



Logic: From Greeks to philosophers to circuits.

COS 116: 3/15/2005

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Question from Last Lecture

Boolean Expression $E = S \text{ AND } \bar{D}$

Can we deduce an expression for D?

Let's look at truth table...

D	S	E
0	0	0
0	1	1
1	0	0
1	1	0

→

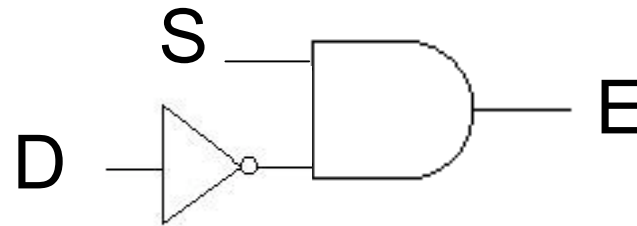
E	S	D
0	0	?
0	1	?
1	0	?
1	1	?

Three Equivalent Representations

Boolean Expression

$$E = S \text{ AND } \bar{D}$$

Boolean Circuit



Truth table:

Value of E for every possible D, S.

TRUE=1; FALSE= 0.

D	S	E
0	0	0
0	1	1
1	0	0
1	1	0

Boolean “algebra”

A **AND** B written as $A \cdot B$

A **OR** B written as $A + B$

$$0 \cdot 0 = 0$$

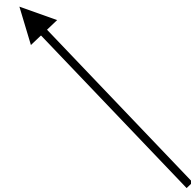
$$0 + 0 = 0$$

$$0 \cdot 1 = 0$$

$$1 + 0 = 1$$

$$1 \cdot 1 = 1$$

$$1 + 1 = 1$$



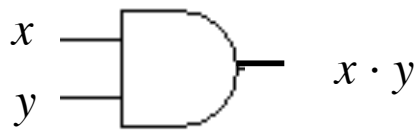
Will provide readings on this...

Funny arithmetic

Boolean gates

Shannon (1939)

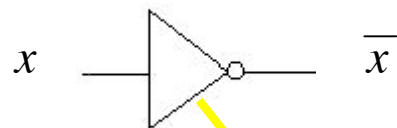
High voltage = 1
Low voltage = 0



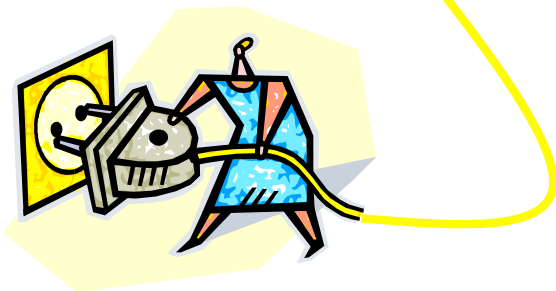
Output voltage is high
if **both** of the input voltages are high;
otherwise output voltage low.



Output voltage is high
if **either** of the input voltages are high;
otherwise output voltage low.



Output voltage is high
if the input voltage is low;
otherwise output voltage high.

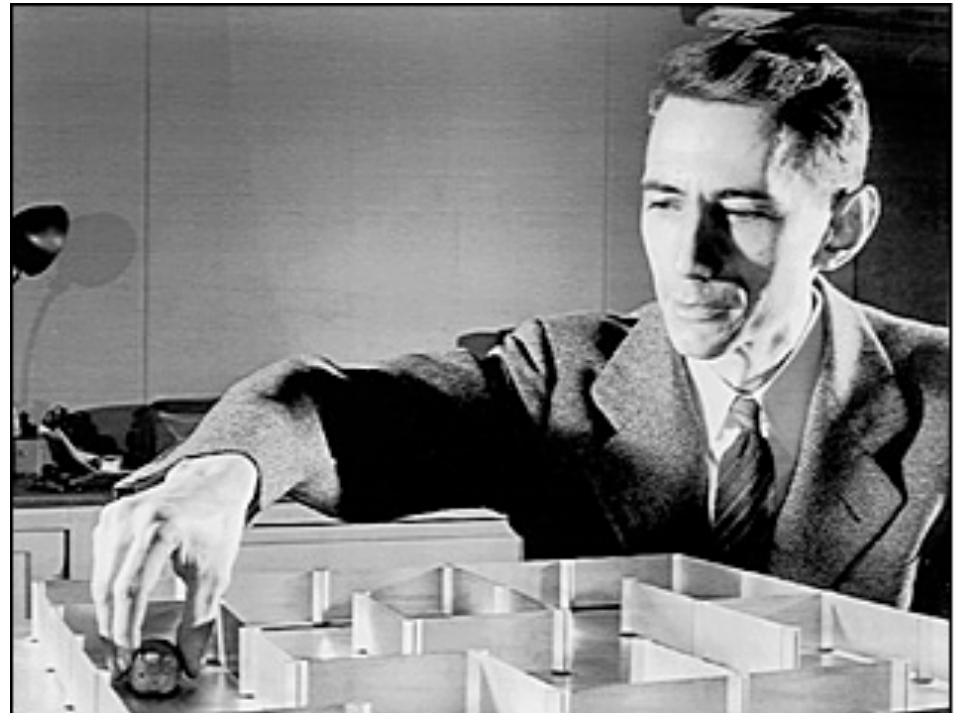


(implicit extra wires for power)

Claude Shannon (1916-2001)

Founder of many fields

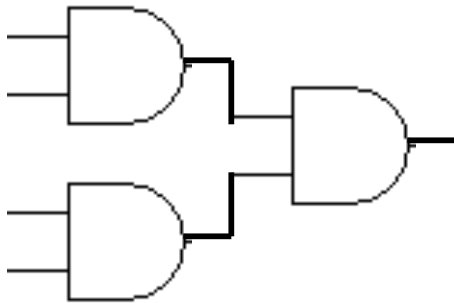
(circuits, information theory, artificial intelligence...)



With “Theseus” mouse

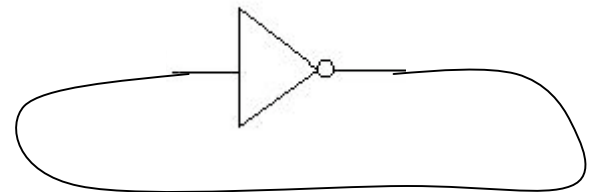
Combinational circuit

- Boolean gates connected by wires



Wires: transmit voltage
(and hence value)

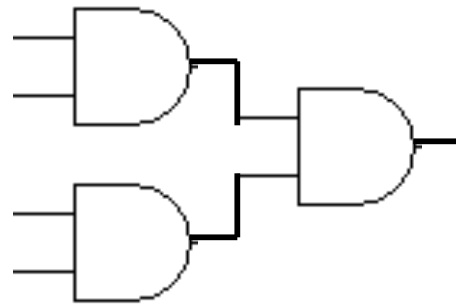
- Important: no cycles allowed



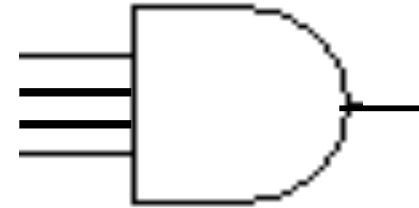


Examples

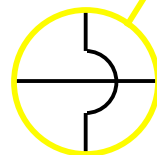
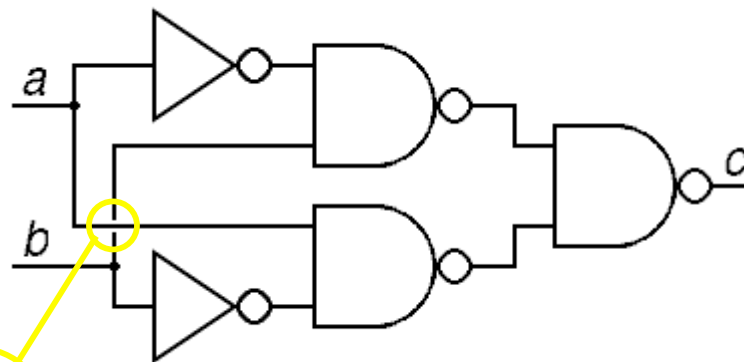
4-way AND



(Sometimes we use this for shorthand)



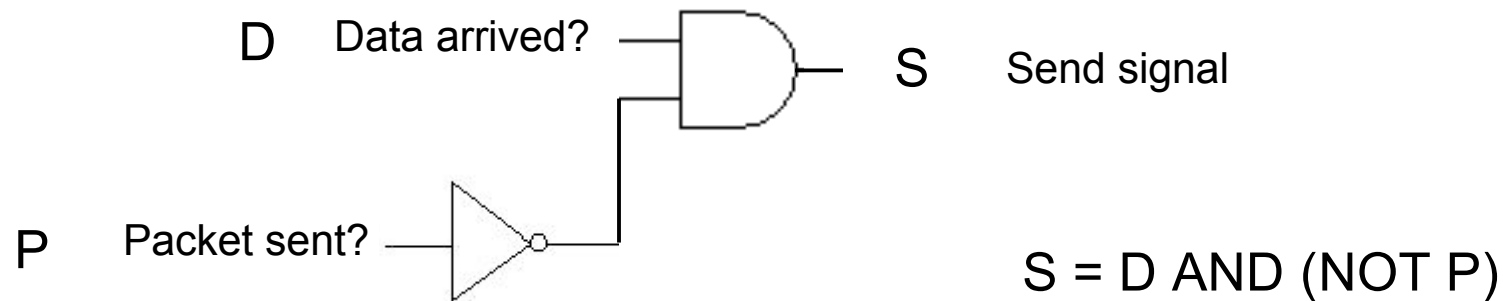
More complicated example



← Crossed wires that are not connected are sometimes drawn like this.

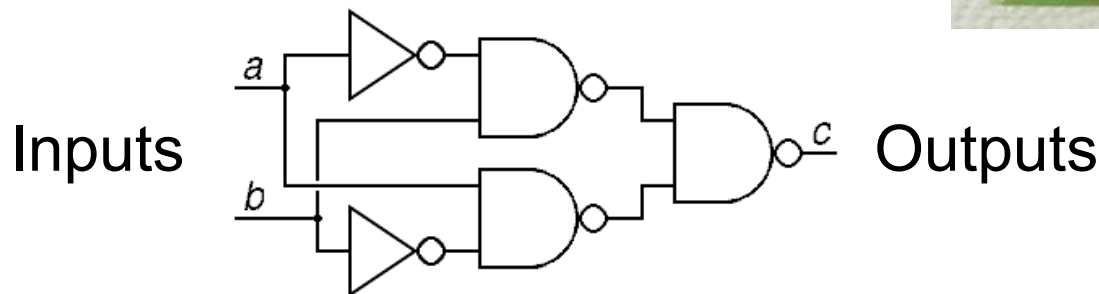
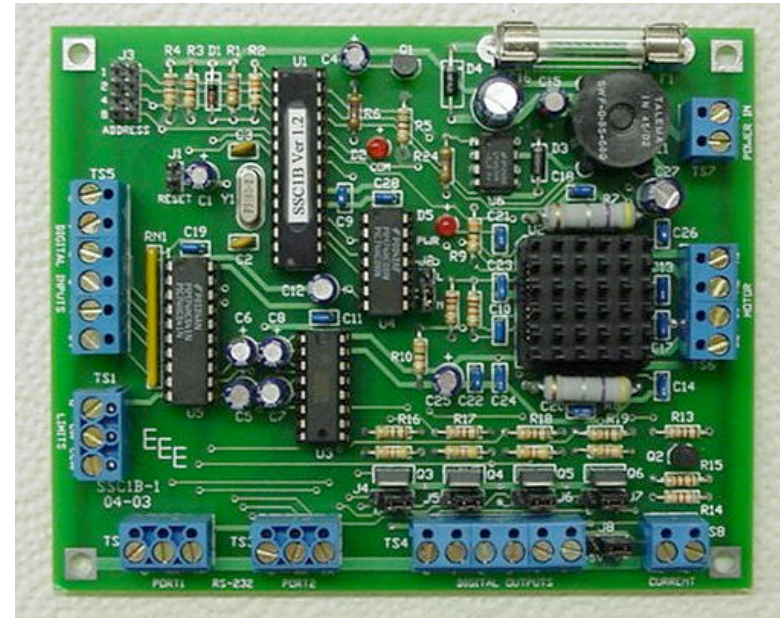
Combinational circuits and control

- “If data has arrived and packet has not been sent, send a signal”



Circuits compute functions

- Every combinational circuit computes a Boolean function of its inputs





Ben Revisited

Ben only rides to class if he overslept, but even then if it is raining he'll walk and show up late (he hates to bike in the rain). But if there's an exam that day he'll bike if he overslept, even in the rain.

B: Ben Bikes

R: It is raining

E: There is an exam today

O: Ben overslept



Ben's truth table

O	R	E	B
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Truth table → Boolean expression

Use **OR** of all input combinations that lead to **TRUE**

$$B = O \cdot \bar{R} \cdot \bar{E} + O \cdot \bar{R} \cdot E + O \cdot R \cdot E$$

O	R	E	B
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Note:

AND, OR, and NOT gates suffice to implement every Boolean function!

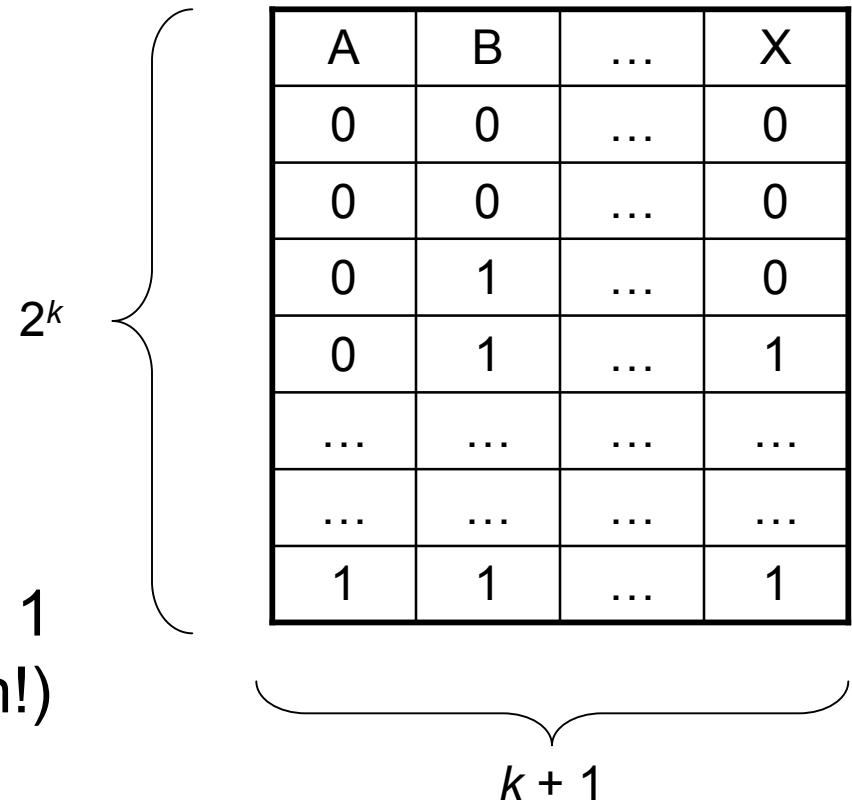
Sizes of representations

- For k variables:

k	10	20	30
2^k	1024	1048576	1073741824

For an arbitrary function,
expect roughly half of X's to be 1
(for 30 inputs roughly 1/2 billion!)

Tools for reducing size:
(a) circuit optimization (b) modular design



Expression simplification

- Some simple rules:

$$x + \bar{x} = 1$$

$$x \cdot 1 = x$$

$$x \cdot 0 = 0$$

$$x + 0 = x$$

$$x + 1 = 1$$

$$x + x = x \cdot x = x$$

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

$$x + (y \cdot z) = (x+y) \cdot (x+z)$$

$$\begin{aligned} x \cdot y + x \cdot \bar{y} \\ &= x \cdot (y + \bar{y}) \\ &= x \cdot 1 \\ &= x \end{aligned}$$

De Morgan's Laws:

$$\overline{x \cdot y} = \bar{x} + \bar{y}$$

$$\overline{x + y} = \bar{x} \cdot \bar{y}$$



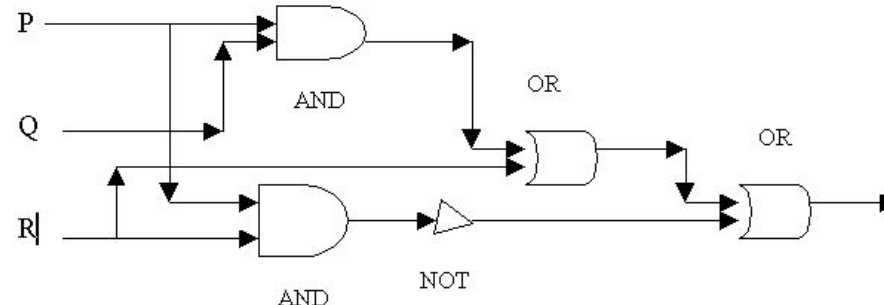
Simplifying Ben's circuit

- $$\begin{aligned} B &= O \cdot \bar{R} \cdot \bar{E} + O \cdot \bar{R} \cdot E + O \cdot R \cdot E \\ &= O \cdot (\bar{R} \cdot \bar{E} + \bar{R} \cdot E + R \cdot E) \\ &= O \cdot (\bar{R} \cdot (\bar{E} + E) + R \cdot E) \\ &= O \cdot (\bar{R} + R \cdot E) \\ &\dots \\ &= O \cdot (\bar{R} + E) \end{aligned}$$

Something to think about: How hard is Circuit Verification?



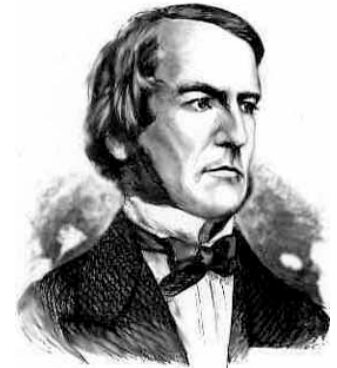
- Given a circuit, decide if it is “trivial” (no matter the input, it either always outputs 1 or always outputs 0)



- Alternative statement: Decide if there is any setting of the inputs that makes the circuit evaluate to 1.

Time required?

Boole's reworking of Clarke's "proof" of existence of God (see handout)

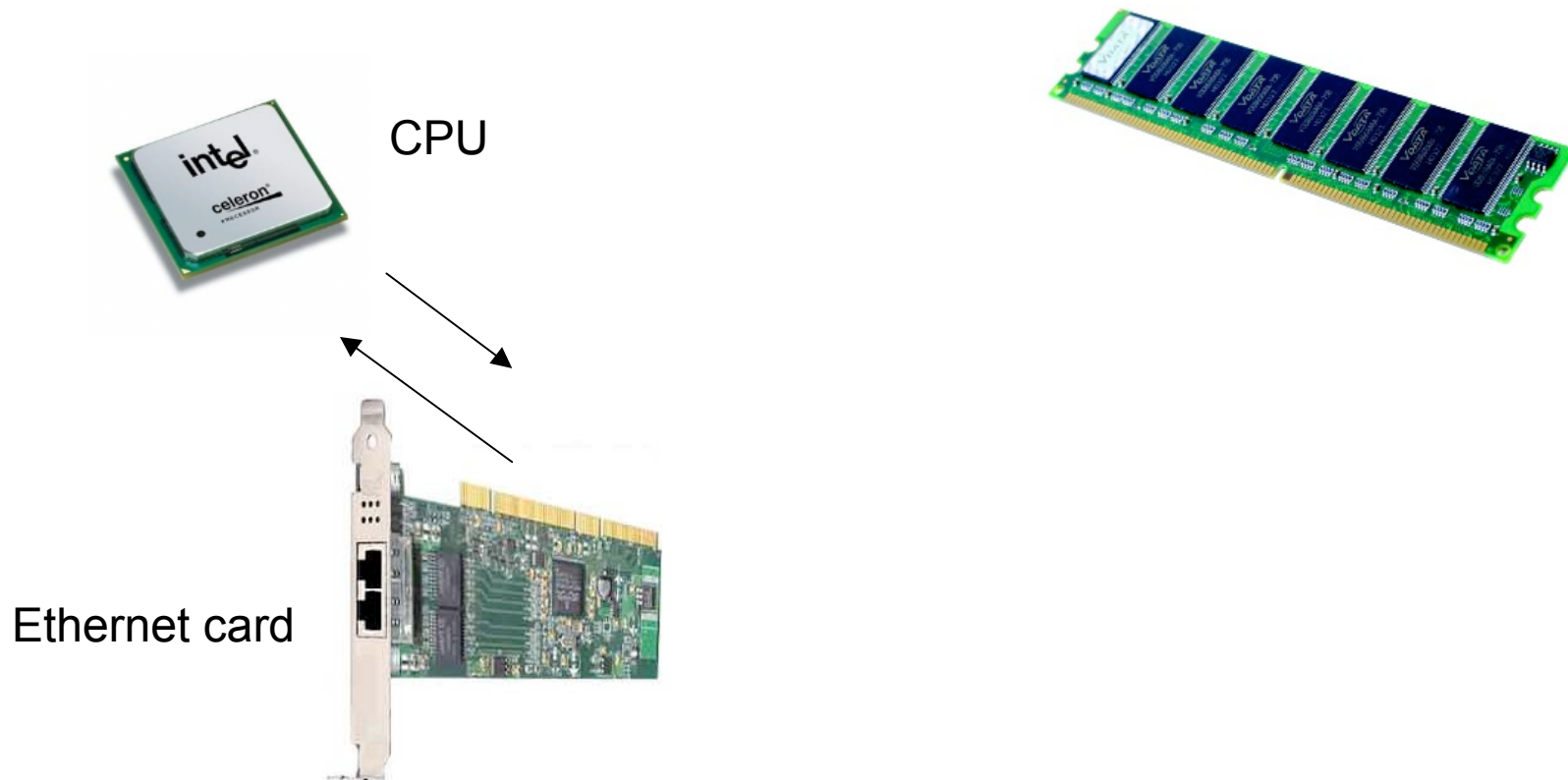


- General idea: Try to prove that Boolean expressions E_1, E_2, \dots, E_k cannot simultaneously be true
- Method: Show $E_1 \cdot E_2 \cdot \dots \cdot E_k = 0$
- Discussion for next time: What exactly does Clarke's "proof" prove? How convincing is such a proof to you?

Also: Do Google search for "Proof of God's Existence."

Beyond combinational circuits ...

- Need 2-way communication (must allow cycles!)
- Need memory (scratchpad)





Circuit for Binary Addition?

$$\begin{array}{r} 25 = 11001 \\ 29 = 11101 \\ \hline 110110 \end{array}$$

Read handout: Will discuss next time.