# Analysis of Algorithms

- overview
- case study
- formulating hypotheses

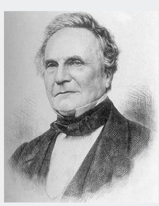
References:
Algorithms in Java, Chapter 2
Intro to Programming in Java, Section 4.1





## Running time

> As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - Charles Babbage



Charles Babbage (1864)

Analytic Engine

how many times do you have to turn the crank?



## Overview

Analysis of algorithms: framework for comparing algorithms and predicting performance.

Scientific method.
- Observe some feature of the universe.
- Hypothesize a model that is consistent with observation.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate the theory by repeating the previous steps until the hypothesis agrees with the observations.

Universe = computer itself.



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## Case study: Sorting

**Sorting problem:**

- Given N items, rearrange them in ascending order.
- Applications: commercial databases, statistics, databases, data compression, computational biology, computer graphics, scientific computing, ...

| | |
|---|---|
| ... | ... |
| Hauser | Hanley |
| Hong | Haskell |
| Hsu | Hauser |
| Hayes | Hayes |
| Haskell | Hong |
| Hanley | Hornet |
| Hornet | Hsu |
| ... | ... |



## Insertion sort

**Insertion sort.**

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

```
S O R T E X A M P L E

O S R T E X A M P L E
O R S T E X A M P L E
O R S T E X A M P L E
E O R S T X A M P L E
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A E L M O P R S T X E
A E E L M O P R S T X

A E E L M O P R S T X
```



## Insertion sort

**Insertion sort.**

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

```java
public static void InsertionSort(double[] a)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0; j--)
         if (a[j] < a[j-1])
             exch(a, j, j-1);
         else break;
}
```



## Insertion sort: Observation

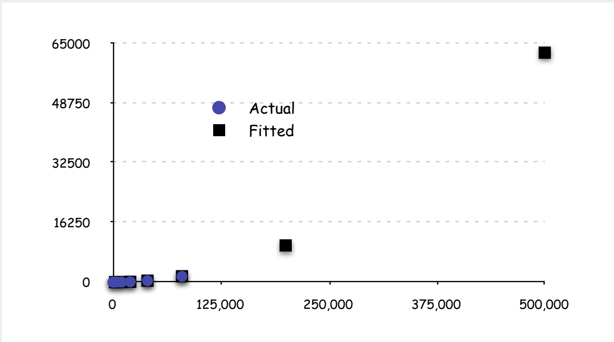
**Observe and tabulate operation counts for various values of N.**

- concentrate on most frequently performed operation (comparisons for sorting)
- Data source: N random numbers between 0 and 1.

| N | Comparisons |
|---|---|
| 5,000 | 6 million |
| 10,000 | 25 million |
| 20,000 | 99 million |
| 40,000 | 398 million |
| 80,000 | 1600 million |

## Insertion sort: Experimental hypothesis

Data analysis. Plot # comparisons vs. input size on log-log scale.




Regression. Fit line through data points $\approx$ a $N^b$.

slope

Hypothesis. # comparisons grows quadratically with input size $\approx N^2/4$.



## Insertion sort: Prediction and verification

Experimental hypothesis. # comparisons $\approx N^2/4$.

Prediction. 400 million comparisons for N = 40,000.

Observations.

| N | Comparisons |
|---|---|
| 40,000 | 401.3 million |
| 40,000 | 399.7 million |
| 40,000 | 401.6 million |
| 40,000 | 400.0 million |

Agrees.

Prediction. 10 billion comparisons for N = 200,000.

Observation.

| N | Comparisons |
|---|---|
| 200,000 | 9.997 billion |

Agrees.



## Experimental vs. theoretical hypotheses

Experimental hypothesis.
- Measure running times, plot, and fit curve.
- Model useful for predicting.

Theoretical hypothesis.
- Analyze algorithm to estimate # comparisons as a function of:
  - number of elements N to sort
  - average or worst case input
- Model useful for predicting and explaining.

Difference. Theoretical model is independent of a particular machine or compiler; applies to machines not yet built.



## Insertion sort: Theoretical hypothesis

Worst case. [descending]
- Iteration i requires i comparisons.
- Total = 0 + 1 + 2 + ... + N-2 + N-1 $\approx N^2/2$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| E | F | G | H | I | J | D | C | B | A |
| | | | | | | i | | | |

Average case. [random]
- Iteration i requires i/2 comparisons on average.
- Total = 0 + 1/2 + 2/2 + ... + (N-1)/2 $\approx N^2/4$.

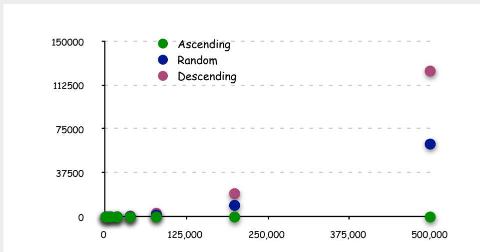
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | C | D | F | H | J | E | B | I | G |
| | | | | | | i | | | |

## Insertion sort: Theoretical hypothesis

Theoretical hypothesis.

| Analysis | Input | Comparisons | Stddev |
|---|---|---|---|
| Worst | Descending | $1/2\ N^2$ | - |
| Average | Random | $1/4\ N^2$ | $1/6\ N^{3/2}$ |
| Best | Ascending | $N$ | - |

Validation. Theory agrees with observations.






## Insertion sort: Observation

Observe and tabulate actual running time for various values of N.

- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.

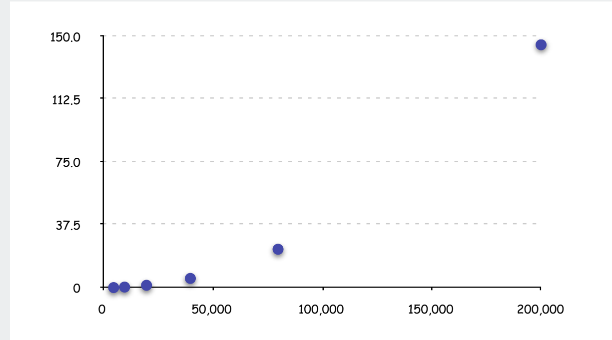
| N | Comparisons | Time |
|---|---|---|
| 5,000 | 6.2 million | 0.13 seconds |
| 10,000 | 25 million | 0.43 seconds |
| 20,000 | 99 million | 1.5 seconds |
| 40,000 | 400 million | 5.6 seconds |
| 80,000 | 1.6 billion | 23 seconds |
| 200,000 | 10 billion | 145 seconds |

Goal: use models to predict running time.



## Insertion sort: A last check

Data analysis. Plot total running time vs. input size on log-log scale.




Regression fit validates hypothesis that total running time is $\sim cN^2$

A scientific connection between program and natural world.



## Timing in Java

Wall clock. Measure time between beginning and end of computation.

- Manual: Skagen wristwatch.
- Automatic: `Stopwatch.java` library.

```
Stopwatch.tic();
. . .
double elapsed = StopWatch.toc();
```

```
public class Stopwatch {
   private static long start;
   public static void tic() {
      start = System.currentTimeMillis();
   }
   public static double toc() {
      long stop = System.currentTimeMillis();
      return (stop - start) / 1000.0;
   }
}
```

## Measuring running time

Factors that affect running time.

- Machine.
- Compiler.
- Algorithm.
- Input data.

More factors.

- Caching.
- Garbage collection.
- Just-in-time compilation.
- CPU used by other processes.

Bottom line. Often difficult to get precise measurements.



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Remaining question. How to formulate a hypothesis?



## Types of hypotheses

Worst case running time. Obtain bound on largest possible running time of algorithm on any input of a given size N.

- Easy to obtain an initial estimate, harder to refine
- Draconian view: real instances may not come close to worst case

Average case running time. Obtain bound on running time of algorithm on random input as a function of input size N.

- Hard to accurately model real instances by random distributions.
- Randomized algorithm: create random distribution.

Amortized running time. Worst-case bound on running time of any sequence of N operations.

## Estimating the Running Time

Total running time: sum of cost × frequency for all of the basic ops.

- Cost depends on machine, compiler.
- Frequency depends on algorithm, input.

Cost for sorting.

- A = # exchanges.
- B = # comparisons.
- Cost on a typical machine = 11A + 4B.

Frequency of sorting ops.

- N = # elements to sort.
- Selection sort: A = N-1, B = N(N-1)/2.

Donald Knuth
1974 Turing Award



## Asymptotic growth

An easier alternative.

(i) Analyze asymptotic growth as a function of input size N.
(ii) For medium N, run and measure time.
(iii) For large N, use (i) and (ii) to predict time.

Asymptotic growth rates.

- Estimate as a function of input size N.
  - N, N log N, $N^2$, $N^3$, $2^N$, N!
- Ignore lower order terms.
  - $6N^3 + 17N^2 + 56$ is approximately $6N^3$
- Formulate hypotheses that cancel constants.
  - doubling hypothesis

$$6*(2\ N)^3 / 6*N^3 = 8$$



## Big Oh Notation

Big Theta, Oh, and Omega notation.

- $\Theta(N^2)$ means { $N^2$, $17N^2$, $N^2 + 17N^{1.5} + 3N$, . . . }
  - ignore lower order terms and leading coefficients
- $O(N^2)$ means { $N^2$, $17N^2$, $N^2 + 17N^{1.5} + 3N$, $N^{1.5}$, 100N, . . . }
  - $\Theta(N^2)$ and smaller
  - use for upper bounds
- $\Omega(N^2)$ means { $N^2$, $17N^2$, $N^2 + 17N^{1.5} + 3N$, $N^3$, $100N^5$, . . . }
  - $\Theta(N^2)$ and larger
  - use for lower bounds

Never use O-notation to predict performance or to compare algorithms.

Little Oh and Tilde notation.

- $o(N^2)$ means { $17N^{1.5} + 3N$, N log N . . . }
  - lower order terms and leading coefficients
- $\sim 6N^2$ means { $6N^2$, $6N^2 + 17N^{1.5} + 3N$, $6N^2 + N^{1.5}$, $6N^2 + 100N$, . . . }
  - leading term
  - use to predict performance and compare algorithms



## Predictions and guarantees

Research literature: The running time of an algorithm is (O(f(N))

worst case

advantages

- guaranteed performance
- can ignore constants

problems

- worst-case running time, cannot predict performance
- constants could play a significant role

This course: The running time of an algorithm is ~c f(N)

"expected"

advantages

- can use to predict performance
- can use to compare algorithms

problems

- need to model actual input
- no guarantees

## Why asymptotic growth rate matters

| Run time in nanoseconds --> | | $1.3\ N^3$ | $10\ N^2$ | $47\ N \log_2 N$ | $48\ N$ |
|---|---|---|---|---|---|
| Time to solve a problem of size | 1000 | 1.3 seconds | 10 msec | 0.4 msec | 0.048 msec |
| | 10,000 | 22 minutes | 1 second | 6 msec | 0.48 msec |
| | 100,000 | 15 days | 1.7 minutes | 78 msec | 4.8 msec |
| | million | 41 years | 2.8 hours | 0.94 seconds | 48 msec |
| | 10 million | 41 millennia | 1.7 weeks | 11 seconds | 0.48 seconds |
| Max size problem solved in one | second | 920 | 10,000 | 1 million | 21 million |
| | minute | 3,600 | 77,000 | 49 million | 1.3 billion |
| | hour | 14,000 | 600,000 | 2.4 trillion | 76 trillion |
| | day | 41,000 | 2.9 million | 50 trillion | 1,800 trillion |
| N multiplied by 10, time multiplied by | | 1,000 | 100 | 10+ | 10 |

Reference: More Programming Pearls by Jon Bentley



## Orders of magnitude

| Seconds | Equivalent |
|---|---|
| 1 | 1 second |
| 10 | 10 seconds |
| $10^2$ | 1.7 minutes |
| $10^3$ | 17 minutes |
| $10^4$ | 2.8 hours |
| $10^5$ | 1.1 days |
| $10^6$ | 1.6 weeks |
| $10^7$ | 3.8 months |
| $10^8$ | 3.1 years |
| $10^9$ | 3.1 decades |
| $10^{10}$ | 3.1 centuries |
| . . . | forever |
| $10^{17}$ | age of universe |

| Meters Per Second | Imperial Units | Example |
|---|---|---|
| $10^{-10}$ | 1.2 in / decade | Continental drift |
| $10^{-8}$ | 1 ft / year | Hair growing |
| $10^{-6}$ | 3.4 in / day | Glacier |
| $10^{-4}$ | 1.2 ft / hour | Gastro-intestinal tract |
| $10^{-2}$ | 2 ft / minute | Ant |
| 1 | 2.2 mi / hour | Human walk |
| $10^2$ | 220 mi / hour | Propeller airplane |
| $10^4$ | 370 mi / min | Space shuttle |
| $10^6$ | 620 mi / sec | Earth in galactic orbit |
| $10^8$ | 62,000 mi / sec | 1/3 speed of light |

| | | |
|---|---|---|
| Powers of 2 | $2^{10}$ | thousand |
| | $2^{20}$ | million |
| | $2^{30}$ | billion |

Reference: More Programming Pearls by Jon Bentley



## Constant Time

Constant time. Running time is O(1).

Elementary operations.

- Function call.
- Boolean operation.
- Arithmetic operation.
- Assignment statement.
- Access array element by index.



## Logarithmic Time

Logarithmic time. Running time is O (log N).

Searching in a sorted list. Given a sorted array of items, find index of query item.

O(log N) solution. Binary search.

```
public static int binarySearch(String[] a, String key) {
   int left  = 0;
   int right = a.length - 1;
   while (left <= right) {
      int mid = left + (right - left) / 2;
      int cmp = key.compareTo(a[mid]);
      if      (cmp < 0) right = mid - 1;
      else if (cmp > 0) left  = mid + 1;
      else return mid;
   }
   return -1;
}
```

## Linear Time

Linear time. Running time is O(N).

Find the maximum. Find the maximum value of N items in an array.

```
double max = Double.NEGATIVE_INFINITY;
for (int i = 0; i < N; i++) {
   if (a[i] > max)
      max = a[i];
}
```



## Linearithmic Time

Linearithmic time. Running time is O(N log N).

Sorting. Given an array of N elements, rearrange in ascending order.

~c N log N solution. Mergesort. [stay tuned]

Remark. Ω(N log N) comparisons required. [stay tuned]



## Quadratic Time

Quadratic time. Running time is $O(N^2)$.

Closest pair of points. Given N points in the plane, find closest pair.

~c $N^2$ solution. Enumerate all pairs of points.

```
double min = Double.POSITIVE_INFINITY;
for (int i = 0; i < N; i++){
   for (int j = i+1; j < N; j++) {
      double dx = (x[i] - x[j]);
      double dy = (y[i] - y[j]);
      if (dx*dx + dy*dy < min)
         min = dx*dx + dy*dy;
   }
}
```

Remark. $\Omega(N^2)$ seems inevitable, but this is just an illusion.



## Exponential Time

Exponential time. Running time is $O(a^N)$ for some constant a > 1.

Finbonacci sequence: 1 1 2 3 5 8 13 21 34 55 …

$O(\phi^N)$ solution. Spectacularly inefficient!

$\phi = \frac{1}{2}\left(1+\sqrt{5}\right) = 1.618034....$

```
public static int F(int N) {
   if (n == 0 || n == 1) return n;
   else                  return F(n-1) + F(n-2);
}
```

Efficient solution. $F(N) = \left[\frac{\phi^N}{\sqrt{5}}\right]$

nearest integer function

## Summary of Common Hypotheses

| Complexity | Description | When N doubles, running time |
|---|---|---|
| 1 | Constant algorithm is independent of input size. | does not change |
| $\log N$ | Logarithmic algorithm gets slightly slower as N grows. | increases by a constant |
| $N$ | Linear algorithm is optimal if you need to process N inputs. | doubles |
| $N \log N$ | Linearithmic algorithm scales to huge problems. | slightly more than doubles |
| $N^2$ | Quadratic algorithm practical for use only on relatively small problems. | quadruples |
| $2^N$ | Exponential algorithm is not usually practical. | squares! |