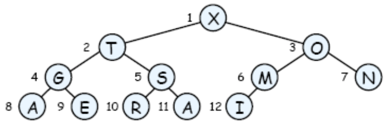
- Keys in nodes.
- No smaller than children's keys.

Array representation.

- Take nodes in level order.
- No explicit links needed since tree is complete.



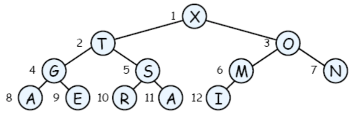
complete tree: balanced except for bottom level



1	2	3	4	5	6	7	8	9	10	11	12
X	T	O	G	S	M	N	A	E	R	A	I

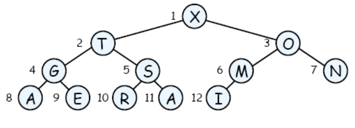
Binary Heap Properties

Property A. Largest key is at root.



Binary Heap Properties

Property A. Largest key is at root.



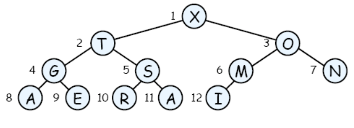
Property B. Can use array indices to move through tree.

- Note: indices start at 1.
- Parent of node at k is at k/2.
- Children of node at k are at 2k and 2k+1.

1	2	3	4	5	6	7	8	9	10	11	12
X	T	O	G	S	M	N	A	E	R	A	I

Binary Heap Properties

Property A. Largest key is at root.



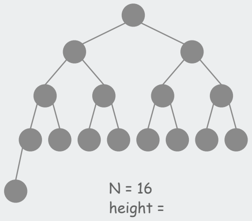
Property B. Can use array indices to move through tree.

- Note: indices start at 1.
- Parent of node at k is at k/2.
- Children of node at k are at 2k and 2k+1.

1	2	3	4	5	6	7	8	9	10	11	12
X	T	O	G	S	M	N	A	E	R	A	I

Property C. Height of N node heap is $1 + \lfloor \lg N \rfloor$.

height only increases when N is a power of 2



Promotion In a Heap

Scenario. Exactly one node has a larger key than its parent.

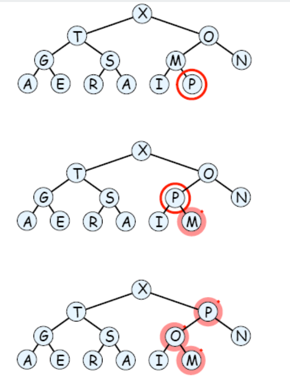
To eliminate the violation:

- Exchange with its parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2

Peter principle: node promoted to level of incompetence.

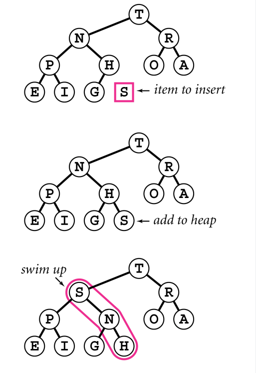


1	2	3	4	5	6	7	8	9	10	11	12	13
X	T	O	G	S	M	N	A	E	R	A	I	P
X	T	P	G	S	O	N	A	E	R	A	I	M

Insert

Insert. Add node at end, then promote.

```
public void insert(Item x)
{
   pq[++N] = x;
   swim(N);
}
```



Demotion In a Heap

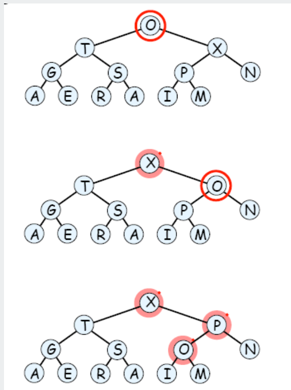
Scenario. Exactly one node has a smaller key than does a child.

To eliminate the violation:

- Exchange with larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2k and 2k+1



1	2	3	4	5	6	7	8	9	10	11	12	13
O	T	X	G	S	P	N	A	E	R	A	I	M
X	T	P	G	S	O	N	A	E	R	A	I	M

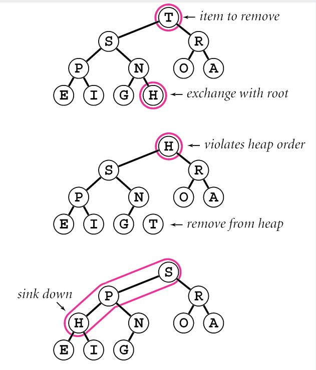
Power struggle: better subordinate promoted.

Remove the Maximum

Remove max. Exchange root with node at end, then demote.

```
public Item delMax()
{
   Item max = pq[1];
   exch(1, N--);
   sink(1);
   pq[N+1] = null;
   return max;
}
```

prevent loitering



Binary heap implementation summary

```
public class MaxPQ<Item extends Comparable>
{
   private Item[] pq;
   private int N;

   public MaxPQ(int maxN)
   {   . . .   }
   public boolean isEmpty()
   {   . . .   }

   public void insert(Item x)
   {   . . .   }
   public Item delMax()
   {   . . .   }

   private void swim(int k)
   {   . . .   }
   private void sink(int k)
   {   . . .   }

   private boolean less(int i, int j)
   {   . . .   }
   private void    exch(int i, int j)
   {   . . .   }
}
```

same as array-based PQ, but allocate one extra element

PQ ops

heap helper functions

array helper functions

Binary heap considerations

Minimum oriented priority queue. Replace `less()` with `greater()` and implement `greater()`.

Array resizing. Add no-arg constructor, and apply repeated doubling.

O(log N) amortized time per op

Immutability of keys. We assume client does not change keys while they're on the PQ. Best practice: make keys immutable.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.
- Can implement using **`sink()`** and **`swim()`** abstractions, but we defer.

Priority Queues Implementation Cost Summary

Operation	Insert	Remove Max	Find Max
ordered array	N	1	1
ordered list	N	1	1
unordered array	1	N	N
unordered list	1	N	N
binary heap	lg N	lg N	1

worst-case asymptotic costs for PQ with N items

Hopeless challenge. Make all ops O(1). Why hopeless?

API
elementary implementations
binary heaps
heapsort
event-driven simulation

Digression: Heapsort

First pass: build heap.

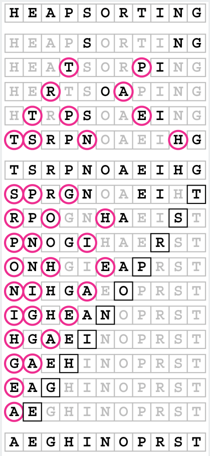
- Insert items into heap, one at at time.
- Or can use faster bottom-up method; see book.

```
for (int k = N/2; k >= 1; k--)
   sink(a, k, N);
```

Second pass: sort.

- Remove maximum items, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
   exch(a, 1, N--);
   sink(a, 1, N);
}
```



H E A P S O R T I N G

H E A P S O R T I N G
H E A T S O R P I N G
H E R T S O A P I N G
H T R P S O A E I N G
T S R P N O A E I H G

T S R P N O A E I H G
S P R G N O A E I H T
R P O G N H A E I S T
P N O G I H A E R S T
O N H G I E A P R S T
N I H G A E O P R S T
I G H E A N O P R S T
H G A E I N O P R S T
G A E H I N O P R S T
E A G H I N O P R S T
A E G H I N O P R S T

A E G H I N O P R S T

Property D. At most 2 N lg N comparisons.

Significance of Heapsort

Q. Sort in O(N log N) worst-case without using extra memory?
A. Yes. Heapsort.

Not mergesort? Linear extra space.

in-place merge possible, not practical

Not quicksort? Quadratic time in worst case.

O(N log N) worst-case quicksort possible, not practical.

Heapsort is optimal for both time and space, but:

- inner loop longer than quicksort's.
- makes poor use of cache memory.

Sorting Summary

	In-Place	Stable	Worst	Average	Best	Remarks
Bubble sort	X	X	$N^2 / 2$	$N^2 / 2$	N	never use it
Selection sort	X		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
Insertion sort	X	X	$N^2 / 2$	$N^2 / 4$	N	use as cutoff for small N
Shellsort	X		$N^{1+1/k}$	$N^{1+1/k}$	N	can do better
Quicksort	X		$N^2 / 2$	2N ln N	N lg N	fastest in practice
Mergesort		X	N lg N	N lg N	N lg N	N log N guarantee, stable
Heapsort	X		2 N lg N	2 N lg N	N lg N	N log N guarantee, in-place

key comparisons to sort N distinct randomly-ordered keys

API
elementary implementations
binary heaps
heapsort
event-driven simulation

Molecular dynamics simulation of hard spheres

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.

Hard sphere model.

- Moving particles interact via elastic collisions with each other, and with fixed walls.
- Each particle is a sphere with known position, velocity, mass, and radius.
- No other forces are exerted.

temperature, pressure, diffusion constant

motion of individual atoms and molecules

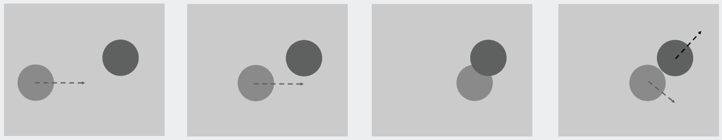
Significance. Relates macroscopic observables to microscopic dynamics.

- Maxwell and Boltzmann: derive distribution of speeds of interacting molecules as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

Time-driven simulation

Time-driven simulation.

- Discretize time in quanta of size dt.
- Update the position of each particle after every dt units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.



t

t + dt

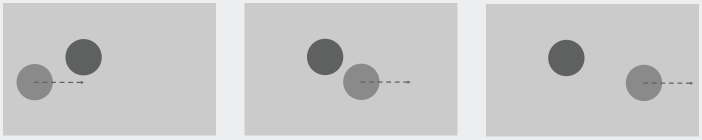
t + 2 dt
(collision detected)

t + Δt
(roll back clock)

Time-driven simulation

Main drawbacks.

- N^2 overlap checks per time quantum.
- May miss collisions if dt is too large and colliding particles fail to overlap when we are looking.
- Simulation is too slow if dt is very small.



t

t + dt

t + 2 dt

Event-driven simulation

Change state only when something happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain priority queue of collision events, prioritized by time.
- Remove the minimum = get next collision.

Collision prediction. Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

Collision resolution. If collision occurs, update colliding particle(s) according to laws of elastic collisions.

Note: Same approach works for a broad variety of systems

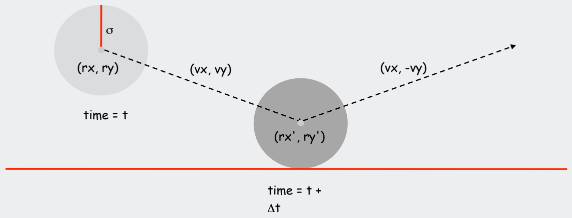
Particle-wall collision

Collision prediction.

- Particle of radius σ at position (rx, ry), moving with velocity (vx, vy).
- Will it collide with a horizontal wall? If so, when?

$$\Delta t \;=\; \begin{cases} \infty & \text{if } vy = 0 \\ (\sigma - ry)/vy & \text{if } vy < 0 \\ (1 - \sigma - ry)/vy & \text{if } vy > 0 \end{cases}$$

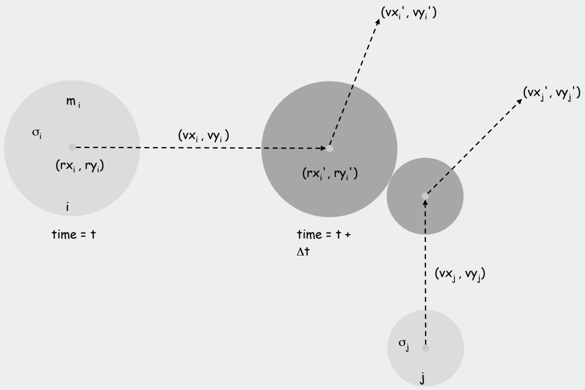
Collision resolution. (vx', vy') = (vx, -vy).



Particle-particle collision prediction

Collision prediction.

- Particle i: radius σ_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius σ_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?



Particle-particle collision prediction

Collision prediction.

- Particle i: radius σ_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius σ_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \ge 0 \\ \infty & \text{if } d < 0 \\ -\dfrac{\Delta v \cdot \Delta r + \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d = (\Delta v \cdot \Delta r)^2 - (\Delta v \cdot \Delta v)\,(\Delta r \cdot \Delta r - \sigma^2) \qquad \sigma = \sigma_i + \sigma_j$$

$$\Delta v = (\Delta vx,\ \Delta vy) = (vx_i - vx_j,\ vy_i - vy_j)$$
$$\Delta r = (\Delta rx,\ \Delta ry) = (rx_i - rx_j,\ ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + (\Delta vy)^2$$
$$\Delta r \cdot \Delta r = (\Delta rx)^2 + (\Delta ry)^2$$
$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + (\Delta vy)(\Delta ry)$$

Particle-particle collision prediction implementation

Particle has method to predict collision with another particle

```
public double dt(Particle b)
{
   Particle a = this;
   if (a == b) return INFINITY;
   double dx  = b.rx - a.rx;
   double dy  = b.ry - a.ry;
   double dvx = b.vx - a.vx;
   double dvy = b.vy - a.vy;
   double dvdr = dx*dvx + dy*dvy;
   if(dvdr > 0) return INFINITY;
   double dvdv = dvx*dvx + dvy*dvy;
   double drdr = dx*dx + dy*dy;
   double sigma = a.radius + b.radius;
   double d = (dvdr*dvdr) - dvdv * (drdr - sigma*sigma);
   if (d < 0) return INFINITY;
   return -(dvdr + Math.sqrt(d)) / dvdv;
}
```

and methods `dtX()` and `dtY()` to predict collisions with walls

Particle-particle collision prediction implementation

CollisionSystem has method to predict all collisions

```
private void predict(Particle a, double limit)
{
   if (a == null) return;
   for(int i = 0; i < N; i++)
   {
      double dt = a.dt(particles[i]);
      if(t + dt <= limit)
         pq.insert(new Event(t + dt, a, particles[i]));
   }
   double dtX = a.dtX();
   double dtY = a.dtY();
   if (t + dtX <= limit)
      pq.insert(new Event(t + dtX, a, null));
   if (t + dtY <= limit)
      pq.insert(new Event(t + dtY, null, a));
}
```

Particle-particle collision resolution

Collision resolution. When two particles collide, how does velocity change?

$$
\begin{aligned}
vx_i' &= vx_i + Jx / m_i \\
vy_i' &= vy_i + Jy / m_i \\
vx_j' &= vx_j - Jx / m_j \\
vy_j' &= vx_j - Jy / m_j
\end{aligned}
$$

Newton's second law
(momentum form)

$$Jx = \frac{J\,\Delta rx}{\sigma},\quad Jy = \frac{J\,\Delta ry}{\sigma},\quad J = \frac{2\,m_i\,m_j\,(\Delta v \cdot \Delta r)}{\sigma(m_i + m_j)}$$

impulse due to normal force
(conservation of energy, conservation of momentum)

Particle-particle collision resolution implementation

Particle has method to resolve collision with another particle

```
public void bounce(Particle b)
{
    Particle a = this;
    double dx  = b.rx - a.rx;
    double dy  = b.ry - a.ry;
    double dvx = b.vx - a.vx;
    double dvy = b.vy - a.vy;
    double dvdr = dx*dvx + dy*dvy;
    double dist = a.radius + b.radius;
    double F = 2 * a.mass * b.mass * dvdr / ((a.mass + b.mass) * dist);
    double Fx = F * dx / dist;
    double Fy = F * dy / dist;
    a.vx += Fx / a.mass;
    a.vy += Fy / a.mass;
    b.vx -= Fx / b.mass;
    b.vy -= Fy / b.mass;
    a.count++;
    b.count++;
 }
```

and methods `bounceX()` and `bounceY()` to resolve collisions with walls

Collision system: event-driven simulation main loop

Initialization.

- Fill PQ with all potential particle-wall collisions
- Fill PQ with all potential particle-particle collisions.

"potential" since collision may not happen if
some other collision intervenes

Main loop.

- Delete the impending event from PQ (min priority = t).
- If the event in no longer valid, ignore it.
- Advance all particles to time t, on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

Collision system: main event-driven simulation loop implementation

```
public void simulate(double limit)
{
    pq = new MinPQ<Event>();
    for(int i = 0; i < N; i++)
        predict(particles[i], limit);
    pq.insert(new Event(0, null, null));
    while(!pq.isEmpty())
    {
        Event e = pq.delMin();
        if(!e.isValid()) continue;
        Particle a = e.a();
        Particle b = e.b();

        for(int i = 0; i < N; i++)
            particles[i].move(e.time() - t);
        t = e.time();

        if      (a != null && b != null) a.bounce(b);
        else if (a != null && b == null) a.bounceX()
        else if (a == null && b != null) b.bounceY();
        else if (a == null && b == null)
        {
            StdDraw.clear(StdDraw.WHITE);
            for(int i = 0; i < N; i++) particles[i].draw();
            StdDraw.show(20);
            if (t < limit)
              pq.insert(new Event(t + 1.0 / Hz, null, null));
        }
        predict(a, limit);
        predict(b, limit);
     }
 }
```

initialize PQ with collision events and redraw event

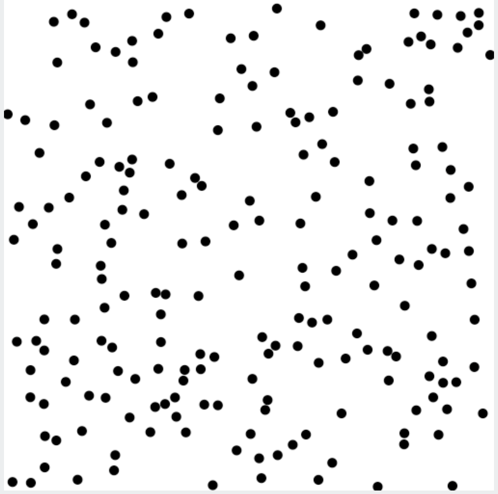
main event-driven simulation loop

update positions and time

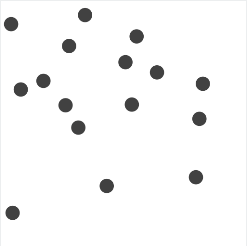
process event

predict new events based on changes

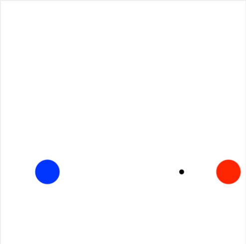
```
java CollisionSystem 200
```



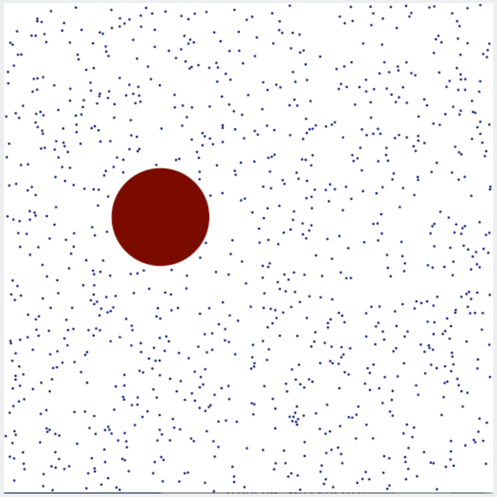
```
java CollisionSystem < billiards5.txt
```



```
java CollisionSystem < squeeze2.txt
```



```
java CollisionSystem < brownianmotion.txt
```



```
java CollisionSystem < diffusion.txt
```